TEXT FLY IN THE BOOK

PLANE AND SPHERICAL TRIGONOMETRY

BOOKS BY

LYMAN M. KELLS, WILLIS F. KERN,
and JAMES R. BLAND

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Without tables, 401 pages.

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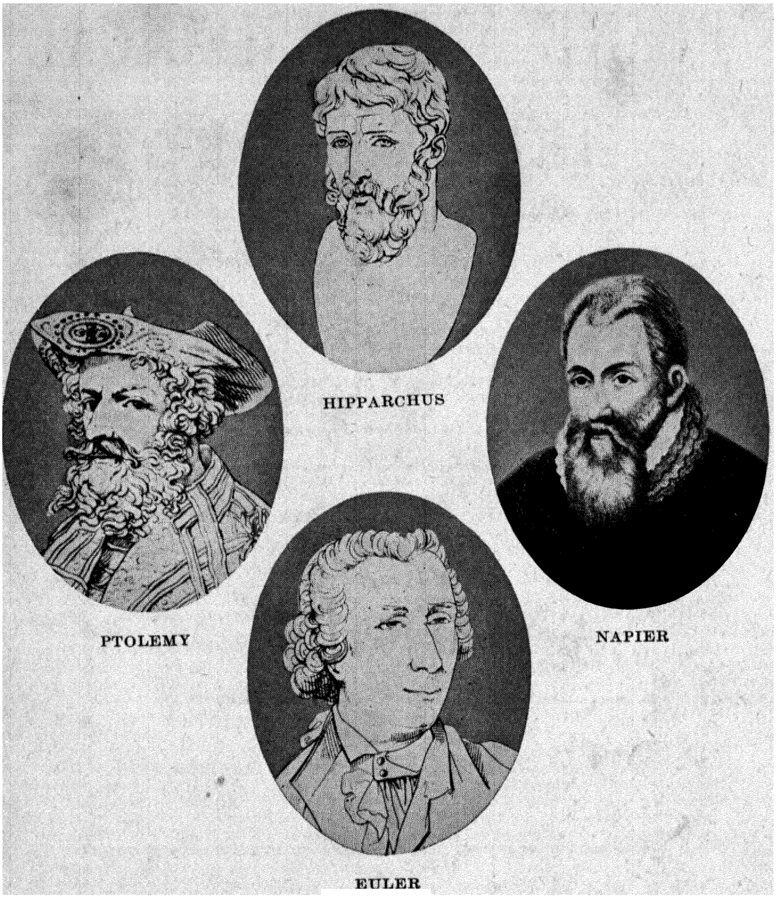

Hipparchus (*c.* 140 B. C.) definitely began the science of trigonometry by working out a table of chords, that is, of double sines of half the angle.

Claude Ptolemy (*c.* 150) did for astronomy what Euclid did for plane geometry. His work on astronomy was a standard of excellence for many centuries.

John Napier (1550–1617) invented logarithms. This remarkable invention affects the whole world with constantly increasing power.

Leonard Euler (1707–1783) was, in a sense, the creator of modern mathematical expression. The equation $e^{ix} = \cos x + i \sin x$ is called by his name.

PLANE AND SPHERICAL TRIGONOMETRY

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PREFACE

The improvements attempted in this revision fall roughly into three main categories, namely: those obtained by enlarging the old lists of problems and by supplying new lists; those obtained by employing a psychological approach to trigonometry and to each of its main branches; and those obtained by using freely suggestions and criticisms derived from classroom experience.

Each original list of problems has been greatly amplified, and new review lists have been introduced. These are supplemented by numerous pictures which are interesting in themselves, and which serve the purpose of visually calling the student's attention to the direct nature of the applications. They suggest to his mind the actual situation and the reality of the problem. These problems and pictures will provide both teacher and student with a wide range of motivating and interesting material.

The greatest of care has been exercised in presenting an introductory chapter that will at once grip the student's interest and give him a firm foundation for the trigonometrical superstructure. A number of elementary applications of fundamental ideas to familiar everyday situations illustrate both principle and application; they make the definitions appear natural and useful and thus furnish initial motivation. These lead to practical problems with figures and to exercises in which the right triangle appears in various positions and others in which it appears as part of a rectilinear figure. Solving these exercises teaches the student the practical value and power of trigonometry while giving that thorough working knowledge of the definitions which enables the student to grasp easily the deductions flowing from them. The same care has been used to follow closely the laws of learning in presenting each new phase of the subject.

A number of the users of the text have given constructive criticisms of many special topics, and the treatment of various ideas has been discussed almost daily by the teachers of mathe-

matics at the Naval Academy. Criticisms and suggestions have been freely employed to make many minor improvements.

The authors gladly take this opportunity to thank all those who have helped with constructive ideas. We are especially indebted to Commander W. P. O. Clarke, who furnished us with many of our newest applications, and to Professor James B. Scarborough, who read the manuscript completely.

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ANNAPOLIS, MD.,
July, 1940.

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GREEK ALPHABET

Letters	Names	Letters	Names	Letters	Names
α	Alpha	ι	Iota	ρ	Rho
β	Beta	κ	Kappa	σ s	Sigma
γ	Gamma	λ	Lambda	τ	Tau
δ	Delta	μ	Mu	υ	Upsilon
ϵ	Epsilon	ν	Nu	φ	Phi
ζ	Zeta	ξ	Xi	χ	Chi
η	Eta	o	Omicron	ψ	Psi
θ	Theta	π	Pi	ω	Omega

LIST OF SYMBOLS

$\equiv$, read *is identical with.*
$\neq$, read *is not equal to.*
$<$, read *is less than.*
$>$, read *is greater than.*
$\leqq$, read *is less than or equal to.*
$\geqq$, read *is greater than or equal to.*
(x, y), read *point whose coordinates are x and y.*

PLANE TRIGONOMETRY

CHAPTER I

TRIGONOMETRIC FUNCTIONS OF AN ACUTE ANGLE

1. Introduction. A cadet who was 6 ft. tall found that his shadow was 3 ft. long (see Fig. 1). He argued that since his height was twice the length of his shadow, the height of a near-by flagpole must be twice the length of its shadow. He then measured the shadow of the flagpole and found that it was 7 ft. long. He concluded that the height of the flagpole was twice the

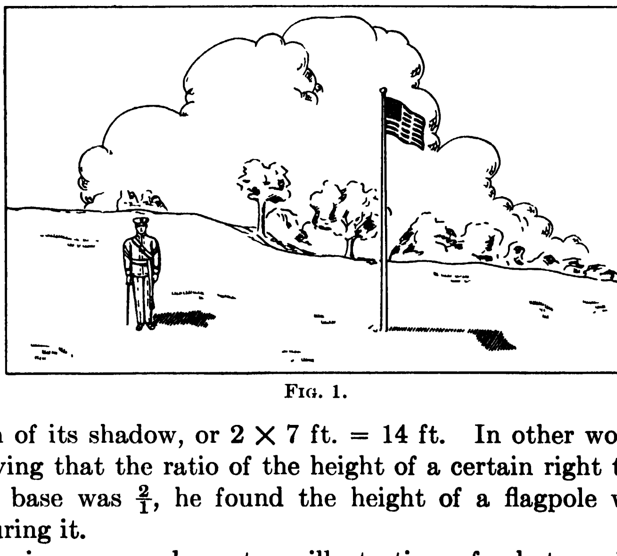


FIG. 1.

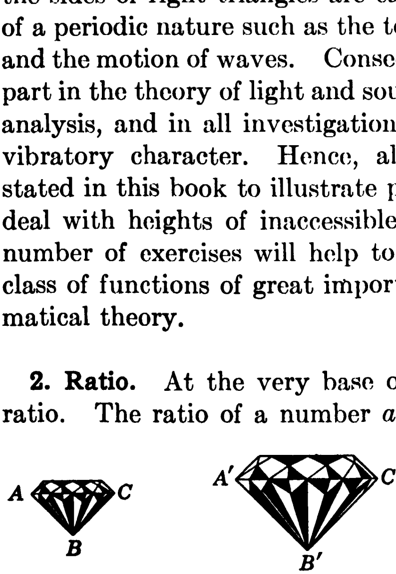
length of its shadow, or 2×7 ft. $= 14$ ft. In other words, by observing that the ratio of the height of a certain right triangle to its base was $\frac{2}{1}$, he found the height of a flagpole without measuring it.

This is a very elementary illustration of what navigators, surveyors, engineers, and others do with trigonometry. By applying the complete theory of the ratios of the sides of a right triangle (that is, trigonometry) to data obtained by measurements, they find inaccessible heights of mountains and distances through them; distances across lakes, rivers, and inaccessible swamps; boundaries of fields and countries; and positions at sea. Engineers use trigonometry every day in their work of constructing large buildings, bridges, and roads; astronomers use it to

determine the time by which clocks are regulated; surveyors use it constantly to find all sorts of heights, distances, and directions; and navigators use it to compute latitude, longitude, and course at sea.

Trigonometry has other very important uses. The ratios of the sides of right triangles are capable of describing phenomena of a periodic nature such as the to-and-fro motion of a pendulum and the motion of waves. Consequently, they play an important part in the theory of light and sound, in electrical theory, in wave analysis, and in all investigations dealing with phenomena of a vibratory character. Hence, although most of the problems stated in this book to illustrate practical phases of trigonometry deal with heights of inaccessible objects and distances, a large number of exercises will help to familiarize the student with a class of functions of great importance in more advanced mathematical theory.

2. Ratio. At the very base of trigonometry lies the idea of ratio. The ratio of a number a to a number b is the quotient a divided by b, that is, a/b; the ratio of two line segments is the ratio of the length of one segment to the length of the other and is independent of the unit of measure; the ratio of a line segment 1 mile long to another 2 miles long is $\frac{1}{2}$, whether the lengths be expressed in miles or in feet.



$\angle ABC = \angle A'B'C'$

FIG. 2.

One of the main reasons for the usefulness of trigonometry is that it furnishes a method of finding ratios associated with angles. One gets some idea of the importance of a knowledge of these ratios by considering the usefulness of models of machines, of blueprints of buildings, and of various kinds of maps. The plane angle made by two straight lines in the model is the same as the angle made by the corresponding lines in the actual structure; therefore the ratios associated with the angles in the model will be the same as those in the corresponding angles in the structure represented. Thus the angles made by corresponding lines in the similar diamonds represented in Fig. 2 are equal. The cadet mentioned in §1 found the height of the flagpole by using the ratio

of the length of an object to that of its shadow. A traveler can find distances approximately by using the fact that map distances have the same ratio as actual distances.

Three important ratios, the fundamental quantities of trigonometry, will be considered in the next article. If A represents any angle, the three ratios are called the tangent of A, the sine of A, and the cosine of A, respectively.

3. The tangent, the sine, and the cosine. If every value of a variable x within a certain interval is associated with a value of another variable y in such a way that when x is given y is determined, then y is a function of x. Thus the area of a square is a function of its side, since when the side is given the area is determined; the distance covered by a car running at a constant speed is a function of the time. Later we shall find that certain ratios of lengths of line segments are functions of angles.

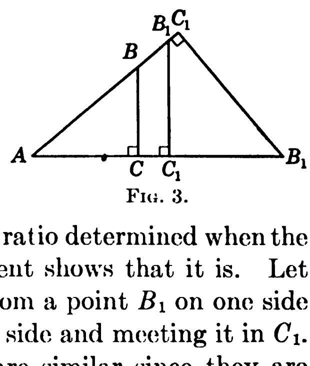


FIG. 3.

Consider any acute angle such as angle A of Fig. 3. From any point B on one side of the angle drop a perpendicular to the other side, meeting it in C, and consider the ratio CB/AC. The question arises: Is the value of this ratio determined when the angle is given? The following argument shows that it is. Let B_1C_1 represent any other line drawn from a point B_1 on one side of the angle perpendicular to the other side and meeting it in C_1. Then the triangles ABC and AB_1C_1 are similar since they are right triangles having an acute angle of one equal to an acute angle of the other. Since corresponding sides of similar triangles are proportional, we have

$$\frac{CB}{AC} = \frac{C_1B_1}{AC_1}. \tag{1}$$

Thus the value of the ratio CB/AC is determined when an acute angle is given. Consequently, in accordance with the definition just given, this ratio is a function of the acute angle. The ratio CB/AC in Fig. 3 is named the *tangent of angle A*, and we write

$$\tan A = \frac{CB}{AC}. \tag{2}$$

Also, two acute angles that have the same tangent are equal. Let A and A' in Fig. 4 be two angles such that

$$\tan A = \tan A'. \tag{3}$$

Construct the right triangles shown in Fig. 4. Then, from (3)

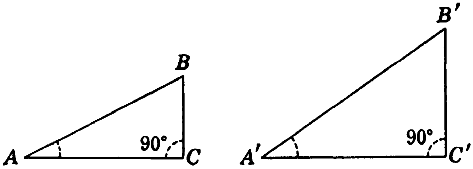


FIG. 4.

and the definition (2),

$$\frac{CB}{AC} = \tan A = \tan A' = \frac{C'B'}{A'C'}. \tag{4}$$

Hence the two triangles in Fig. 2 are similar, having an angle (90°) of one equal to an angle of the other and the including sides proportional. Therefore angle A and angle A', being corresponding angles of similar triangles, are equal.

For convenience, we shall indicate that an angle is a right angle by drawing a small square at its vertex. Thus the small square at C in Fig. 5 shows that angle C is a right angle.

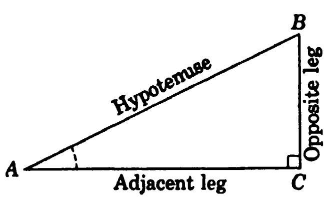


FIG. 5.

Two other ratios, besides the tangent of an angle, are very important. The ratio CB/AB in Fig. 5 is called the *sine* of angle A, and the ratio AC/AB is called the *cosine* of angle A. Using the abbreviations *cos* for *cosine* and *sin* for *sine*, we have from Fig. 5

$$\left.\begin{aligned} \sin A &= \frac{\text{opposite leg}}{\text{hypotenuse}}, \\ \cos A &= \frac{\text{adjacent leg}}{\text{hypotenuse}}, \\ \tan A &= \frac{\text{opposite leg}}{\text{adjacent leg}}. \end{aligned}\right\} \tag{5}$$

These ratios are called trigonometric functions. By using the same line of reasoning applied in the case of the tangent, we can show that *the value of each of the three trigonometric functions of an acute angle is determined when the acute angle is given. Furthermore, it can be shown that if the value of any one of the three trigonometric functions of an acute angle is equal to the value of the same function of a second acute angle, the two acute angles are equal.*

Example 1. Find the values of the three trigonometric functions of an angle A if its sine is $\frac{3}{5}$.

Solution. Draw a right triangle having its hypotenuse 5 units long and one leg 3 units long (see Fig. 6). The acute angle opposite the 3-unit leg is angle A, since its sine is $\frac{3}{5}$. Also, the side $AC = \sqrt{25 - 9} = 4$. Then, from Fig. 6, we read in accordance with the definitions (5)

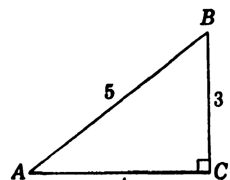


Fig. 6.

$$\sin A = \tfrac{3}{5},$$
$$\cos A = \tfrac{4}{5},$$
$$\tan A = \tfrac{3}{4}.$$

Example 2. A surveyor wishing to find the height of a lighthouse measures the angle A at a point 120 ft. from its base. His findings are represented in Fig. 7, where $\tan A = \frac{2}{3}$. What is the height of the lighthouse?

Solution. From triangle ABC we read

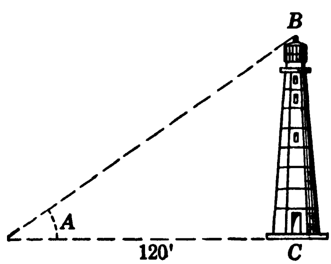


Fig. 7.

$$\tan A = \frac{CB}{AC}, \quad \text{or} \quad \tan A = \frac{CB}{120}.$$

Solving this equation for CB and replacing $\tan A$ by its value $\frac{2}{3}$, we obtain

$$CB = 120 \tan A = 120(\tfrac{2}{3}) = \mathbf{80\ ft.}^*$$

* Throughout this book the answers to illustrative examples will be printed in **boldface** characters.

EXERCISES

1. From each of the Figs. 8, 9, 10, 11, 12, and 13 read tan A and tan B.

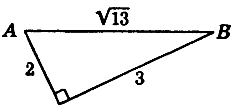


FIG. 8.

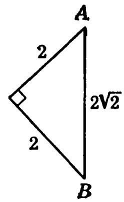


FIG. 9.

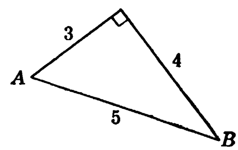


FIG. 10.

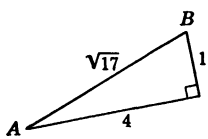


FIG. 11.

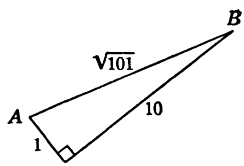


FIG. 12.

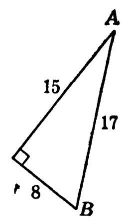


FIG. 13.

2. From each of Figs. 14, 15, 16, and 17 obtain sin A, cos A, and tan A.

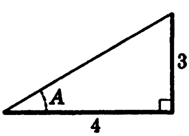


FIG. 14.

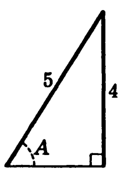


FIG. 15.

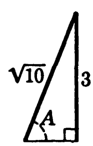


FIG. 16.

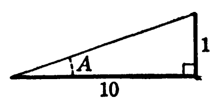


FIG. 17.

3. If $\sin A = \frac{5}{13}$, find $\cos A$ and $\tan A$.

4. If $\cos A = \frac{7}{25}$, find $\sin A$ and $\tan A$.

5. If $\tan A = \frac{8}{15}$, find $\sin A$ and $\cos A$.

6. If $\sin A = \frac{8}{17}$, find $\cos A$ and $\tan A$.

7. If $\cos A = \frac{24}{25}$, find $\sin A$ and $\tan A$.

8. If $\cos A = \frac{15}{17}$, find $\sin A$ and $\tan A$.

9. If $\sin A = \dfrac{1}{\sqrt{2}}$, show that $\sin A = \cos A$.

10. For angle A in Fig. 14, show that

(*a*) $\sin A \cos A = \frac{12}{25}$, (*c*) $(\sin A)^2 + (\cos A)^2 = 1$,

(*b*) $\dfrac{\sin A}{\cos A} \tan A = \frac{9}{16}$, (*d*) $\dfrac{1}{(\cos A)^2} - (\tan A)^2 = 1$.

11. An observer at A (see Fig. 18), 1110 ft. from and on a level with the base of the Washington Monument, sights its top and finds that the angle A is such that $\tan A = \frac{1}{2}$. Find the height of the monument.

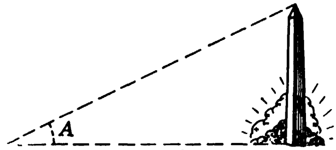


FIG. 18.

12. A base line AC 350 ft. in length is laid along one bank of a river. On the opposite bank a point B is located so that CB is perpendicular to AC. The tangent of the angle CAB is then measured and found to be $\frac{16}{5}$. Find the width of the river.

13. Figure 19 represents a ladder leaning against the side of a house. If the ladder is 36 ft. long and $\cos A = \frac{1}{4}$, how far is the foot of the ladder from the house?

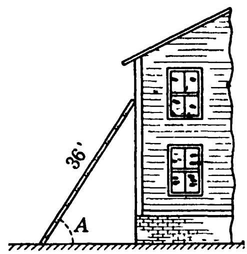


FIG. 19.

14. The length of string between a kite and a point on the ground is 225 ft. If the string is straight and makes with the level ground an angle whose tangent is $\frac{15}{8}$, how high is the kite?

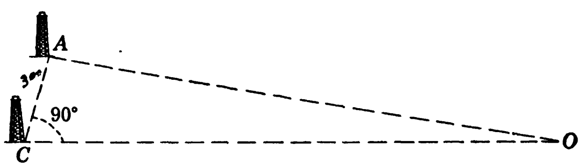


FIG. 20.

15. Figure 20 shows the relative positions of a point O and two oil wells, A and C, 300 ft. apart. An observer at O finds that the sine of angle AOC is $\frac{1}{5}$. What is his distance from the well at A?

4. The cotangent, the secant, and the cosecant. Besides the three ratios (5) of pairs of sides of a right triangle, there are three others got by writing the reciprocals of the ratios in (5). The reciprocals of $\tan A$, $\cos A$, and $\sin A$ are called, respectively, *cotangent A*, *secant A*, and *cosecant A*, and are represented by *cot A*, *sec A*, and *csc A*.

Referring to the right triangle in Fig. 21, we make the following definitions:

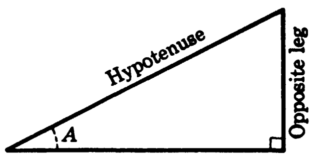


FIG. 21.

$$\left.\begin{aligned} \cot A &= \frac{\text{adjacent leg}}{\text{opposite leg}}, \\ \sec A &= \frac{\text{hypotenuse}}{\text{adjacent leg}}, \\ \csc A &= \frac{\text{hypotenuse}}{\text{opposite leg}}. \end{aligned}\right\} \qquad (6)$$

Just as before, the value of each trigonometric function is determined when the acute angle is given; and if the value of any one of the six trigonometric functions of an acute angle is equal to the value of the same function of a second acute angle, the two acute angles are equal.

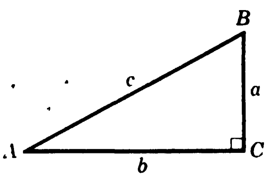


FIG. 22.

Since $y/x = 1 \div (x/y)$, it appears from the definitions (5) and (6) and Fig. 22 that

$$\left.\begin{aligned} \csc A &= \frac{c}{a} = \frac{1}{a/c} = \frac{1}{\sin A}, \\ \sec A &= \frac{c}{b} = \frac{1}{b/c} = \frac{1}{\cos A}, \\ \cot A &= \frac{b}{a} = \frac{1}{a/b} = \frac{1}{\tan A}. \end{aligned}\right\} \qquad (7)$$

It will be well for the student to think of csc A, sec A, and cot A as reciprocals of sin A, cos A, and tan A, respectively; thus, to find csc A, think of the fraction for sin A and then write its reciprocal.

Use is sometimes made of the trigonometric functions defined as follows:

$$\left.\begin{aligned} \text{versed sine of } \theta \text{ (written vers } \theta) &= 1 - \cos\theta, \\ \text{haversine of } \theta \text{ (written hav } \theta) &= \tfrac{1}{2}(1 - \cos\theta), \\ \text{coversed sine of } \theta \text{ (written covers } \theta) &= 1 - \sin\theta. \end{aligned}\right\} \qquad (8)$$

EXERCISES

1. In each of the Figs. 23, 24, 25, 26, 27, and 28 write the six trigonometric functions of angle A.

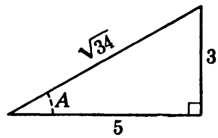


FIG. 23.

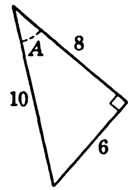


FIG. 24.

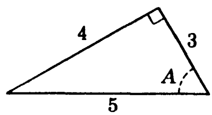


FIG. 25.

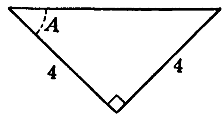


FIG. 26.

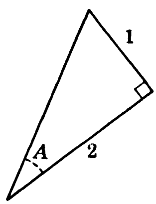


FIG. 27.

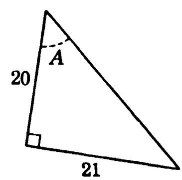


FIG. 28.

2. The sides of a right triangle are 5, 12, and 13, respectively. Read the values of the trigonometric functions of the angle opposite the 5-unit leg. Also read the functions of the angle opposite the 12-unit leg.

3. Find the values of all the trigonometric functions of an acute angle having (*a*) its sine equal to $\frac{4}{5}$; (*b*) its tangent equal to $\frac{8}{15}$; (*c*) its cosine equal to $\frac{1}{2}$.

4. If $\sin A = \frac{6}{7}$, find the value of

(*a*) $(\sin A)^2 + (\cos A)^2$. (*b*) $(\csc A)^2 - (\cot A)^2$.

5. Given that $\sin D = \frac{4}{5}$, $\tan E = \frac{5}{12}$, $\cos F = \frac{8}{17}$, $\cot G = \frac{24}{7}$, show that the following equations are true:

(*a*) $(\cos D)^2 \sec G \cos E = \frac{9}{26}$.
(*b*) $(\csc D)^2 \cot F \cot E = 2$.
(*c*) $\sec E \tan F \cot G \sin G \tan D = \frac{13}{5}$.
(*d*) $\sin D \csc E \sec G \cos E = 2$.
(*e*) $\csc D \cot F \csc G \cos E = \frac{200}{91}$.

6. The relative positions of the point A at the bow of a ship 300 ft. long, C at its stern, and B on a near-by submarine are shown in Fig. 29.

If the tangent of angle *ABC* is $\frac{5}{3}$ and angle *ACB* is 90°, about how far is the submarine from the ship?

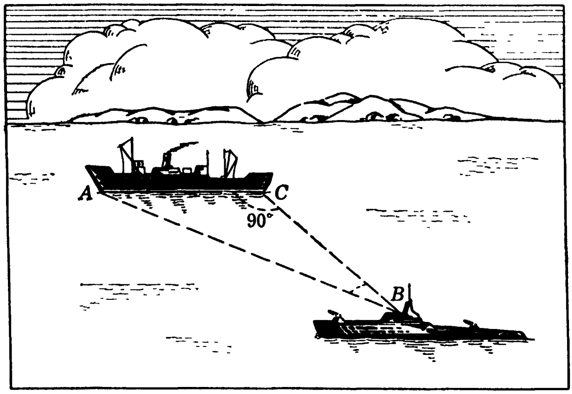


FIG. 29.

7. The central pole of a circular tent is 30 ft. high and is fastened at the top by ropes to stakes set in the ground. Each rope makes an angle *A* with the ground such that csc $A = \frac{3}{2}$. Find the length of each rope.

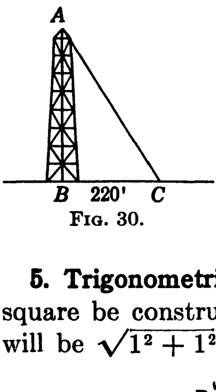


FIG. 30.

8. Figure 30 represents a radio tower. *AC* is a cable anchored at point *C* on a level with the base of the tower. The angle *C* made by the cable with the horizontal is such that sec $C = \frac{9}{5}$. If the distance from *C* to the center *B* of the base is 220 ft., find the length of the cable.

5. Trigonometric functions of 45°, 30°, 60°, 0°, 90°. If a square be constructed with sides 1 unit in length, its diagonal will be $\sqrt{1^2 + 1^2} = \sqrt{2}$ units long and will make a 45° angle with a side (see Fig. 31). Then, from the triangle *ABC* (Fig. 31), we read in accordance with definitions (5) and (6)

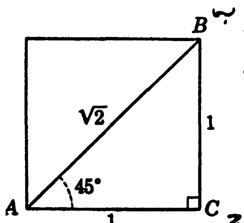


FIG. 31.

$$\begin{aligned}
\sin 45° &= 1/\sqrt{2} = 0.7071,\\
\cos 45° &= 1/\sqrt{2} = 0.7071,\\
\tan 45° &= 1/1 = 1.0000,\\
\csc 45° &= \sqrt{2}/1 = 1.4142,\\
\sec 45° &= \sqrt{2}/1 = 1.4142,\\
\cot 45° &= 1/1 = 1.0000.
\end{aligned}$$

If an equilateral triangle be constructed with sides 2 units in length and if the bisector of one of its angles be drawn, this bisector will have a length of $\sqrt{3}$ units, will make a 30° angle with each of two sides, and will be perpendicular to the third side (see Fig. 32). Hence, from the triangle ABC of Fig. 32, we read

$$\begin{aligned}
\sin 30° &= 1/2 = 0.5000,\\
\cos 30° &= \sqrt{3}/2 = 0.8660,\\
\tan 30° &= 1/\sqrt{3} = 0.5774,\\
\csc 30° &= 2/1 = 2.0000,\\
\sec 30° &= 2/\sqrt{3} = 1.1547,\\
\cot 30° &= \sqrt{3}/1 = 1.7321.
\end{aligned}$$

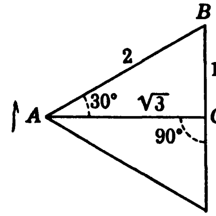


FIG. 32.

Placing the triangle of Fig. 32 in the position shown in Fig. 33, we read from triangle ABC

$$\begin{aligned}
\sin 60° &= \sqrt{3}/2 = 0.8660,\\
\cos 60° &= 1/2 = 0.5000,\\
\tan 60° &= \sqrt{3}/1 = 1.7321,\\
\csc 60° &= 2/\sqrt{3} = 1.1547,\\
\sec 60° &= 2/1 = 2.0000,\\
\cot 60° &= 1/\sqrt{3} = 0.5774.
\end{aligned}$$

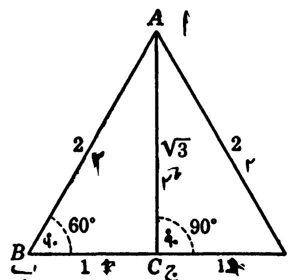


FIG. 33.

The trigonometric functions of 0° are, by definition, the results obtained by placing opposite leg equal to zero and adjacent leg equal to the hypotenuse in the definitions (5) and (6). Hence they may be read from Fig. 34. Since $BC = 0$ and since division by zero is excluded from algebraic operations, it appears that csc 0° and cot 0° are undefined. Nevertheless, we write csc 0° $= \infty$, cot 0° $= \infty$, and mean by these symbols that, as an acute angle θ varies and approaches zero as a limit, the values of csc θ and cot θ vary and become greater and greater without limit. Hence, from Fig. 34, we write

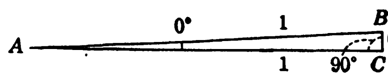


FIG. 34.

$$\begin{aligned}
\sin 0° &= 0, & \csc 0° &= \infty,\\
\cos 0° &= 1, & \sec 0° &= 1,\\
\tan 0° &= 0, & \cot 0° &= \infty.
\end{aligned}$$

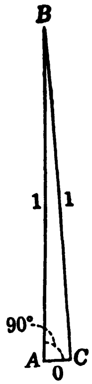


FIG. 35.

Similarly, from Fig. 35, we write

$$\begin{aligned} \sin 90^\circ &= 1, & \csc 90^\circ &= 1, \\ \cos 90^\circ &= 0, & \sec 90^\circ &= \infty, \\ \tan 90^\circ &= \infty, & \cot 90^\circ &= 0. \end{aligned}$$

جا ۹۰ = ۱ ،　　قتا ۹۰ = ۱ ،
جتا ۹۰ = ۰ ،　　قا ۹۰ = ∞ ،
ظا ۹۰ = ∞ ،　　ظتا ۹۰ = ۰ ،

EXERCISES

1. Draw a right triangle, one of whose acute angles is 30°. Assign appropriate lengths to the sides of this right triangle, and from it read the values of the trigonometric functions of 30° and of 60°.

2. Find approximately the values of the trigonometric functions of 1′ by reading them from Fig. 36. From this same figure read the approximate values of the trigonometric functions of 89°59′.

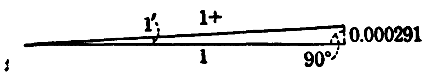


FIG. 36.

3. From Fig. 34 read the values of the trigonometric functions of 0° and of 90°.

4. Draw a triangle from which may be read the values of the trigonometric functions of an angle A whose sine is $\frac{9}{41}$. From this figure read the values of the trigonometric functions of A and of $90^\circ - A$.

5. If $\sec A = 2$, write the trigonometric functions of A.

6. If $\tan A = 1$, write the trigonometric functions of A.

7. Prove that $\cos 60^\circ = 2\cos^2 30^\circ - 1$.

8. Prove that $\tan 30^\circ = \dfrac{\sec 60^\circ}{(\sec 60^\circ + 1)\csc 60^\circ}$.

9. Find the values of each of the following:

(*a*) $\tan 30^\circ \sin 60^\circ \sec 30^\circ \cot 45^\circ$.
(*b*) $\csc 45^\circ \sin 90^\circ \tan 60^\circ \cos 0^\circ$.
(*c*) $\cos 45^\circ \csc 45^\circ - \tan 45^\circ \tan 0^\circ$.
(*d*) $\sin 30^\circ \sin 45^\circ \cos 0^\circ \csc 60^\circ \cot 60^\circ$.

10. Show that

(*a*) $\sin 90° = \sin 30° \cos 60° + \cos 30° \sin 60°.$
(*b*) $\cos 30° = \cos 60° \cos 30° + \sin 60° \sin 30°.$
(*c*) $\sin 30° = \sin 60° \cos 30° - \cos 60° \sin 30°.$

11. If $\tan A = \tan 45° \cos 30° \tan 60°$, find the trigonometric functions of A.

12. That the formulas

$$\sin (A + B) = \sin A \cos B + \cos A \sin B$$
$$\cos (A - B) = \cos A \cos B + \sin A \sin B$$

are true for all values of A and B will be proved in Chap. VI. In these formulas substitute $A = 45°$, $B = 30°$, and evaluate the resulting right-hand members to obtain $\sin 75°$ and $\cos 15°$, respectively..

13. A tree stands at a certain distance from a straight road on which two milestones are located. The tree was observed from each milestone, and the angles between the lines of sight and the road were found to be 30° and 90°, respectively. Find the distance from the tree to the road.

14. The ladder leaning against the wall in Fig. 37 is 45 ft. long. If it makes an angle of 60° with the horizontal, how far is the foot of the ladder from the wall?

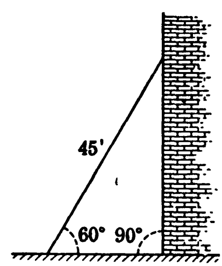


FIG. 37.

15. A farmer wishes to fence a field in the form of a right triangle. If one angle of the triangle is 45° and the hypotenuse is 200 yd., find the amount of fencing needed.

6. Table of values of trigonometric functions. Approximate values of the trigonometric functions of certain angles have been computed and arranged in tabular form. The small table printed here gives, accurate to three decimal places, the values of six trigonometric functions for each of the angles 0°, 5°, 10°, . . . , 90°.

The value of a desired function of an angle is found in the column headed by the name of the function and in the row aving as its first entry the number of degrees in the angle. For

example, in the column headed tan (tangent) and in the row having 25° as its first entry, read tan 25° = 0.466.

Table of Trigonometric Functions

Degrees	sin	cos	tan	cot	sec	csc
0	0 000	1.000	0 000	∞	1.000	∞
5	0 087	0.996	0.087	11.430	1.004	11.474
10	0.174	0 985	0 176	5 671	1 015	5 759
15	0.259	0 966	0 268	3 732	1 035	3 864
20	0.342	0 940	0 364	2.747	1 064	2 924
25	0 423	0 906	0 466	2 145	1 103	2 366
30	0 500	0 866	0 577	1 732	1.155	2 000
35	0 574	0 819	0 700	1.428	1 221	1 743
40	0 643	0 766	0 839	1 192	1.305	1 556
45	0.707	0 707	1.000	1 000	1 414	1 414
50	0 766	0 643	1.192	0 839	1.556	1 305
55	0 819	0 574	1 428	0 700	1 743	1.221
60	0.866	0 500	1 732	0 577	2 000	1 155
65	0 906	0.423	2 145	0 466	2 366	1 103
70	0 940	0.342	2 747	0 364	2 924	1 064
75	0.966	0 259	3 732	0 268	3.864	1 035
80	0.985	0 174	5 671	0 176	5 759	1 015
85	0 996	0 087	11 430	0 087	11.474	1.004
90	1.000	0 000	∞	0 000	∞	1.000

EXERCISES

1. Use the table of this article to verify the following equations:

(*a*) sin 35° = 0.574. (*f*) cot 65° = 0.466.
(*b*) cos 70° = 0.342. (*g*) sin 45° = 0.707.
(*c*) tan 40° = 0.839. (*h*) cos 85° = 0.087.
(*d*) sec 15° = 1.035. (*i*) tan 85° = 11.430.
(*e*) csc 75° = 1.035. (*j*) cos 5° = 0.996.

2. Compute, accurate to three decimal places, sin 45°, tan 45°, sin 30°, sec 30°, csc 30°, sin 60°, sec 45°, and compare with the values of these functions found from the table.

7. Finding heights and distances by means of trigonometric functions. To find an unknown height or distance, one generally draws a figure representing the situation and then finds the part

of it corresponding to the unknown distance. The method of this article for finding the parts of a right triangle differs from the method used in preceding articles only in the way of getting the desired value of a trigonometric function; in preceding problems the function was given; here it must be found in the table of §6. The following rule will be helpful at first.

Rule. *To find an unknown part of a right triangle when a side and another part are given:*

(**a**) *Draw a figure on which are written the values of the known parts and a letter for the unknown part.*

(**b**) *Read from the figure a formula connecting the known parts and the unknown part.*

(**c**) *Replace any trigonometric function of a known angle in the result from step* (**b**) *by its value from the table of* §6.

(**d**) *Solve the result from step* (**c**) *for the unknown part.*

The following example will illustrate the method.

Example. An angle of a right triangle is 55°, and the adjacent leg is 58 units. Find the remaining parts.

Solution. In Fig. 38 the known parts of the right triangle are shown, and the letters B, a, c represent the unknown parts. Evidently $B = 90° - 55° =$ **35°**. From the figure read

$$\frac{a}{58} = \tan 55°. \qquad (a)$$

From the table in §6, $\tan 55° = 1.428$. Substitute this value in (a), and solve the result for a to obtain

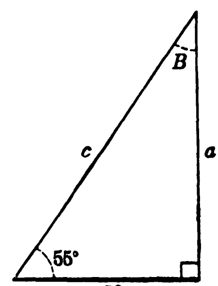


FIG. 38.

$$a = 58(1.428) = \mathbf{82.8.}$$

Repeat the procedure to find c. From Fig. 38,

$$\frac{c}{58} = \sec 55°. \qquad (b)$$

Replace sec 55° by 1.743, its value from the table of §6, in (b), and solve the result for c to obtain

$$c = 58(1.743) = \mathbf{101.1.}$$

A number is rounded off to three significant figures when it is expressed as nearly as possible by means of a first digit different from zero, two digits immediately following the first, and enough zeros to place the decimal point. Thus the figures 84321, 0.05436, 0.5985, 0.5996, when rounded off to three significant figures, become 84300, 0.0544, 0.598, 0.600, respectively.

In order to avoid indicating more accuracy than is warranted when a table accurate to three decimal places is used, round all answers off to three significant figures unless the first significant digit is 1; in this latter case round the answer to four significant figures.

EXERCISES

1. Find the unknown parts of the triangles of Figs. 39 to 42:

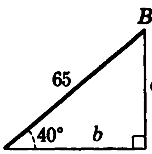


FIG. 39.

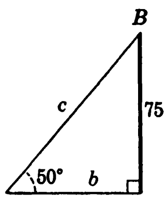


FIG. 40.

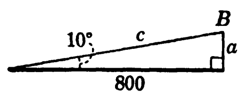


FIG. 41.

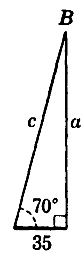


FIG. 42

2. In each of the following exercises, c refers to the hypotenuse of a right triangle, a to the leg opposite the acute angle A, and b to the leg opposite the acute angle B. Solve each of the right triangles in which the known parts are,

(a) $c = 85$, $A = 35°$.
(b) $a = 200$, $B = 80°$.
(c) $a = 500$, $A = 55°$.
(d) $B = 75°$, $c = 20$.
(e) $c = 100$, $A = 25°$.
(f) $b = 60$, $B = 70°$.

3. The hypotenuse of a right triangle is 800 ft., and $\sin A = \frac{12}{13}$. Find the legs of the triangle.

4. The following data refer to right triangles. In each case find the unknown sides.

(a) $c = 520$, $\sin A = \frac{3}{5}$.
(b) $a = 880$, $\cos A = \frac{8}{17}$.
(c) $b = 34$, $\tan B = \frac{1}{2}$.
(d) $c = 250$, $\cot B = \frac{12}{5}$.
(e) $a = 173$, $\csc B = 3$.
(f) $b = 284$, $\sin B = \frac{1}{3}$.

5. A surveyor wishing to find the height of a tower, represented by MN in Fig. 43, stands 90 ft. from its base, measures the angle A, and finds it to be 35°. If the surveyor's eye is 5 ft. above the ground, find the height of the tower.

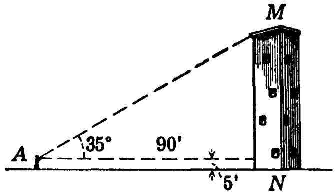


FIG. 43.

6. A city block is in the form of a right triangle with a hypotenuse of 300 ft. If one angle is 35°, find the lengths of the other two sides.

7. In order to find the distance from C to an inaccessible point A (see Fig. 44), line CB, 100 ft. long, was laid off perpendicular to CA, and angle CBA was found to be 70°. Find the distance CA.

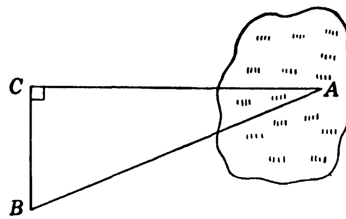


FIG. 44

8. At a point 55 ft. from the base of a flagpole that is standing on level ground the angle of elevation of the top of the pole is 50°. Find the height of the flagpole, correct to the nearest foot.

9. A guy wire from a point 5 ft. from the top of a telephone pole makes an angle of 65° with the level ground and is anchored 15 ft. from the base of the pole, as shown in Fig. 45. How high is the pole?

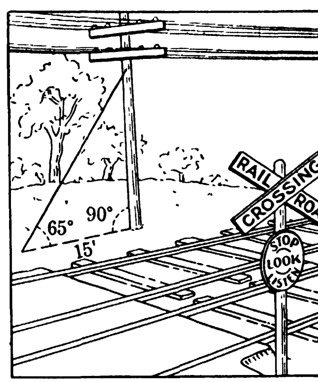


FIG. 45.

10. An airplane starts from a station and rises at an angle of 10° with the horizontal. By how many feet will it clear a vertical wall 100 ft. high and 900 ft. from the station?

11. An observer in a captive balloon is 985 yd. above level ground. The line of direction of the enemy's outpost makes an angle of 80° with the vertical. How far away is the outpost?

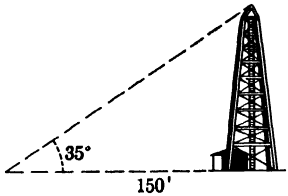


FIG. 46.

12. When the direction of the sun makes an angle of 35° with the horizontal, an oil derrick casts a shadow 150 ft. long. How high is the derrick (see Fig. 46)?

13. In a certain quartz crystal two of the plane faces of the crystal meet at an angle of 50°. If the perpendicular distance from a point A in one face to the other face is 3 cm., find the distance of A from the intersection of the two faces.

14. A plot of ground is in the form of a right triangle, with one leg 10 yd. long and its adjacent angle 20°. Find the length of a fence surrounding the plot.

15. An observer in the airplane shown in Fig. 47 measures the angle ABC and finds it to be 35°. He reads from his altimeter the altitude BC to be 3467 ft. What is the width AC of the island?

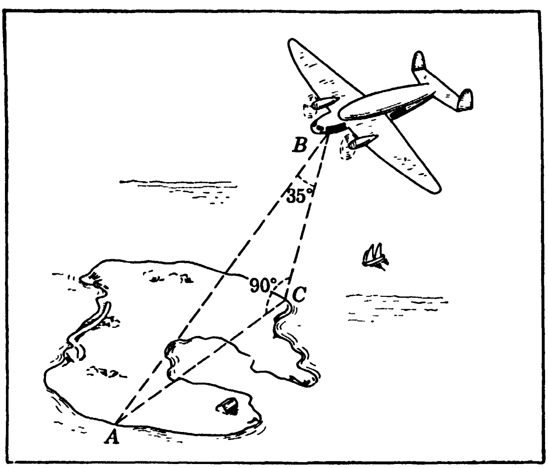


FIG. 47.

16. The shortest side of a field in the form of a right triangle is 300 ft. long. If the angle opposite this side is 40°, find the area of the field.

8. Solving rectilinear figures. If all lines in a figure are straight, the figure is said to be rectilinear. By applying repeatedly the method of solving right triangles explained in §7, all parts of a rectilinear figure can often be found in terms of given

parts. In simple cases, the method consists in locating a right triangle that can be solved and solving it, then finding the parts of a second right triangle that can be solved after the parts of the first one are obtained, then solving a third right triangle, etc. The following example will illustrate the method.

Example. In Fig. 48, $OD = 35$ units, $AB = 29$ units, $\csc x = \frac{5}{4}$, $\tan y = \frac{12}{5}$. Find the lengths of all line segments.

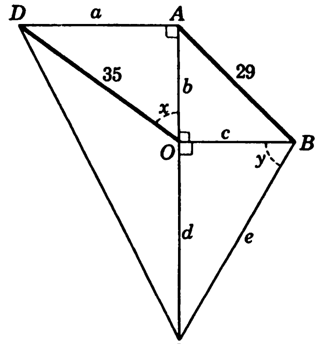


FIG. 48.

Solution. Since $\csc x = \frac{5}{4}$, Fig. 49 may be used to find any function of x; similarly, Fig. 50 may be used to find any function of y. From triangle ODA, $\sin x = a/35$; and from Fig. 49, $\sin x = \frac{4}{5}$. Therefore

$$\frac{a}{35} = \frac{4}{5}, \qquad \text{or} \qquad a = \mathbf{28}.$$

Also from triangle ODA, $\cos x = b/35$, and from Fig. 49, $\cos x = \frac{3}{5}$. Therefore

$$\frac{b}{35} = \frac{3}{5}, \qquad \text{or} \qquad b = \mathbf{21}.$$

Applying the Pythagorean theorem to triangle AOB, if $b = 21$, we have

$$c^2 + 21^2 = 29^2, \qquad \text{or} \qquad c = \sqrt{29^2 - 21^2} = \mathbf{20}.$$

From triangle BOC, $\tan y = d/c = d/20$, and, from Fig. 50, $\tan y = \frac{12}{5}$. Therefore

$$\frac{d}{20} = \frac{12}{5}, \qquad \text{or} \qquad d = \mathbf{48}.$$

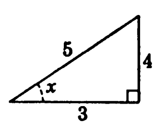


FIG. 49.

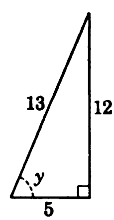


FIG. 50.

From triangle BOC, $\sec y = e/20$, and, from Fig. 50, $\sec y = \frac{13}{5}$. Therefore

$$\frac{e}{20} = \frac{13}{5}, \qquad \text{or} \qquad e = \frac{(13)(20)}{5} = \mathbf{52}.$$

From triangle ADC,

$$DC = \sqrt{\overline{AC}^2 + \overline{AD}^2} = \sqrt{(b + d)^2 + a^2}.$$

Replacing a, b, and d by their values found above, we have

$$DC = \sqrt{(48 + 21)^2 + 28^2} = \mathbf{74.46}.$$

EXERCISES

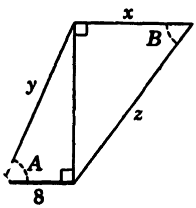


FIG. 51.

1. If, in Fig. 51, $\tan A = \frac{9}{4}$ and $\sec B = \frac{5}{3}$, find x, y, and z.

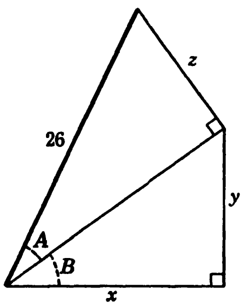


FIG. 52.

2. If, in Fig. 52, $\tan A = \frac{5}{12}$ and $\tan B = \frac{3}{4}$, find x, y, and z.

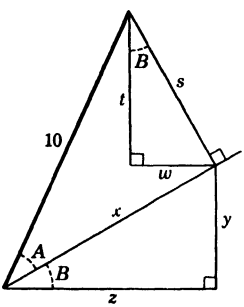


FIG. 53.

3. If, in Fig. 53, $\sin A = \frac{3}{5}$ and $\tan B = \frac{5}{12}$, find s, t, w, x, y, and z.

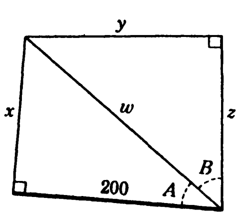


FIG. 54.

4. If, in Fig. 54, $\sin A = \frac{3}{5}$ and $\tan B = \frac{8}{15}$, find the lengths of all the line segments.

5. Find the length of line segment y in Fig. 55.

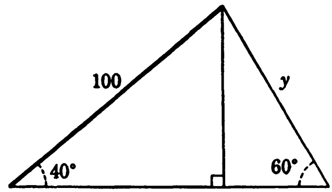


FIG. 55.

6. Find length BD in Fig. 56.

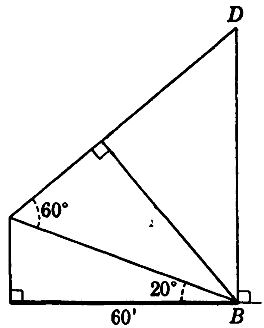


FIG. 56.

7. Find all unknown lengths of line segments in Fig. 57.

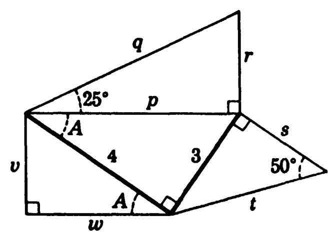


FIG. 57.

9. MISCELLANEOUS EXERCISES

1. In each of the Figs. 58 and 59 read the six trigonometric functions of angle A.

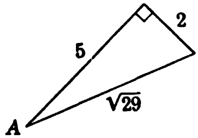


FIG. 58.

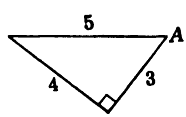


FIG. 59.

2. If $\sec A = \frac{17}{8}$, find $\sin A$, $\cos A$, and $\cot A$.

3. If $\sin A = \frac{3}{5}$, show that

(a) $\cos A \cot A = \frac{16}{15}$.
(b) $\sin^2 A + \cos^2 A = 1$.
(c) $1 + \tan^2 A = \sec^2 A$.
(d) $1 + \cot^2 A = \csc^2 A$.

4. Find the values of the trigonometric functions of an acute angle having (*a*) its sine equal to $\frac{4}{5}$; (*b*) its tangent equal to $\frac{8}{15}$; (*c*) its cosine equal to $\frac{12}{13}$.

5. If $\sin B = \frac{24}{25}$, find the value of

(*a*) $2 \sin B \cos B$. (*b*) $\cos^2 B - \sin^2 B$.

6. If $\sin A = \dfrac{1}{\sqrt{2}}$, find $\sin 2A$ by means of the formula (to be derived later)

$$\sin 2A = 2 \sin A \cos A.$$

7. If $\sin A = \frac{1}{2}$ and $\cos B = \frac{3}{4}$, find the value of $\sin (A + B)$ by means of the formula (to be derived later)

$$\sin (A + B) = \sin A \cos B + \cos A \sin B.$$

8. The base of an isosceles triangle is 30 units, and each of its base angles has $\frac{5}{13}$ as the value of its cosine. Find the lengths of the altitudes and of the sides of the triangle.

9. For a certain triangle ABC, $\sin A = \frac{12}{13}$, $\tan B = \frac{15}{8}$, and the altitude to side AB is 60 units. Find the lengths of the sides and of the altitudes of the triangle.

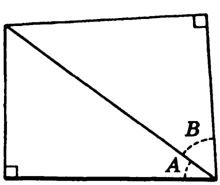


FIG. 60.

10. Find all unknown line segments in Fig. 60 if $\sin A = \frac{3}{5}$, $\tan B = \frac{6}{5}$.

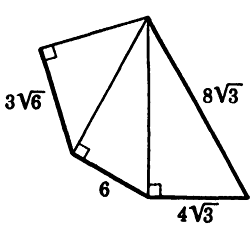


FIG. 61.

11. Find all unknown sides in radical form and all unknown angles in Fig. 61.

12. In Fig. 62 $\tan \alpha = \frac{3}{4}$, $\sin \gamma = \frac{1}{2}$, and $\sin \beta = \frac{24}{25}$. Compute the lengths of the sides of triangle ABC, and write the trigonometric functions of angle ABC.

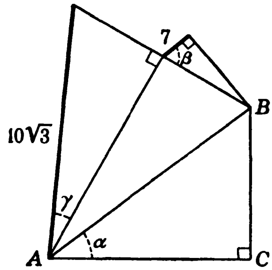


FIG. 62.

13. If, in Fig. 63, $\csc \alpha = \frac{5}{4}$, $AB = 29$ units, $BC = 25$ units, and $OD = 35$ units, find the lengths of all line segments in the figure, and write the values of the trigonometric functions of β, of γ, and of δ. Also find the length of the perpendicular from O to the line DC.

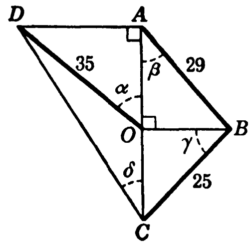


FIG. 63.

14. At a point A in a horizontal plane through the base of a flagpole the angle of elevation of its top is 35°. If the flagpole is 40 ft. high, find the distance from A to the pole.

15. In Fig. 64 CE is the median to side AB of the triangle ABC, $\tan A = \frac{12}{35}$, $AC = 37$ units, and $BD = 5$ units. Find the lengths of all line segments in the figure, and write the trigonometric functions of angle DCE.

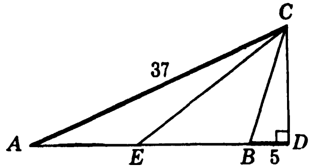


FIG. 64.

16. If, in Fig. 65, $\sin \theta = \frac{3}{5}$, $\cos \varphi = \frac{3}{5}$, $AB = 20$ ft., and $CA = 16$ ft., find the lengths of all line segments in the figure. Also find the values of the trigonometric functions of angle AED.

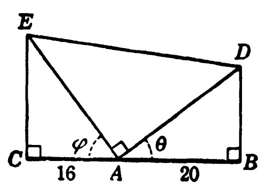


FIG. 65.

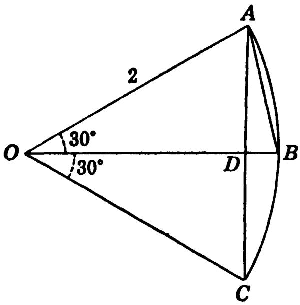


FIG. 66.

17. In Fig. 66 ABC is an arc of a circle with center at O. Prove that angle DAB is 15°. Compute the lengths DB, DA, and AB in radical form, and then write the trigonometric functions of 15°.

18. Construct a figure like Fig. 66 but with 45° in place of 30°. Use the figure to find the trigonometric functions of $22\frac{1}{2}°$.

19. If the map distance BC is 2.5 cm. (see Fig. 67) and if angle $ABC = 25°$, find the map distance AB.

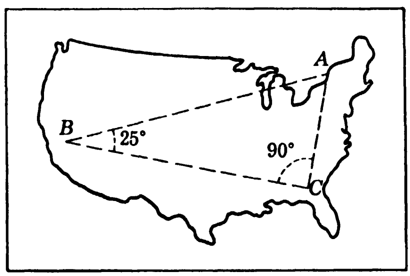


FIG. 67.

20. At a point midway between two trees on a horizontal plane the angles of elevation of their tips are 30° and 60°, respectively. Show that one tree is three times as high as the other.

21. An observer in an airplane (see Fig. 68) 2000 ft. above the sea sights two ships A and B and finds their angles of depression to be 44°

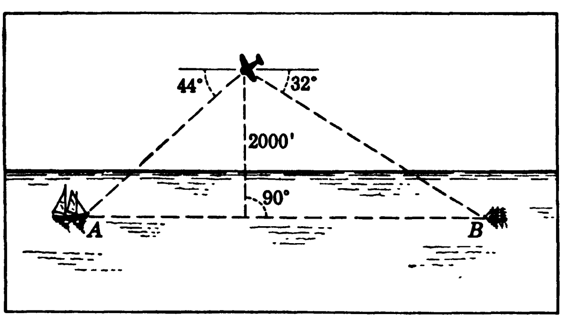


FIG. 68.

and 32°, respectively. If the observer is in the same vertical plane with the ships, find the distance AB (cot 44° = 1.036; cot 32° = 1.600).

22. The mine A in Fig. 69 is attached to the fixed point B by means of the 800-ft. cable AB. When the cable is vertical, the mine is 15 ft. below the surface of the water. How far from the surface is it when the tidal current has swung it to the position A' ($\cos 38° = 0.788$)?

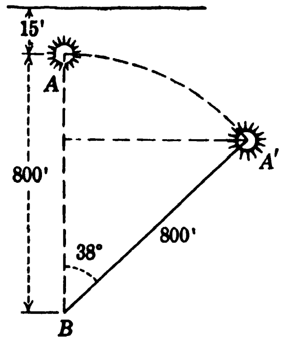


FIG. 69.

23. The ship represented in Fig. 70 steams at a uniform speed due east. At 7 A.M. its captain observes a lighthouse 10 miles away bearing due north, and at 7:30 A.M. he finds that it bears 40° west of north. Find the speed.

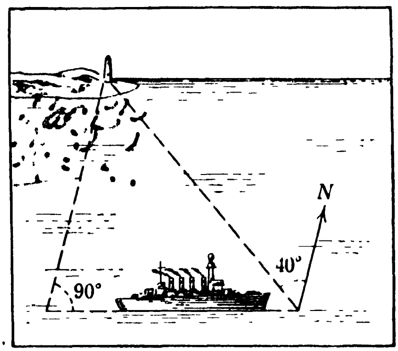


FIG. 70.

24. Prove that the area K of a right triangle (see Fig. 71) may be expressed by

$$K = \tfrac{1}{2}a \times b = \tfrac{1}{2}ac \cos A = \tfrac{1}{2}bc \sin A,$$
$$K = \tfrac{1}{2}b^2 \tan A = \tfrac{1}{2}a^2 \tan B,$$
$$K = \tfrac{1}{2}c^2 \sin A \cos A = \tfrac{1}{2}c^2 \sin B \cos B.$$

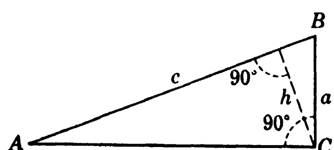


FIG. 71.

CHAPTER II

FUNDAMENTAL RELATIONS AMONG THE TRIGONOMETRIC FUNCTIONS

10. Introduction. Since one value of a trigonometric function of an acute angle determines the angle and since there are six of these trigonometric functions, we naturally expect to find many relations connecting them. Among the forms of expressing a quantity there is usually one best adapted to our purposes. To obtain this one it is often convenient to use a number of elementary identities. The main object of this chapter is to familiarize the student with these important elementary relations and give him the ability to use them with facility.

11. Simple relations. For convenience of reference, we shall write again the reciprocal relations

$$\csc A = \frac{1}{\sin A},\quad \sec A = \frac{1}{\cos A},\quad \cot A = \frac{1}{\tan A}. \tag{1}$$

Referring to triangle ABC in Fig. 1, we see that

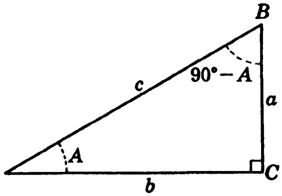


Fig. 1.

$$\tan A = \frac{a}{b} = \frac{a/c}{b/c} = \frac{\sin A}{\cos A},$$

$$\cot A = \frac{b}{a} = \frac{b/c}{a/c} = \frac{\cos A}{\sin A}.$$

Therefore

$$\tan A = \frac{\sin A}{\cos A},\qquad \cot A = \frac{\cos A}{\sin A}. \tag{2}$$

Another set of equations has reference to complementary angles. Referring to Fig. 1, we read from triangle ABC

$$\sin A = \frac{a}{c} \quad \text{and} \quad \cos(90° - A) = \frac{a}{c}.$$

Since $\sin A$ and $\cos(90° - A)$ are both equal to a/c, we have

$$\sin A = \cos(90° - A).$$

By using the same kind of argument in connection with each of the trigonometric functions, the student may prove the following equations:

$$\left.\begin{array}{ll} \cos(90° - A) = \sin A, & \sin(90° - A) = \cos A, \\ \cot(90° - A) = \tan A, & \tan(90° - A) = \cot A, \\ \csc(90° - A) = \sec A, & \sec(90° - A) = \csc A, \end{array}\right\} \quad (3)$$

or, stated in other words, **any trigonometric function of an acute angle is equal to the co-function of its complement.** This statement shows the significance of the prefix **co-** in the names of the trigonometric functions; it has reference to the word **complement.**

The relations (1), (2), and (3) are easily derived and recalled from a figure. First we construct Fig. 2 and from it read

$$\frac{a}{1} = \sin A, \quad \text{or} \quad a = \sin A,$$

$$\frac{b}{1} = \cos A, \quad \text{or} \quad b = \cos A.$$

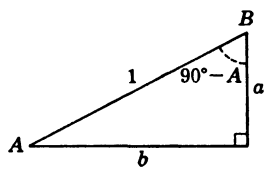


FIG. 2.

By replacing a by $\sin A$ and b by $\cos A$ in Fig. 2, we obtain Fig. 3. Now apply the definitions of the trigonometric functions to read, from Fig. 3,

$$\tan A = \frac{\sin A}{\cos A}, \qquad \cot A = \frac{\cos A}{\sin A}, \quad (4)$$

$$\sec A = \frac{1}{\cos A}, \qquad \csc A = \frac{1}{\sin A}. \quad (5)$$

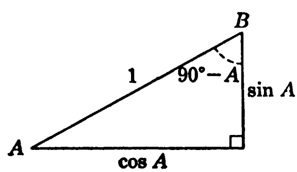


FIG. 3.

Using (4) we obtain

$$\cot A = \frac{\cos A}{\sin A} = 1 \div \frac{\sin A}{\cos A} = \frac{1}{\tan A}. \quad (6)$$

Next read the functions of $(90° - A)$ from Fig. 3 to get $\sin(90° - A) = \cos A$, $\cos(90° - A) = \sin A$, and the other relations of (3). Since one may obtain the relations (1), (2), and (3) directly from Fig. 3, it is only necessary to draw the figure to recall them.

12. Identities and conditional equations. An identity is an equation that is true for all values of the variables for which its members are defined. Thus the equations

$$1 - x^{2*} \equiv (1 - x)(1 + x), \qquad \csc x \equiv \frac{1}{\sin x}$$

are true for all values of x for which they are defined and are therefore identities. The equation $x^2 = 1$ is not an identity, since it is true only when $x = 1$ or -1. Similarly $\sin x = \cos x$ is a conditional equation, since 45° is the only acute angle for which it is true. Equations (1), (2), and (3) of this article are identities. Familiarity with these identities will be obtained by using them to simplify expressions, to verify identities, to find solutions of equations of condition, and to solve various kinds of problems.

Example 1. Simplify

$$\sin A \cos (90° - A) \csc A \cot A - \sin (90° - A). \qquad (a)$$

Solution. From equations (3), we have

$$\cos (90° - A) = \sin A, \qquad \sin (90° - A) = \cos A, \qquad (b)$$

and from equations (1) and (2)

$$\csc A = \frac{1}{\sin A}, \qquad \cot A = \frac{\cos A}{\sin A}. \qquad (c)$$

Replacing $\cos (90° - A)$, $\sin (90° - A)$, $\cot A$, and $\csc A$ in (a) by their values from (b) and (c), we obtain

$$\sin A \cdot \sin A \cdot \frac{1}{\sin A} \cdot \frac{\cos A}{\sin A} - \cos A. \qquad (d)$$

Since $\sin A$ is a number it may be canceled with $\sin A$. Hence (d) simplifies to

$$\cos A - \cos A = 0.$$

Example 2. Find an acute angle x which satisfies the equation

$$\sin (3x - 30°) = \cos (2x + 10°). \qquad (a)$$

* The symbol $\equiv$ is frequently used to mean "is identically equal to." However, for convenience, we shall use the ordinary symbol of equality throughout the book.

Solution. Using the first equation of (3) to replace $\cos(2x + 10°)$ of (a) by $\sin(90° - 2x - 10°)$, we obtain

$$\sin(3x - 30°) = \sin(90° - 2x - 10°).$$

This equation is satisfied if

$$3x - 30° = 90° - 2x - 10°.$$

Solving this equation for x, we get $\boldsymbol{x = 22°}$.

EXERCISES

1. Express as trigonometric functions of angles less than 45°

(a) $\sin 75°$. (c) $\tan 89°30'$. (e) $\cot 45°50'$.
(b) $\cos 87°$. (d) $\sec 49°20'$. (f) $\csc 70°20'16''$.

2. Find for each of the following equations an acute angle that satisfies it:

$$\sin(2x - 20°) = \cos(3x + 10°).$$
$$\cos(5\theta - 10°) = \sin(3\theta + 20°).$$
$$\tan(65° - 3\theta) = \cot(5° + 7\theta).$$
$$\csc(2\theta + 70°) = \sec(4\theta - 36°).$$

3. Simplify

(a) $\sin\theta \cot\theta$.
(b) $\cos\theta \tan\theta$.
(c) $\sec\theta \cot\theta$.
(d) $\cos(90° - \theta) \sec\theta \cot\theta$.
(e) $\csc\theta \cot(90° - \theta)$.
(f) $\sin\theta \cos(90° - \theta) \csc\theta \tan(90° - \theta)$.
(g) $(\tan\theta)^2 (\cos\theta)^2 (\csc\theta)^2$.
(h) $(\cot\theta)^2 [\cos(90° - \theta)]^2 (\sec\theta)^2$.
(i) $\sin\theta \cos(90° - \theta) \tan(90° - \theta) (\sec\theta)^2$.

4. Draw Fig. 3, and apply the definitions of the trigonometric functions to read from it all six functions of A and of $90° - A$. Compare the result with equations (1), (2), and (3).

5. Verify each of the following identities by transforming the left-hand member, the right-hand member, or both members until they have the same form:

(a) $1 + \sin\alpha \cot\alpha = \sin\alpha \csc\alpha + \cos\alpha$.
(b) $\tan\alpha + \sec\alpha = \sin\alpha \csc(90° - \alpha) + \tan\alpha \csc\alpha$.
(c) $(\sin\alpha)^2 \csc\alpha \cot\alpha - \cos\alpha = (\cos\alpha)^2 \sec\alpha \tan\alpha - \sin\alpha$.
(d) $\dfrac{(\sin\theta)^2}{(\cos\theta)^2} = (\sin\theta)^4 (\sec\theta)^2 (\csc\theta)^2$.

(e) $\dfrac{\cot \theta}{\csc \theta} = \sin (90^\circ - \theta).$

(f) $\cos \varphi \csc \varphi \tan \varphi = 1.$

(g) $(\sin A)^2 (\csc A)^2 + (\cos A)^2 (\sec A)^2 = 2.$

(h) $\dfrac{\cos A \tan A}{\tan (90^\circ - A)} = (\sin A)^2 \sec A.$

(i) $\tan \theta (\cos \theta)^2 - \tan (90^\circ - \theta) (\sin \theta)^2 = 0.$

(j) $\sin \theta \tan \theta \sec \theta = \sec \theta \cot (90^\circ - \theta) \sin \theta.$

(k) $\sec \theta \cot \theta \cot (90^\circ - \theta) - \sin \theta \csc (90^\circ - \theta) = \sec \theta - \tan \theta.$

(l) $\tan (3\theta) = \dfrac{\sec (3\theta)}{\csc (3\theta)}.$

(m) $\tan (3\theta) \tan (90^\circ - 3\theta) + \sin (2\theta) \csc (2\theta) + \cos \theta \sec \theta = 3.$

6. For each of the following equations find an acute angle that satisfies it:

$$\tan (6\theta - 50^\circ) \tan (57^\circ + \theta) = 1.$$
$$\sin (9\theta + 10^\circ 12') \sec (2\theta + 8^\circ 40') = 1.$$
$$\csc (4\theta + 43^\circ 29') \cos (5\theta + 5^\circ 13') = 1.$$
$$\tan (8\theta - 35^\circ) \sin (2\theta - 22^\circ) = \cos (2\theta - 22^\circ).$$

13. Relations derived from the Pythagorean theorem. From the right triangle ABC of Fig. 4 we have, by the well-known Pythagorean theorem,

$$a^2 + b^2 = c^2. \tag{7}$$

Dividing both members of this equation first by c^2, then by b^2, and finally by a^2, we obtain

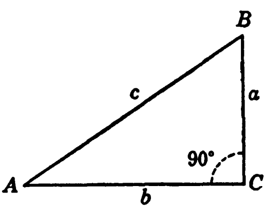


FIG. 4.

$$\left.\begin{aligned} \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 &= \left(\frac{c}{c}\right)^2, \\ \left(\frac{a}{b}\right)^2 + \left(\frac{b}{b}\right)^2 &= \left(\frac{c}{b}\right)^2, \\ \left(\frac{a}{a}\right)^2 + \left(\frac{b}{a}\right)^2 &= \left(\frac{c}{a}\right)^2. \end{aligned}\right\} \tag{8}$$

Expressing the quantities inside the parentheses in terms of trigonometric functions of the angle A, we have

$$\left.\begin{aligned} \sin^2 A + \cos^2 A &= 1, \\ \tan^2 A + 1 &= \sec^2 A, \\ 1 + \cot^2 A &= \csc^2 A, \end{aligned}\right\} \tag{9}$$

where $\sin^2 A$ means $(\sin A)^2$, $\cos^2 A$ means $(\cos A)^2$, etc.

Equations (1), (2), (3), and (9) should be memorized.

Another method of deriving these formulas consists of applying the Pythagorean theorem to Fig. 5 to obtain

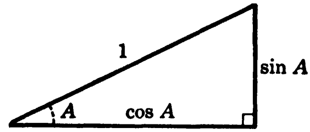


FIG. 5.

$$\sin^2 A + \cos^2 A = 1$$

and then dividing this equation first by $\cos^2 A$ and then by $\sin^2 A$ to obtain

$$\frac{\sin^2 A}{\cos^2 A} + \frac{\cos^2 A}{\cos^2 A} = \frac{1}{\cos^2 A},$$

or

$$\tan^2 A + 1 = \sec^2 A,$$

and

$$\frac{\sin^2 A}{\sin^2 A} + \frac{\cos^2 A}{\sin^2 A} = \frac{1}{\sin^2 A},$$

or

$$1 + \cot^2 A = \csc^2 A.$$

EXERCISES

1. By using relations (9) simplify

(*a*) $1 - \sin^2 \beta$. (*d*) $\sec^2 \beta - \tan^2 \beta$. (*g*) $\dfrac{(\sin^2 A + \cos^2 A)}{(\sec^2 A - \tan^2 A)}$.

(*b*) $1 - \cos^2 \beta$. (*e*) $1 - \csc^2 \beta$. (*h*) $\dfrac{1 - \cos^2 \theta}{1 - \csc^2 \theta}$.

(*c*) $\sec^2 \beta - 1$. (*f*) $\csc^2 \beta - \cot^2 \beta$.

2. Use equations (1), (2), (3), and (9) to simplify

(*a*) $\dfrac{\sin^2 \varphi + \cos^2 \varphi}{\sec \varphi \cos \varphi}$. (*d*) $\tan \varphi + \cot \varphi$.

(*b*) $(\sec^2 \varphi - 1)(\csc^2 \varphi - 1)$. (*e*) $\dfrac{\sin \varphi}{\csc \varphi} + \dfrac{\cos \varphi}{\sec \varphi}$.

(*c*) $\dfrac{(1 - \sin \varphi)(1 + \sin \varphi)}{(1 - \cos \varphi)(1 + \cos \varphi)}$. (*f*) $(\sin \varphi + \cos \varphi)^2 - 2 \sin \varphi \cos \varphi$.

3. Transform each of the following expressions so that the equivalent expression will contain only sines and cosines of θ, then replace $\cos \theta$ by $\sqrt{1 - \sin^2 \theta}$ so that the final expression will contain no trigonometric functions except $\sin \theta$:

(*a*) $2 \sin \theta \cos^4 \theta \tan^2 \theta$. (*d*) $(\tan \theta - \cot \theta) \sin \theta \cos \theta$.

(*b*) $\dfrac{\tan^2 \theta - 1}{\tan^2 \theta + 1}$. (*e*) $\sec \theta - \sin^2 \theta \sec^2 \theta$.

(*c*) $\cos^4 \theta - \sin^4 \theta$. (*f*) $\tan \theta \sec^2 \theta - \cot (90° - \theta)$.

4. **Transform each of the expressions in the left-hand column into the one written to the right of it.**

(*a*) $\csc^2\theta + \sec^2\theta$	$\sec^2\theta\csc^2\theta$
(*b*) $\dfrac{1}{\tan^2 A + 1} + \dfrac{1}{\cot^2 A + 1}$	1
(*c*) $\cos\theta\tan\theta$	$\sin\theta$
(*d*) $\sin^2\theta \div \csc^2\theta$	$\sin^4\theta$
(*e*) $\dfrac{\cot^2 A}{1 + \cot^2 A}$	$\cos^2 A$
(*f*) $\cos^2 A\tan^2 A + \sin^2 A\cot^2 A$	1
(*g*) $1 + \dfrac{\tan^2 A}{1 + \sec A}$	$\sec A$

14. Verification of identities. There are two methods of procedure for verifying identities. By means of the fundamental identities* and suitable algebraic operations, (*a*) *the more complicated member of the identity may be transformed into the other member of the identity;* (*b*) *both members may be transformed into the same expression. It may be advisable, as a last resort, to transform both members into expressions that contain only one trigonometric function.* The following examples will illustrate methods of procedure:

Example 1. Verify the identity

$$(\tan\theta + \cot\theta)^2 = \sec^2\theta + \csc^2\theta.$$

Verification. Expansion of the left-hand member gives

$$\tan^2\theta + 2\tan\theta\cot\theta + \cot^2\theta.$$

Since $\cot\theta \cdot \tan\theta = 1$, we may write this in the form

$$(\tan^2\theta + 1) + (1 + \cot^2\theta).$$

From the last two equations of (9), this expression is

$$\sec^2\theta + \csc^2\theta.$$

* Although we have proved the identities (1), (2), (3), and (9) only for acute angles, they will be found to be true, as soon as we have defined the trigonometric functions of the general angle, for all angles for which the functions are defined. A similar statement applies to all the identities of this article.

Example 2. Verify the identity

$$1 - \cot^4 \theta = 2 \csc^2 \theta - \csc^4 \theta.$$

Verification. In the following outline, the work on the left of the vertical line gives the steps for reducing the left-hand member to a function of $\sin \theta$; the work on the right of the vertical line applies to the right-hand member:

$1 - \cot^4 \theta =$	$2 \csc^2 \theta - \csc^4 \theta =$
$1 - \dfrac{\cos^4 \theta}{\sin^4 \theta} =$	$\dfrac{2}{\sin^2 \theta} - \dfrac{1}{\sin^4 \theta} =$
$\dfrac{\sin^4 \theta - \cos^4 \theta^*}{\sin^4 \theta} =$	$\dfrac{2 \sin^2 \theta - 1}{\sin^4 \theta}.$
$\dfrac{\sin^4 \theta - (1 - \sin^2 \theta)^2}{\sin^4 \theta} =$	
$\dfrac{-1 + 2 \sin^2 \theta}{\sin^4 \theta}.$	

Thus the identity is verified, since we have shown that both its members are equal to the same expression.

Alternative verification. The steps outlined in the following plan give a more direct verification:

$1 - \cot^4 \theta =$	$2 \csc^2 \theta - \csc^4 \theta =$
$(1 + \cot^2 \theta)(1 - \cot^2 \theta) =$	$\csc^2 \theta(2 - \csc^2 \theta) =$
$\csc^2 \theta(1 - \cot^2 \theta).$	$\csc^2 \theta(2 - \cot^2 \theta - 1) =$
	$\csc^2 \theta(1 - \cot^2 \theta).$

EXERCISES

Simplify each of the following expressions:

1. $\tan x \sin x + \cos x.$
2. $\cot A - \sec A \csc A(1 - 2 \sin^2 A).$
3. $(\tan B + \cot B) \sin B \cos B.$
4. $\tan A \sin A \cos A + \sin A \cos A \cot A.$
5. $(\cot^2 A - \csc^2 A)(\sec^2 A - \tan^2 A).$
6. $(\cos^2 \theta - 1) \csc^2 \theta.$

Transform each of the following expressions into the expression written to the right of it:

* Beginning at this point we could have written

$$(\sin^2 \theta - \cos^2 \theta)(\sin^2 \theta + \cos^2 \theta) = \sin^2 \theta - (1 - \sin^2 \theta) = 2 \sin^2 \theta - 1.$$

7.	$\cos\theta \csc\theta \tan\theta.$	1.
8.	$\tan A \sec A \cot A \cos A \tan(90° - A).$	$\cot A.$
9.	$\csc A \cot A \cos A + 1.$	$\csc^2 A.$
10.	$\dfrac{1}{\sin^2 A} + \dfrac{1}{\cos^2 A}.$	$\sec^2 A \csc^2 A.$
11.	$\sec^2 A \csc^2 A.$	$\sec^2 A + \csc^2 A.$
12.	$(\sec\theta - \cos\theta)(\csc\theta - \sin\theta).$	$\sin\theta\cos\theta.$
13.	$(\sec A - \tan A)(\sec A + \tan A).$	1.
14.	$(\csc A - \cot A)(\csc A + \cot A).$	1.
15.	$\sin(90° - B)\cot B \sin B - 1.$	$-\sin^2 B.$
16.	$2\cos^2 A - 1.$	$1 - 2\sin^2 A.$
17.	$\sec^2 A + \tan^2 A.$	$2\sec^2 A - 1.$

Verify the following identities:

18. $\sin\theta \sec\theta \cot\theta = 1.$
19. $(\tan y + \cot y)\cot y = \csc^2 y.$
20. $\tan A = \dfrac{\sec A}{\csc A}.$
21. $(\cos A - 1)(\cos A + 1) = -\sin^2 A.$
22. $\cot C \sin C + \cos C = 2\cos C.$
23. $\tan(90° - A)\tan A - \cos^2(90° - A) = \sin^2(90° - A).$
24. $\sin\theta\cot\theta + \cos^2\theta\sec\theta = 2\cos\theta.$
25. $\cos^2\alpha(1 + \tan^2\alpha) = 1.$
26. $\cot\theta\cos\theta + \sin\theta = \csc\theta.$
27. $\sin^2 A \sec^2 A = \sec^2 A - 1.$
28. $(\sin\varphi - \cos\varphi)^2 = 1 - 2\sin\varphi\cos\varphi.$
29. $\dfrac{\cos\beta}{1 + \sin\beta} + \dfrac{\cos\beta}{1 - \sin\beta} = 2\sec\beta.$
30. $\sin^4 x - \cos^4 x = 2\sin^2 x - 1.$
31. $(1 - \sec^2 A)(1 - \csc^2 A) = 1.$
32. $\dfrac{1 + \tan^2\alpha}{1 + \cot^2\alpha} = \tan^2\alpha.$
33. $\dfrac{\cos x}{1 + \sin x} + \dfrac{1 + \sin x}{\cos x} = 2\sec x.$
34. $\csc^2\varphi - \csc^2\varphi\cos^2\varphi = 1.$
35. $\tan x + \cot x = \sec x \csc x.$
36. $(\cot\alpha - \tan\alpha)^2\sin^2\alpha\cos^2\alpha = 1 - 4\sin^2\alpha\cos^2\alpha.$
37. $\sec^4\alpha - \tan^4\alpha = \sec^2\alpha + \tan^2\alpha.$

38. $\dfrac{\sec A + \csc A}{\sin A + \cos A} = \sec A \csc A.$

39. $\dfrac{\csc \theta + 1}{\cot \theta} = \dfrac{\cot \theta}{\csc \theta - 1}.$

40. $\tan A \sin A + \cos A = \sec A.$

41. $\csc^4 A - \cot^4 A = 2 \cot^2 A + 1.$

42. $\dfrac{\tan x - \cot x}{\sin x - \cos x} = \sec x + \csc x.$

43. $\dfrac{\tan \theta \sin \theta}{\tan \theta - \sin \theta} = \dfrac{\sin \theta}{1 - \cos \theta}.$

44. $\dfrac{\cot B - \cos B}{\cos^3 B} = \dfrac{1 - \sin B}{\cos^2 B \sin B}.$

45. $\tan \varphi - \csc \varphi \sec \varphi(1 - 2 \cos^2 \varphi) = \cot \varphi.$

46. $\cos^6 A + \sin^6 A = 1 - 3 \sin^2 A \cos^2 A.$

47. $\sqrt{\dfrac{1 - \cos x}{1 + \cos x}} = \csc x - \cot x.$

48. $\sqrt{\dfrac{\sec \varphi - \tan \varphi}{\sec \varphi + \tan \varphi}} = \sec \varphi - \tan \varphi.$

49. $\dfrac{\sec y + \tan y}{\cos y + \cot y} = \sec y \tan y.$

50. $(\sec \theta + \tan \theta)^2 = \dfrac{1 + \sin \theta}{1 - \sin \theta}.$

51. $\cot y + \dfrac{\sin y}{1 + \cos y} = \csc y.$

52. $\dfrac{\cos A}{1 + \sin A} + \dfrac{1 - \sin A}{\cos A} = 2(\sec A - \tan A).$

53. $\dfrac{1}{(\cos^2 x - \sin^2 x)^2} - \dfrac{4 \tan^2 x}{(1 - \tan^2 x)^2} = 1.$

54. $\dfrac{\tan \theta + \sec \theta - 1}{\tan \theta - \sec \theta + 1} = \dfrac{1 + \sin \theta}{\cos \theta}.$

15. Formulas from right triangles. It appeared in §11 that we could read formulas (1), (2), and (3) directly from Fig. 3. Other identities may be obtained in the same manner.

For example, we draw the right triangle shown in Fig. 6 with leg AC equal to 1. Then

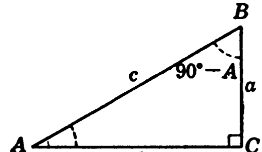


Fig. 6.

$$\frac{a}{1} = \tan A,$$

$$\frac{c}{1} = \sec A.$$

Figure 7 is obtained by replacing a by $\tan A$ and c by $\sec A$ in Fig. 6. Using the definitions of the trigonometric functions on Fig. 7, we get

$$\cot A = \frac{AC}{CB} = \frac{1}{\tan A}, \qquad \cos A = \frac{AC}{AB} = \frac{1}{\sec A},$$

$$\cot (90° - A) = \frac{BC}{AC} = \tan A, \qquad \csc (90° - A) = \frac{AB}{AC} = \sec A.$$

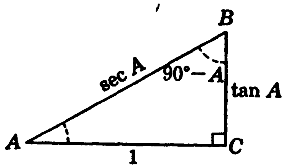


FIG. 7.

By applying the Pythagorean theorem to Fig. 7, we get

$$1 + \tan^2 A = \sec^2 A. \tag{10}$$

Evidently other identities could also be obtained. Thus, from Fig. 7, we read

$$\sin A = \frac{\tan A}{\sec A}, \qquad \cos (90° - A) = \frac{\tan A}{\sec A}, \text{ etc.}$$

Figure 8 was obtained by using the idea underlying the construction of Fig. 7. From it we read

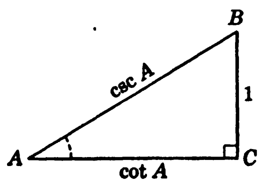


FIG. 8.

$$\tan A = \frac{1}{\cot A}, \qquad \sin A = \frac{1}{\csc A},$$

$$\tan B = \tan (90° - A) = \cot A,$$

$$\sec (90° - A) = \csc A,$$

$$1 + \cot^2 A = \csc^2 A, \tag{11}$$

and others. The fundamental identities can be recalled at any time by reproducing Figs. 3, 7, and 8 and reading the identities directly from these figures.

By means of figures, it is a simple matter to express all of the trigonometric functions in terms of one. Figure 9 is about the same as Fig. 7; instead of replacing AB by *sec A*, we have observed that

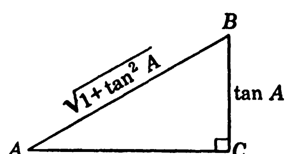


FIG. 9.

$$AB = \sqrt{\overline{AC}^2 + \overline{CB}^2} = \sqrt{1 + \tan^2 A}$$

and have written $\sqrt{1 + \tan^2 A}$ on AB. The definitions of the trigonometric functions may now be used to read from Fig. 9

$$\sin A = \frac{\tan A}{\sqrt{1 + \tan^2 A}}, \qquad \cos A = \frac{1}{\sqrt{1 + \tan^2 A}},$$

$$\sec A = \sqrt{1 + \tan^2 A}, \qquad \csc A = \frac{\sqrt{1 + \tan^2 A}}{\tan A},$$

$$\cot A = \frac{1}{\tan A}.$$

EXERCISES

1. Using Fig. 10, express all the trigonometric functions of angle A in terms of $\sin A$.

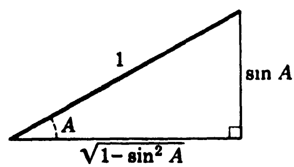


FIG. 10.

2. Using Fig. 11, express all the trigonometric functions of angle A in terms of $\cos A$.

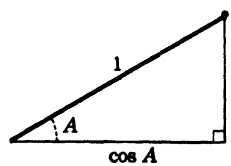


FIG. 11.

3. Express all the trigonometric functions of angle A in terms of (*a*) $\cot A$, (*b*) $\sec A$, (*c*) $\csc A$.

4. In Fig. 12 $AC = 1$. Find the lengths CB, AB, AD, and DC and equate two values of AC to obtain

$$\sin^2 A + \cos^2 A = 1.$$

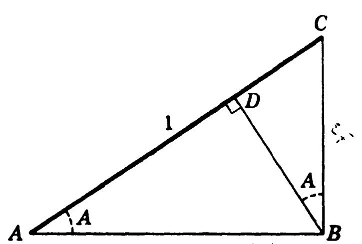


FIG. 12.

5. In Fig. 13 $AD = 1$. Find the lengths of AB, BD, AC, and CD and equate two values of AC to obtain

$$1 + \tan^2 A = \sec^2 A.$$

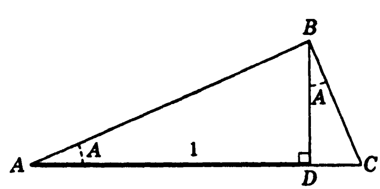


FIG. 13.

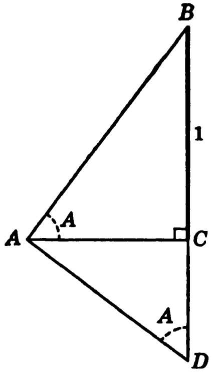


FIG. 14.

6. In Fig. 14 $BC = 1$. Find AB, BD, AC, and CD and equate two values of BD to obtain

$$1 + \cot^2 A = \csc^2 A.$$

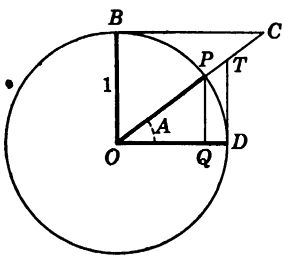


FIG. 15.

7. The radius of the circle in Fig. 15 is 1. Find the lengths of the line segments PQ, OQ, TD, OT, OC, BC, write them on the figure, and read from the figure the following identities:

$$\sin^2 A + \cos^2 A = 1,$$
$$1 + \tan^2 A = \sec^2 A,$$
$$1 + \cot^2 A = \csc^2 A.$$

16. Length of line segments. The same ideas employed in §7 may be used in connection with more complicated figures. The ability to express all parts of a rectilinear figure simply in terms of given parts is one of the most important values obtained from a study of trigonometry. It enables one to derive and recall the important formulas of trigonometry and to derive simple formulas for heights and distances.

Consider the right triangle ABC shown in Fig. 16. The given parts A and c are encircled. First let us try to express x, h, y,

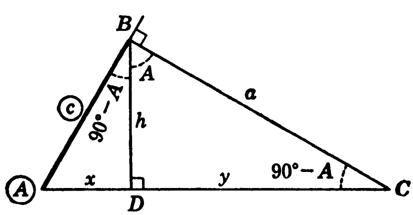


FIG. 16.

and a in terms of the given parts. From triangle ABD, we write

$$\frac{x}{c} = \cos A; \qquad \therefore\ x = c \cos A. \tag{12}$$

$$\frac{h}{c} = \sin A; \qquad \therefore\ h = c \sin A. \tag{13}$$

Similarly, from triangle BDC, we have

$$\frac{y}{h} = \tan A; \qquad \therefore\ y = h \tan A. \tag{14}$$

Replacing h in this formula by its value $c \sin A$ from (13), we have

$$y = c \sin A \tan A. \tag{15}$$

Also from triangle BDC, we get

$$\frac{a}{h} = \sec A; \qquad \therefore\ a = h \sec A. \tag{16}$$

Replacing h in this formula by its value $c \sin A$ from (13), we have

$$a = c \sin A \sec A. \tag{17}$$

Figure 17 is obtained from Fig. 16 by replacing x, y, h, and a by their values from (12), (14), (13), and (17), respectively.

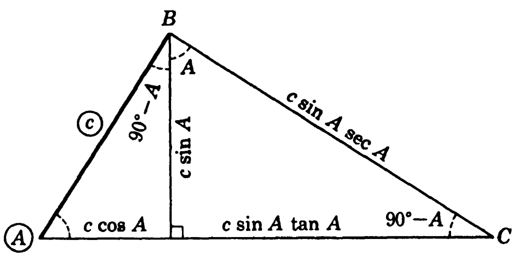


FIG. 17.

It is to be observed that when there are given only enough parts of a rectilinear figure to determine it and when all parts of the figure have been expressed in terms of the given ones, then any relation obtained by reading an equation from the figure, either by applying a proposition from geometry or by using the definitions of the trigonometric functions, is an identity. Thus an identity may be formed from Fig. 17 by using the Pythagorean theorem. In accordance with it,

$$\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2. \tag{18}$$

Replacing the lengths of the line segments in (18) by their values from Fig. 17, we get the identity

$$c^2 + c^2 \sin^2 A \sec^2 A = (c \cos A + c \sin A \tan A)^2.$$

That this is an identity may be verified in the usual way.

The student will find the following statement helpful while he is becoming familiar with the method.

To find the lengths of line segments of a rectilinear figure in terms of specified parts and to obtain identities:

(*a*) *Draw a figure, encircle each symbol representing a specified part, and put a letter on each of the other parts.*

(*b*) *Find all angles of the figure in terms of encircled angles.*

(*c*) *Use the definitions of the trigonometric functions to express all parts in terms of specified parts.*

(*d*) *Form identities by using the definitions of the trigonometric functions, by equating two expressions for the same length or area, and by using theorems from geometry.*

EXERCISES

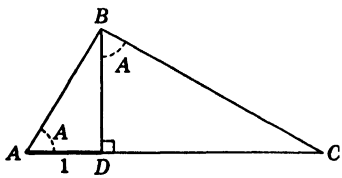


FIG. 18.

1. In Fig. 18 show that $AB = \sec A$, $BD = \tan A$, $BC = \tan A \sec A$, $DC = \tan^2 A$. Write each of these values on the appropriate line of the figure and then apply the Pythagorean theorem to triangle ABC to obtain an identity.

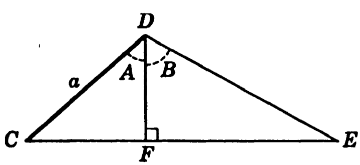


FIG. 19.

2. In Fig. 19 find DE and CE in terms of a, A, and B.

Hint. Find in order the lengths DF, DE, FE, CF, CE.

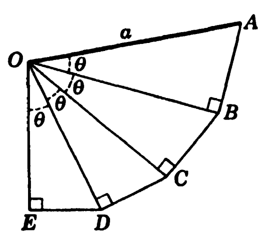


FIG. 20.

3. In Fig. 20 find the length of OE.

Hint. Find in succession the lengths OB, OC, OD, and OE.

4. In Fig. 20 replace θ by $(90° - \theta)$, and then find the length of OE in the resulting figure.

5. Compute the lengths of AB and AD in Fig. 21.

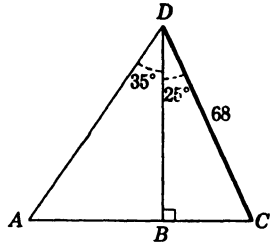


FIG. 21.

6. Compute lengths FE and BC in Fig. 22 (angle $ABE \neq 90°$).

Hint. To find the length of BC, find in succession the lengths x, y, BC.

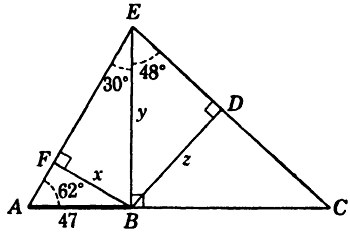


FIG. 22.

7. In Fig. 23 find the lengths DC, BC, and AB, and then read from the figure a formula for $\tan \frac{1}{2}\theta$ in terms of $\sin \theta$ and $\cos \theta$.

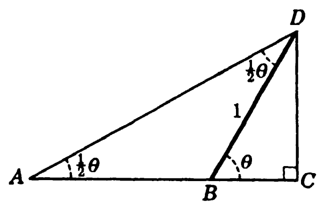


FIG. 23.

8. In Fig. 24 AB is parallel to DE. Find AB and DE in terms of a and θ.

Hint. Find in succession the lengths CB, AB, DB, DE.

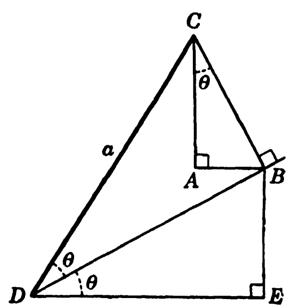


FIG. 24.

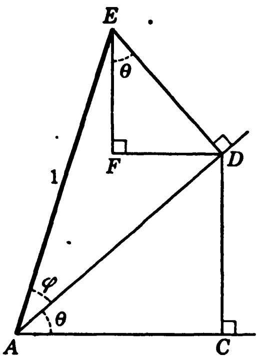


FIG. 25.

9. In Fig. 25 find in succession the lengths ED, FE, FD, AD, CD, AC in terms of θ and φ, and write each of them on the appropriate line segment of the figure.

10. In Fig. 25 erase 1 from AE, take $AC = 1$, and find in succession the lengths CD, AD, DE, FE, FD.

11. Draw an isosceles triangle with vertical angle equal to 2θ; drop a perpendicular from the vertical angle to the side opposite and a perpendicular from a second angle to the side opposite. Find the values of all line segments in the figure thus drawn. Write two expressions for the area of the triangle and equate them to obtain an identity.

17. MISCELLANEOUS EXERCISES

1. Express as trigonometric functions of angles less than 45°:

(*a*) $\sin 65°$. (*b*) $\tan 49°$. (*c*) $\sec 82°$.

2. Simplify:

(*a*) $\cot\theta \tan(90° - \theta)\sin^2\theta$.
(*b*) $\sin\theta\tan\theta\cos\theta + \cos^2\theta$.
(*c*) $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2$.
(*d*) $\sin\theta\csc\theta + \tan^2\theta$.
(*e*) $\left(\dfrac{\sin\theta}{\cos\theta}\right)^2 + \sec\theta\cos\theta$.
(*f*) $\cot(90° - \theta)\sin\theta\cos\theta$.
(*g*) $\cot(90° - A) - \tan A + \sin 90° + \tan 45°$.

3. Transform each of the expressions in the left-hand column into the one written to the right of it.

(*a*) $\sin\theta\cot\theta$.	$\cos\theta$.
(*b*) $\sin\theta\sec\theta$.	$\tan\theta$.
(*c*) $\dfrac{\cos^2 A}{1 - \sin A}$.	$1 + \sin A$.
(*d*) $\dfrac{\csc^2\theta - 1}{\sec^2\theta - 1}$.	$\cot^4\theta$.

(e) $\dfrac{1}{\sec A - \cos A}$.	$\cot A \csc A$.
(f) $\dfrac{1 + \sin A}{1 - \sin A} - \dfrac{1 - \sin A}{1 + \sin A}$.	$4 \tan A \sec A$.
(g) $\csc^4 A - \cot^4 A$.	$\csc^2 A + \cot^2 A$.
(h) $\cos \theta \sqrt{\sec^2 \theta - 1}$.	$\sin \theta$.
(i) $\dfrac{1 + \sin^2 A \sec^2 A}{1 + \cos^2 A \csc^2 A}$.	$\tan^2 A$.
(j) $\dfrac{1 - 2\cos^2 A}{\sin A \cos A}$.	$\tan A - \cot A$.
(k) $\dfrac{1 + \cos A}{\sec A - \tan A} - \dfrac{1 - \cos A}{\sec A + \tan A}$.	$2(1 + \tan^2 \theta)$.

4. Express each of the following in terms of $\sin A$:

(a) $\cos A \cot A$. (c) $\tan A/\sec A$.
(b) $\sin A(\cot^2 A + 1)$. (d) $\cos^4 A - \sin^4 A$.

5. Express each of the following in terms of $\cos A$:

(a) $\sin A \cot A$. (b) $\cot^2 A/(1 + \cot^2 A)$.

6. Express each of the following in terms of $\tan \theta$:

(a) $(\sec^2 \theta - 1) \cot \theta$. (b) $\sec^4 \theta - \sec^2 \theta$.

7. Change each of the following to equivalent forms involving only $\sin \theta$ and $\cos \theta$:

(a) $\tan \theta + \cot \theta$. (b) $\csc \theta - \cot \theta$. (c) $\sec \theta + \tan \theta$.

8. (a) If $x = a \cos \theta$ and $y = b \sin \theta$, show that $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

(b) If $x = a \sec \theta$ and $y = b \tan \theta$, show that $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

(c) If $x = a \cos^3 \theta$ and $y = a \sin^3 \theta$, show that $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

9. In each of the expressions in the left-hand column replace x by its value written opposite, and solve the result for y:

(a) $x^2 + y^2 = a^2$.	$x = a \cos \theta$.
(b) $b^2x^2 + a^2y^2 = a^2b^2$.	$x = a \cos \theta$.
(c) $b^2x^2 - a^2y^2 = a^2b^2$.	$x = a \sec \theta$.
(d) $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.	$x = a \cos^4 \theta$.
(e) $x^{\frac{1}{3}} + y^{\frac{1}{3}} = a^{\frac{1}{3}}$.	$x = a \cos^6 \theta$.
(f) $x^2y^2 = b^2x^2 + a^2y^2$.	$x = a \sec \theta$.
(g) $x^2y^2 = a^2y^2 - b^2x^2$.	$x = a \sin \theta$.
(h) $y^2(2a - x) = x^3$.	$x = 2a \sin^2 \theta$.
(i) $y^2(x^2 + 4a^2) = 16a^4$.	$x = 2a \tan \theta$.

Verify the identities numbered 10 to 37.

10. $\sec x - \cos x = \sin x \tan x.$

11. $\tan^2 x \csc^2 x \cot^2 x \sin^2 x = 1.$

12. $\tan^2 x \cos^2 x + \sin^2 x \cot^2 x = 1.$

13. $(1 + \tan \theta)(1 + \cot \theta) \sin \theta \cos \theta = 1 + 2 \sin \theta \cos \theta.$

14. $(\tan \theta + \cot \theta)^2 = \sec^2 \theta \csc^2 \theta.$

15. $\sec^2 x + \csc^2 x = \sec^2 x \csc^2 x.$

16. $\sec^4 x - \sec^2 x = \tan^4 x + \tan^2 x.$

17. $\sin \theta \cos \theta(\sec \theta + \csc \theta) = \sin \theta + \cos \theta.$

18. $\sin^2 x \sec^2 x = \sec^2 x - 1.$

19. $\dfrac{1 + \tan^2 A}{1 + \cot^2 A} = \dfrac{\sin^2 A}{\cos^2 A}.$

20. $\dfrac{\sin A}{\csc A} + \dfrac{\cos A}{\sec A} = 1.$

✓**21.** $\cot A + \dfrac{\sin A}{1 + \cos A} = \dfrac{1}{\sin A}.$

22. $\sec^4 \theta - 1 = 2 \tan^2 \theta + \tan^4 \theta.$

23. $\dfrac{\csc \theta}{\cot \theta + \tan \theta} = \cos \theta.$

24. $(\tan \theta + \sec \theta)^2 = \left(\dfrac{1 + \sin \theta}{\cos \theta}\right)^2.$

✓**25.** $\sin x(1 + \tan x) + \cos x(1 + \cot x) = \sec x + \csc x.$

26. $\dfrac{\sin x}{1 + \cos x} + \dfrac{1 + \cos x}{\sin x} = 2 \csc x.$

27. $\dfrac{\cos \theta}{1 - \tan \theta} + \dfrac{\sin \theta}{1 - \cot \theta} = \sin \theta + \cos \theta.$

28. $\dfrac{1 - \cos \theta}{1 + \cos \theta} = \dfrac{(1 - \cos \theta)^2}{\sin^2 \theta}.$

29. $\dfrac{\sec x}{1 + \cos x} = \dfrac{\tan x - \sin x}{\sin x(1 - \cos^2 x)}.$

30. $\cot x + \csc x = \dfrac{\sin x}{1 - \cos x}.$

✓**31.** $\dfrac{\sin^3 \theta + \cos^3 \theta}{\sin \theta + \cos \theta} = 1 - \sin \theta \cos \theta.$

✓**32.** $\sec^6 \theta - \tan^6 \theta = 1 + 3 \sec^2 \theta \tan^2 \theta.$

33. $\cos^6 A - \sin^6 A = (2 \cos^2 A - 1)(1 - \sin^2 A \cos^2 A).$

34. $(\cos^2 x - 1)(\cot^2 x + 1) + 1 = 0.$

✓**35.** $2(\sin^6 \theta + \cos^6 \theta) - 3(\cos^4 \theta + \sin^4 \theta) = -1.$

36. $\tan^2 \theta + \cot^2 \theta = \sec^2 \theta \csc^2 \theta - 2.$

37. $\sec^2 \theta + \cos^2 \theta = \tan^2 \theta \sin^2 \theta + 2.$

38. In Fig. 26 compute the length of x.

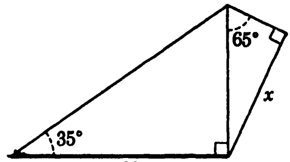


Fig. 26.

39. Compute the lengths of AB and AD in Fig. 27.

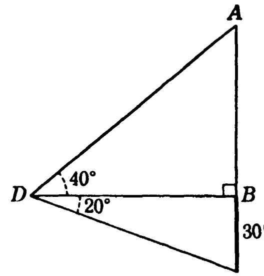


Fig. 27

40. Compute the length of each line segment in Fig. 28.

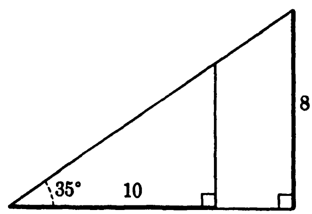


Fig. 28.

41. In Fig. 29 compute y by first finding x.

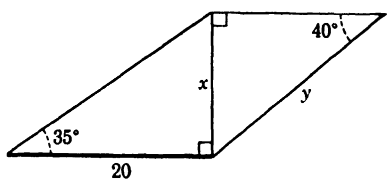


Fig. 29.

42. In Fig. 30 find the lengths of AC and AB in terms of a, θ, ϕ, and α.

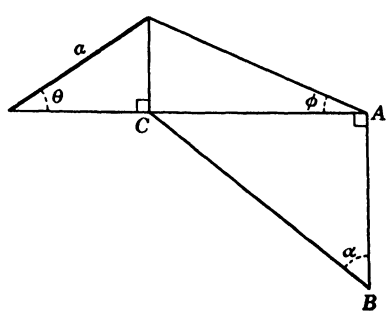


Fig. 30.

CHAPTER III

GENERAL DEFINITIONS OF TRIGONOMETRIC FUNCTIONS

18. Definition of angle. Only trigonometric functions of angles no greater than 90° have been considered in the first two chapters. This chapter will be concerned with functions of angles that may have any magnitude.

A half line or ray is the part of a straight line lying on one side of a point of the line. It is designated by naming its end point and another point on it. Thus OA in Fig. 1 is the ray beginning at O and extending through A. If a half line or ray beginning at point O rotates about O in a plane from an initial position OA to a terminal position OB, it is said to generate the angle AOB (see Fig. 1). When the legs of a compass are drawn apart an angle is generated; the hands of a clock rotate and generate angles.

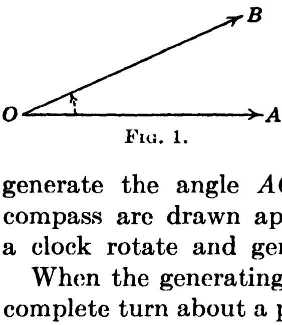
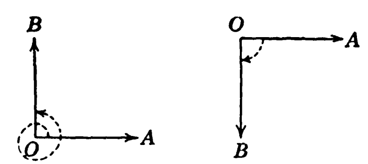


FIG. 1.

When the generating ray is turned through one-fourth of the complete turn about a point, the angle generated is called a right angle; a degree is $\frac{1}{90}$ of a right angle, a minute is $\frac{1}{60}$ of a degree, and a second is $\frac{1}{60}$ of a minute. Although either direction of rotation may be considered positive, it is customary in trigonometry to call angles generated by *counterclockwise* rotation *positive* angles and those generated by *clockwise* rotation *negative* angles. In Fig. 2 (*a*) the curved arrow indicates counterclock-wise or positive rotation through five right angles; in Fig. 2(*b*) a negative right angle is indicated.



Counter clockwise or positive rotation (*a*)

Clockwise or negative rotation (*b*)

FIG. 2.

EXERCISES

1. Construct the following angles:

(*a*) 6 right angles.
(*b*) -6 right angles.
(*c*) 5 right angles.
(*d*) -3 right angles.
(*e*) $3\frac{1}{3}$ right angles.
(*f*) $-2\frac{1}{2}$ right angles.

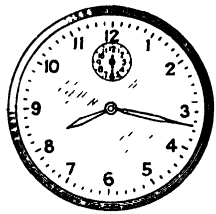


FIG. 3*a*.

2. Through how many right angles does the minute hand of a clock turn from 12:15 P.M. to 2 P.M. of the same day [see Fig. 3(*a*)]?

3. What are the magnitude and sense of the angles generated by the hour hand of a clock between 3 A.M. and the next 8 A.M.?

4. Through what part of a right angle does the minute hand of a clock move in 1 min. of time?

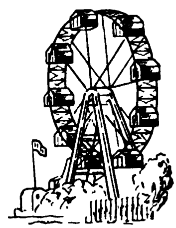


FIG. 3*b*.

5. A Ferris wheel is turning through 3 revolutions in each minute. Through how many right angles will it turn in 2 min. [see Fig. 3(*b*)]?

6. An imaginary line connecting the center of the earth's orbit to the center of the earth makes one complete revolution each year. Assuming that this line turns in a plane at a constant rate, find the number of right angles described by this line in (*a*) 3 months; (*b*) 7 months; (*c*) 25 months; (*d*) 2000 years; (*e*) 1 day; (*f*) 1 hr.

19. Rectangular coordinates. This article is designed to recall the essential conceptions of rectangular coordinates; they are used in the definitions of the trigonometric functions of any angle.

In Fig. 4, $X'X$ represents a straight line, and O is any point on it. If we choose a unit of measure, any point to the right of O will be designated by a positive number telling its distance from O in terms of the chosen unit, and any point to the left of O will be designated by a negative number whose magnitude gives the distance of the point from O. Thus a point 5 units to the right

of O is designated by 5, whereas a point $3\frac{1}{2}$ units to the left of O is designated by -3.5.

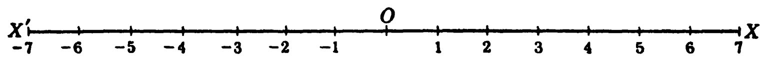


FIG. 4.

By means of a system called rectangular coordinates, the position of any point in the plane is defined by two numbers. In this system two mutually perpendicular lines, referred to as *axes*, are required. In Fig. 5, $X'X$ and $Y'Y$ represent two perpendicular lines intersecting at O. The four parts into which the plane is divided by these lines are called the first, second, third, and fourth quadrants, respectively, as indicated in the figure. Let P be any point in the plane of $X'X$ and $Y'Y$. Drop a perpendicular from P to the x-axis, meeting it in Q, and another from P to the y-axis, meeting it in R. Let x, considered as positive when P is to the right of $Y'Y$ and as negative when P is to the left of $Y'Y$, be the measure of OQ in terms of a given unit of measure; let y, considered as positive when P is above $X'X$ and negative when P is below $X'X$, be the measure of OR in terms of the given unit. Then any point in the plane will be represented by a pair of numbers, x and y.

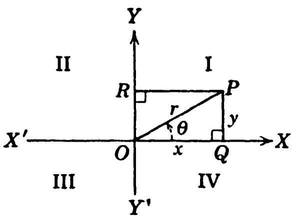


FIG. 5.

The first number x is called the *abscissa* of the point P, and the second number y is called its *ordinate*. The two numbers x and y are called the *coordinates* of P, and the point is designated (x, y). Thus in Fig. 6 the abscissa of P_1 is 2, its ordinate is 3, its coordinates are 2 and 3, and it is designated (2, 3). Similarly, P_2 is designated $(-3, 3)$, P_3 is designated $(-2, -3)$, and P_4 is designated $(1, -2)$.

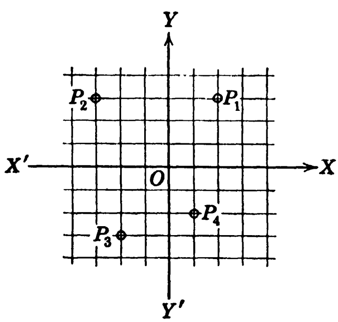


FIG. 6.

EXERCISES

1. Plot the points $(2, 4)$, $(-2, 4)$, $(2, -4)$, $(-2, -4)$, $(4, 2)$, $(4, -2)$, $(-4, 2)$, $(-4, -2)$. Why do all these points lie in a circle?

2. Plot the points (0, 1), (0, 5), (1, 0), (5, 0), (0, −1), (0, −5), (−1, 0), (−5, 0), (0, 0).

3. Read the trigonometric functions of the angle subtended at O by the line connecting (*a*) (12, 0) to (12, 5); (*b*) $(x, 0)$ to (x, y), assuming x and y to be positive numbers.

4. Where are all the points for which (*a*) $x = 3$? (*b*) $y = -3$? (*c*) $x = -4$? (*d*) $y = 5$? (*e*) $x = 0$? (*f*) $y = 0$? (*g*) $r = 3$?

5. What is the abscissa of all points on the y-axis? What is the ordinate of all points on the x-axis?

6. Determine the quadrant in which (*a*) the abscissa and ordinate are both positive; (*b*) the abscissa is negative and the ordinate is positive; (*c*) the abscissa is positive and the ordinate is negative; (*d*) the abscissa and ordinate are both negative.

7. Assuming that r is always positive, in which quadrants are each of the following ratios positive? in which negative?

(*a*) y/r. (*b*) x/r. (*c*) x/y. (*d*) y/x. (*e*) r/x. (*f*) r/y.

20. Definitions of the trigonometric functions of any angle. Appropriate definitions of the trigonometric functions of any angle are desired. Consider the obtuse angle XOP in Fig. 7. The point P on the terminal side of the angle has coordinates $x = -3$ and $y = 4$ as shown. Evidently $OP = 5$ is the hypotenuse. Previously the side along the initial line was called the adjacent leg. Hence $OR = x = -3$, the initial line produced, should be called the adjacent leg. Also, $RP = y = 4$ does not lie along a side of the angle and should be called the opposite leg.

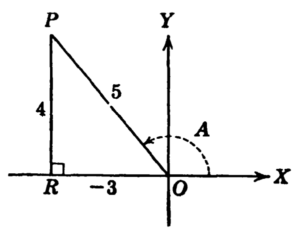


FIG. 7.

Therefore, using the definitions previously given in §§3 and 4, we would naturally write

$$\sin A = \frac{\text{opposite leg}}{\text{hypotenuse}} = \frac{4}{5}, \qquad \csc A = \frac{\text{hypotenuse}}{\text{opposite leg}} = \frac{5}{4},$$

$$\cos A = \frac{\text{adjacent leg}}{\text{hypotenuse}} = \frac{-3}{5}, \qquad \sec A = \frac{\text{hypotenuse}}{\text{adjacent leg}} = \frac{5}{-3},$$

$$\tan A = \frac{\text{opposite leg}}{\text{adjacent leg}} = \frac{4}{-3}, \qquad \cot A = \frac{\text{adjacent leg}}{\text{opposite leg}} = \frac{-3}{4}.$$

In Fig. 8, the coordinates of P are $x = -3$, and $y = -4$. Calling 5 the hypotenuse, −3 the adjacent leg, and −4 the

opposite leg, we would naturally write in accordance with the definitions of §§3 and 4

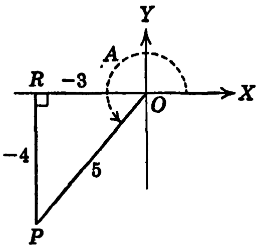


FIG. 8.

$$\sin A = \frac{-4}{5}, \qquad \csc A = \frac{5}{-4},$$
$$\cos A = \frac{-3}{5}, \qquad \sec A = \frac{5}{-3},$$
$$\tan A = \frac{-4}{-3} = \frac{4}{3}, \qquad \cot A = \frac{-3}{-4} = \frac{3}{4}.$$

In Fig. 9 the coordinates of P are $x = 3$, $y = -4$. Taking 5 as hypotenuse, $x = 3$ as adjacent leg, and $y = -4$ as opposite leg, we write, in accordance with the definitions of §§3 and 4,

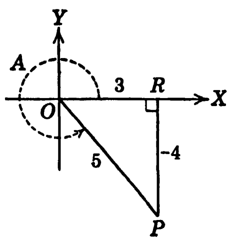


FIG. 9.

$$\sin A = \frac{-4}{5}, \qquad \csc A = \frac{5}{-4},$$
$$\cos A = \frac{3}{5}, \qquad \sec A = \frac{5}{3},$$
$$\tan A = \frac{-4}{3}, \qquad \cot A = \frac{3}{-4}.$$

The foregoing discussion suggests definitions of the trigonometric functions of any angle. Draw the axes for a set of rectangular coordinates and consider the angle A generated by a ray in turning about the origin O from the positive x-axis as initial position to any terminal position. Let P be a point on the terminal ray, let r, considered as positive, be the distance along this ray from O to P, let x be the abscissa of P and y its ordinate, as shown in Figs. 10(a), (b), (c), (d). We then define the trigonometric functions of angle A as follows:

$$\left.\begin{aligned}
\sin A &= \frac{\text{ordinate}}{\text{distance}} = \frac{y}{r}, & \csc A &= \frac{\text{distance}}{\text{ordinate}} = \frac{r}{y},\\
\cos A &= \frac{\text{abscissa}}{\text{distance}} = \frac{x}{r}, & \sec A &= \frac{\text{distance}}{\text{abscissa}} = \frac{r}{x},\\
\tan A &= \frac{\text{ordinate}}{\text{abscissa}} = \frac{y}{x}, & \cot A &= \frac{\text{abscissa}}{\text{ordinate}} = \frac{x}{y}.
\end{aligned}\right\} \qquad (1)$$

The student will perceive that the definitions (1) are natural extensions of the definitions given in §§3 and 4 if he will associate *side adjacent* with *abscissa x*, *side opposite* with *ordinate y*, and

hypotenuse with *distance r*. Note that the definitions (1) include as a special case the definitions given in §§3 and 4.

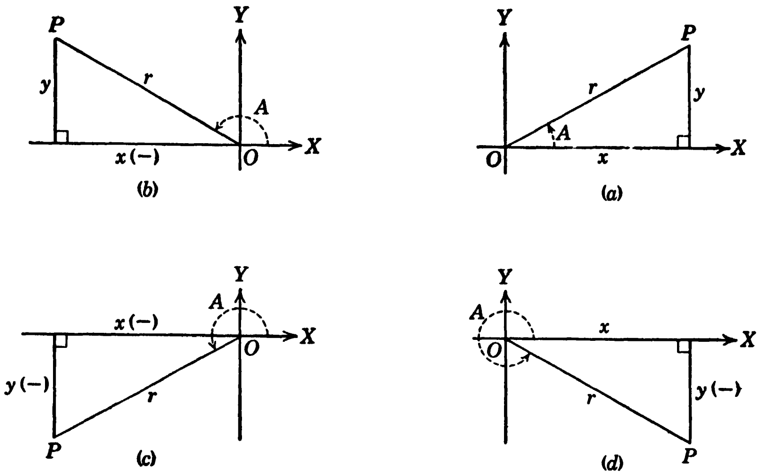


FIG. 10.

EXERCISES

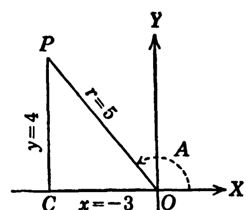


FIG. 11.

1. Read the values of the trigonometric functions of an angle A if its cosine is $-\frac{3}{5}$ and (*a*) if it is a second-quadrant angle (see Fig. 11); (*b*) if it is a third-quadrant angle.

2. Write the appropriate signs, $+$ or $-$, in the blank spaces of the following form:

	sin	cos	tan	cot	sec	csc
1st quad	+	+	+	+	+	+
2d quad	+	−	−	−	−	+
3d quad	−	−	+	+	−	−
4th quad	−	+	−	−	+	−

3. The sine of a certain angle is $-\frac{1}{2}$, and its cosine is $\frac{\sqrt{3}}{2}$. Find the values of the other trigonometric functions of this angle.

4. Fill in the blank spaces of the following diagram:

Angle	sin	cos	tan	cot	sec	csc
A	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$				
A			1		$-\sqrt{2}$	
A				$-\sqrt{3}$		-2
A	$\frac{5}{13}$	$-\frac{12}{13}$				

5. The absolute value (numerical value without reference to sign) of the tangent of an angle is $\frac{5}{12}$. Write the values of the six trigonometric functions of this angle (*a*) when it is less than 90°; (*b*) when it is greater than 90° but less than 180°; (*c*) when it is greater than 180° but less than 270°; (*d*) when it is greater than 270° but less than 360°.

6. Each of the following points is on the terminal side of an angle θ, in standard position; find the trigonometric functions of θ.

(*a*) (4, 3). (*b*) (−4, 3). (*c*) (−5, −12). (*d*) (15, −8). (*e*) (24, −7). (*f*) (1, 3). (*g*) (2, −3). (*h*) $(1, \sqrt{3})$. (*i*) (−2, 0).

7. In what quadrants may θ terminate under the following conditions:

(*a*) $\sin\theta$ pos.? (*b*) $\cos\theta$ neg.? (*c*) $\tan\theta$ pos.? (*d*) $\cot\theta$ neg.? (*e*) $\sec\theta$ neg.? (*f*) $\csc\theta$ pos.?

8. In what quadrant must θ terminate under the following conditions:

(*a*) $\sin\theta$ pos. and $\cos\theta$ neg.?
(*b*) $\tan\theta$ neg. and $\sec\theta$ pos.?
(*c*) $\cot\theta$ neg. and $\cos\theta$ pos.?
(*d*) $\cos\theta$ neg. and $\sin\theta$ neg.?
(*e*) $\cos\theta$ neg. and $\csc\theta$ pos.?
(*f*) $\cot\theta$ neg. and $\csc\theta$ neg.?

9. Locate the terminal side of θ and find its other functions, having given:

(*a*) $\cos\theta = \frac{4}{5}$, $\sin\theta$ pos.
(*b*) $\tan\theta = -\frac{12}{5}$, $\sin\theta$ neg.
(*c*) $\sin\theta = -\frac{8}{17}$, $\cot\theta$ neg.
(*d*) $\sec\theta = \frac{4}{3}$, $\tan\theta$ neg.
(*e*) $\csc\theta = -\frac{17}{8}$, $\tan\theta$ pos.
(*f*) $\cot\theta = -\frac{8}{15}$, $\csc\theta$ neg.

(g) $\sin\theta = \frac{1}{2}$, $\cos\theta$ neg.
(h) $\sec\theta = -2$, $\sin\theta$ neg.
(i) $\tan\theta = -\frac{5}{12}$, $\sec\theta$ pos.
(j) $\cot\theta = -\frac{4}{3}$, $\sin\theta$ neg.
(k) $\cos\theta = \frac{5}{13}$, $\cot\theta$ neg.
(l) $\csc\theta = -2$, $\tan\theta$ neg.

10. Find the value of $2\tan\theta/(1-\tan^2\theta)$ when $\cos\theta = -\frac{3}{5}$ and θ is in the third quadrant.

11. Find the value of $(\csc\theta - \cot\theta)(\sin^2\theta + \cos^2\theta)$ when $\sec\theta = -\frac{5}{4}$ and $\tan\theta$ is negative.

12. If $\sin\theta = \frac{3}{5}$, find the values of $(\cos\theta - \csc\theta)/\cot\theta$ for the various quadrants in which θ may terminate.

21. Observations. We have seen in §§3 and 4 that each of the six trigonometric functions of an acute angle has only one value. Similarly, each of the trigonometric functions of an angle, unrestricted in magnitude, has only one value. However, the converse is not true. Since the trigonometric functions are defined in terms of values dependent on an initial ray and a terminal ray, each of them has the same value for a given angle as for any other angle having the same initial position and the same terminal position as the given angle. In other words, *the value of any trigonometric function of a given angle is equal to the value of the same trigonometric function of any angle differing from the given one by a multiple of* 360°. Hence, in finding the value of a trigonometric function of any angle, one may add to the angle or subtract from it any integral multiple of 360°.

Observing that x is negative and that y and r are positive in the second quadrant, we see that the $\sin\theta$ (y/r) and $\csc\theta$ (r/y) are positive and the other four trigonometric functions are negative for second quadrant angles. Similarly, x and y are both negative in the third quadrant, so that the tangent (y/x) and the cotangent (x/y) are both positive, and the other functions are negative for third quadrant angles. Finally, in the fourth quadrant, x and r are positive, so that the cosine (x/r) and the secant (r/x) are positive and the other functions are negative for fourth quadrant angles.

22. Values of trigonometric functions for special angles. In §5 (Chap. I) we were able to read from appropriate figures the trigonometric functions of 0°, 30°, 45°, 60°, and 90°. Now we are able to consider the values of the trigonometric functions of related angles in other quadrants.

For example, to find the trigonometric functions of 240°, draw the line OP (Fig. 12) so that angle XOP is 240°. Therefore angle $COP = 240° - 180° = 60°$. Take the distance OP as 2 units, draw PC perpendicular to the x-axis, and compute $OC = -1$ and $CP = -\sqrt{3}$. From the triangle OPC we read

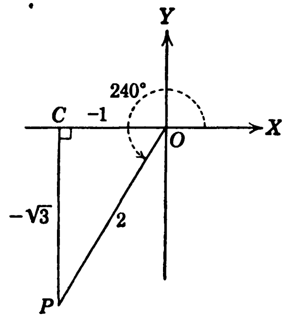


FIG. 12.

$$
\begin{aligned}
\sin 240° &= -\sqrt{3}/2,\\
\cos 240° &= -1/2,\\
\tan 240° &= \sqrt{3},\\
\csc 240° &= -2/\sqrt{3},\\
\sec 240° &= -2/1,\\
\cot 240° &= 1/\sqrt{3}.
\end{aligned}
$$

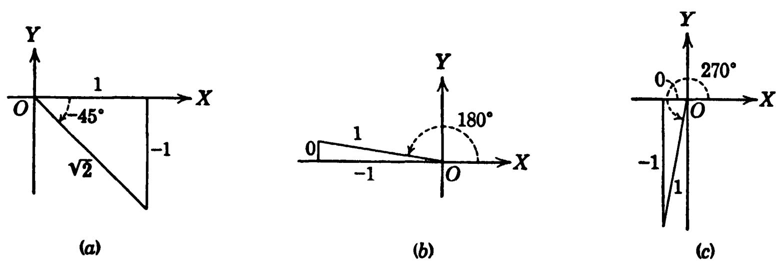


FIG. 13.

To illustrate the procedure further, we devise Figs. 13(a), 13(b), and 13(c) and from them read the values tabulated below.

TABLE A

Angle	sin	cos	tan	cot	sec	csc
$-45°$	$-1/\sqrt{2}$	$1/\sqrt{2}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
180°	0	-1	0	∞	-1	∞
270°	-1	0	∞	0	∞	-1

To find the trigonometric functions of a special angle, the student should *draw the angle, form a right triangle by dropping a perpendicular from a point on the terminal ray to the x-axis, write appropriate numbers on the sides of the right triangle, and read the values of the functions from the figure.*

EXERCISES

1. Draw a figure similar to Fig. 12 but designed for an angle of 210°. From this figure read the values of the trigonometric functions of 210°.

2. Make a tabular form, similar to that of Table A above, containing a blank space for each of the values of the six trigonometric functions of 0°, 60°, 90°, 120°, 135°, −135°, 270°, −60°, 315°. Then fill in the blank spaces of the form from figures prepared for the purpose.

3. Find two positive angles A less than 360° for which

(a) $\sin A = \frac{1}{2}$. (d) $\tan A = -\frac{1}{3}\sqrt{3}$.
(b) $\sin A = -\frac{1}{2}$. (e) $\cos A = 1/\sqrt{2}$.
(c) $\tan A = \frac{1}{3}\sqrt{3}$. (f) $\sec A = -\sqrt{2}$.

4. Find all positive angles less than 360° for which

(a) $\sin A = 1$. (d) $\cos A = 0$. (g) $\cot A = 0$.
(b) $\cos A = -1$. (e) $\sin A = 0$. (h) $\tan A = \infty$.
(c) $\tan A = 0$. (f) $\csc A = -1$. (i) $\cot A = \infty$.

5. Find the values of the trigonometric functions of (a) 165°; (b) 285°; (c) 245°; (d) 205°; (e) 105°.

Hint. Use the table in §6.

6. Evaluate $4\sqrt{3} \tan 150° + 3 \sin 90° \tan 225° - 6 \sin 330° + \cos 270°$.

7. Evaluate (a) $\sin 60° - 2 \sin 330°$; (b) $2 \sin 45° - \sin 690°$; (c) $3 \cos 60° - \cos 180°$; (d) $3 \sin 690° - \sin 90°$.

8. Evaluate $4 \sin 90° \sin 330° \sin 180° + (1/\sqrt{3}) \tan 240°$.

9. Show that $\sin 120° = \sin 180° \cos 60° - \cos 180° \sin 60°$.

10. Show that

$$\tan 210° = \frac{\tan 240° - \tan 30°}{1 + \tan 240° \tan 30°}.$$

11. Show that

$$\cot 330° = \frac{\cos 120° \cos 210° - \sin 120° \sin 210°}{\sin 120° \cos 210° + \cos 120° \sin 210°}.$$

12. Verify that

$$\tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$$

for each of the following values of θ: (a) $\theta = 45°$; (b) $\theta = 135°$; (c) $\theta = 120°$.

13. Verify that $\sin 4\theta = 4 \sin \theta \cos \theta(\cos^2 \theta - \sin^2 \theta)$ for each of the following values of θ: (a) $\theta = 30°$; (b) $\theta = 120°$; (c) $\theta = 210°$.

14. Verify that $\sin (A + B) = \sin A \cos B + \cos A \sin B$ for (*a*) $A = 210°$, $B = 30°$; (*b*) $A = 135°$, $B = 225°$.

15. Verify that $\cos (A + B) = \cos A \cos B - \sin A \sin B$ for (*a*) $A = 120°$, $B = 210°$; (*b*) $A = 315°$, $B = 135°$.

16. Evaluate:

(*a*) $\dfrac{\cos 150° \tan 300°}{\cot 225° + \sin (-30°)}$. (*c*) $\dfrac{\tan^3 315°}{2 \sin^2 240° + \cos 180°}$.

(*b*) $\dfrac{\sec^2 135°}{\cos (-240°) - 2 \sin 210°}$. (*d*) $\dfrac{\sin 90° - 3 \cot 495°}{\cos 510° \csc (-60°)}$.

23. Fundamental identities. The fundamental identities (1), (2), (3), and (9) of Chap. II are true for all angles. The arguments used in Chap. II to prove (1), (2), and (9) for acute angles may be extended to apply to angles of any magnitude, provided no angles are considered for which any function involved is undefined; this may be done by replacing a by x, b by y, and c by r in those arguments. That the relations (3) of §11 are true also for all values of an angle A will be shown in Chap. V. Since only permissible algebraic operations and the identities just referred to were used in the verifications of Chap. II, all these verifications apply whether the angle is acute or not.

24. Expressing a trigonometric function of any angle as a function of an acute angle. When the trigonometric functions

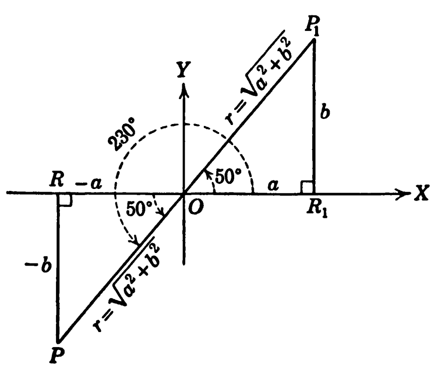


FIG. 14.

of an angle of any magnitude are read from a figure, they are always read from a right triangle, that is, from an acute-angled triangle. Hence it is always possible to express any one of the

six trigonometric functions of an angle as plus or minus a trigonometric function of a positive angle less than 90°; in fact, they can be expressed as functions of an angle no greater than 45°.

Consider, for example, the problem of expressing the six trigonometric functions of 230° in terms of trigonometric functions of angles less than 90°.

In Fig. 14 angle XOP represents 230°. $OP = r$, and line PR is drawn perpendicular to the x-axis. The length of OR is a, that of RP is b, and the coordinates of P are $x = -a$, and $y = -b$ as indicated. PO is prolonged into the first quadrant to P_1 so that $OP_1 = OP = r$, and R_1P_1 is perpendicular to the x-axis. Therefore triangle OR_1P_1 is congruent to triangle ORP and P_1 is the point (a, b). Hence, using the definitions (1), we have

$$\sin 230° = \frac{y(\text{of } P)}{r} = \frac{-b}{r} = -\left(\frac{b}{r}\right).$$

But from triangle R_1OP_1, $\frac{b}{r} = \sin 50°$. Hence

$$\sin 230° = -\left(\frac{b}{r}\right) = -\sin 50°.$$

Similarly, from Fig. 14 we obtain

$$\cos 230° = \frac{x(\text{of } P)}{r} = \frac{-a}{r} = -\left(\frac{a}{r}\right) = -\cos 50,$$

$$\tan 230° = \frac{y(\text{of } P)}{x(\text{of } P)} = \frac{-b}{-a} = \frac{b}{a} = \tan 50°.$$

Continuing the same line of reasoning, we get

$$\cot 230° = \frac{a}{b} = \cot 50°,$$

$$\sec 230° = \frac{r}{-a} = -\sec 50°,$$

$$\csc 230° = \frac{r}{-b} = -\csc 50°.$$

Since for acute angles θ

$$f_n(\theta) = cof_n(90° - \theta)$$

[see (3) §11], we have

$$\sin 230° = -\sin 50° = -\cos 40°,$$
$$\cos 230° = -\cos 50° = -\sin 40°, \text{ etc.}$$

Hence the functions of 230° can be expressed as functions of 40°, an angle less than 45°.

Similarly, to express the functions of $-20°$ in terms of functions of 20°, construct Fig. 15, and from it obtain

$$\sin(-20°) = \frac{-b}{r} = -\sin 20°, \qquad \csc(-20°) = -\csc 20°,$$

$$\cos(-20°) = \frac{a}{r} = \cos 20°, \qquad \sec(-20°) = \sec 20°,$$

$$\tan(-20°) = \frac{-b}{a} = -\tan 20°, \qquad \cot(-20°) = -\cot 20°,$$

It was pointed out in §21 that the values of the six trigonometric functions of $n\ 360° + A$ are respectively identical with those of A, provided n is any integer, positive or negative. Hence, to deal with $-380°$, first add 360° to obtain $-20°$, and then operate with $-20°$ as above. To deal with 950°, first subtract $720° = 2 \times 360°$ to obtain 230°, and then operate with 230° as above.

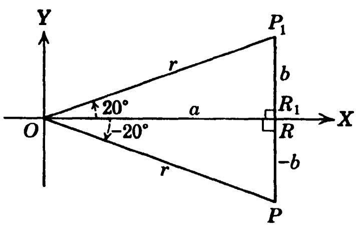


FIG. 15.

EXERCISES

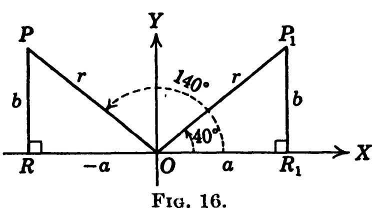


FIG. 16.

1. In Fig. 16, $OP = OP_1$. Use it to express the six trigonometric functions of 140° in terms of functions of 40°.

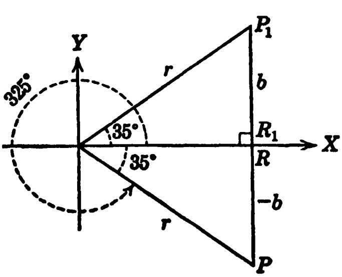


FIG. 17.

2. Use Fig. 17 to express the trigonometric functions of 325° in terms of functions of 35°.

3. Express the trigonometric functions of each of the following angles in terms of functions of an acute angle:

(*a*) 243°.	(*f*) 155°.	(*k*) −200°.
(*b*) 326°.	(*g*) 350°.	(*l*) 99°.
(*c*) 198°.	(*h*) 470°.	(*m*) 200°.
(*d*) 170°.	(*i*) 545°.	(*n*) 130°.
(*e*) 310°.	(*j*) 730°.	(*o*) 925°.

25. Functions of $\pm\theta$ and $180° \pm \theta$ in terms of functions of θ. The process used in §24 may be used to get general formulas to be used in expressing functions of any angles in terms of functions of acute angles. Although the formulas will be derived under the assumption that θ is an acute angle, it will be proved later that they apply to the case when θ represents any angle.

In Fig. 18 angle XOP is $180°$ minus any acute angle θ. P is any point different from O on ray OP, its coordinates are $x = -a$, $y = b$, and it is distant r from the origin. PR is drawn perpendicular to the x-axis, and triangle OP_1R_1 is drawn congruent to triangle OPR as indicated.

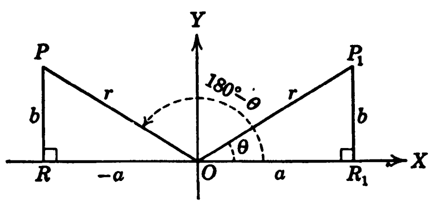


Fig. 18.

Referring to Fig. 18, we find

$$\sin(180° - \theta) = \frac{b}{r},$$

and b/r in triangle OP_1R_1 is $\sin\theta$. Therefore

$$\sin(180° - \theta) = \sin\theta. \tag{2}$$

Similarly

$$\left.\begin{aligned}
\cos(180° - \theta) &= \frac{-a}{r} = -\left(\frac{a}{r}\right) = -\cos\theta,\\
\tan(180° - \theta) &= \frac{b}{-a} = -\left(\frac{b}{a}\right) = -\tan\theta,\\
\cot(180° - \theta) &= \frac{-a}{b} = -\left(\frac{a}{b}\right) = -\cot\theta,\\
\sec(180° - \theta) &= \frac{r}{-a} = -\left(\frac{r}{a}\right) = -\sec\theta,\\
\csc(180° - \theta) &= \frac{r}{b} = \csc\theta.
\end{aligned}\right\} \tag{3}$$

In Fig. 19 angle XOP is equal to $180° + \theta$, where θ is an acute angle. The coordinates of P are $x = -a$, $y = -b$, and the

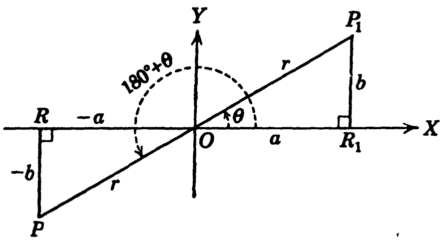


FIG. 19.

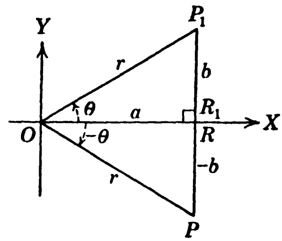


FIG. 20.

congruent triangles OPR and OP_1R_1 have been constructed as indicated. Referring to Fig. 19, we find

$$\left.\begin{aligned}
\sin(180° + \theta) &= \frac{-b}{r} = -\sin\theta,\\
\cos(180° + \theta) &= \frac{-a}{r} = -\cos\theta,\\
\tan(180° + \theta) &= \frac{-b}{-a} = \tan\theta,\\
\cot(180° + \theta) &= \frac{-a}{-b} = \cot\theta,\\
\sec(180° + \theta) &= \frac{r}{-a} = -\sec\theta,\\
\csc(180° + \theta) &= \frac{r}{-b} = -\csc\theta.
\end{aligned}\right\} \tag{4}$$

Similarly, from Fig. 20, we get

$$\left.\begin{aligned}
\sin(-\theta) &= \frac{-b}{r} = -\sin\theta, & \csc(-\theta) &= \frac{r}{-b} = -\csc\theta,\\
\cos(-\theta) &= \frac{a}{r} = \cos\theta, & \sec(-\theta) &= \frac{r}{a} = \sec\theta,\\
\tan(-\theta) &= \frac{-b}{a} = -\tan\theta, & \cot(-\theta) &= \frac{a}{-b} = -\cot\theta.
\end{aligned}\right\} \tag{5}$$

Considering formulas (2), (3), (4), and (5), we may write

$$fn(180° \pm \theta) = \pm fn(\theta), \qquad fn(\pm\theta) = \pm fn(\theta), \tag{6}$$

where fn refers to any one of the six symbols sin, cos, tan, etc., and the plus or minus sign in the right-hand member is to be

used according as the left-hand member is a positive quantity or a negative quantity.

Since any integral multiple of 360° may be added to an angle, equations (6) could be replaced by

$$fn(k180° \pm \theta) = \pm fn\theta \tag{7}$$

where k is an integer and the plus or minus sign in the right-hand member is to be used according as $fn(k180° \pm \theta)$ is positive or negative.

Example. For each of the following expressions write an equivalent expression involving only an acute angle:

(*a*) cos 138°, (*b*) tan 295°, (*c*) sin 235°.

Solution. (*a*) $\cos 138° = \cos (180° - 42°) = -$ **cos 42°.** The minus sign was chosen in the right-hand member because cos 138° is negative.

(*b*) Similarly $\tan 295° = \tan (2 \times 180° - 65°) = -$ **tan 65°.** The minus sign was chosen in the right-hand member because tan 295° is a negative quantity.

(*c*) $\sin 235° = \sin (180° + 55°) = -$ **sin 55°.**

EXERCISES

1. Use the method of this article to express the trigonometric functions of the following angles in terms of trigonometric functions of angles less than 90°; (*a*) 265°; (*b*) 275°; (*c*) 125°.

2. For each of the following expressions use the method of this article to write an equivalent one in terms of an angle no greater than 45°: sin 85°, tan 338°, sec 247°, cos 197°, cot 130°, csc 500°, sin 640°, cos 1280°, tan 2220°.

3. Express as trigonometric functions of θ each of the following:

(*a*) $\sin (360° - \theta)$.
(*b*) $\cos (720° - 2\theta)$.
(*c*) $\tan (180° - \theta)$.
(*d*) $\sec (540° - \theta)$.
(*e*) $\csc (2 \times 180° + \theta)$.
(*f*) $\sin (360° - 2\theta)$.
(*g*) $\cot (30 \times 90° + \theta)$.
(*h*) $\cos (\theta - 360°)$.

4. Using trigonometric functions and positive angles less than 360°, find three expressions equal to

(*a*) sin 20°.
(*b*) cos 50°.
(*c*) tan 75°.
(*d*) csc 87°.
(*e*) sec 132°.
(*f*) cot 247°.
(*g*) sin 328°.
(*h*) tan 432°.
(*i*) cot 550°.
(*j*) cos 635°.
(*k*) sin 740°.

5. Prove that $\sin 20° = \sin 160° = \cos 290° = -\sin 340°$.

6. Simplify:

(*a*) $\dfrac{\sin 335°}{\csc 155°} + \cos 86° \cos 94°$.

(*b*) $\dfrac{\sin 200°}{\cos 20°} \tan 70° - \sec 50° \cos 130°$.

7. Verify:

(*a*) $\dfrac{\sin \theta}{\cos (180° - \theta)} + \tan (360° + \theta) - \sec (180° + \theta) = \sec \theta$.

(*b*) $\dfrac{\cot (180° + A)}{\cot (180° - A)} - \dfrac{\sin (360° - A)}{\cos (360° - A)} = \tan (720° + A) - 1$.

8. Prove that

$$\cos (90° + A) \cos (270° - A) - \sin (180° - A) \sin (360° - A) = 2 \sin^2 A.$$

26. MISCELLANEOUS EXERCISES

1. The tangent of a certain angle is $-\frac{2}{3}$, and its cosine is $3/\sqrt{13}$. Find all the other trigonometric functions of this angle.

2. Find all the trigonometric functions of a third-quadrant angle whose sine is $-\frac{3}{5}$.

3. Find two positive angles A less than 360° for which

(*a*) $\sin A = -\frac{1}{2}$, (*c*) $\cot A = -1/\sqrt{2}$, (*e*) $\csc A = -2$,
(*b*) $\tan A = \sqrt{3}$, (*d*) $\sec A = \sqrt{2}$, (*f*) $\cos A = -\frac{1}{2}$.

4. For each of the following expressions write an equivalent one in terms of an angle less than 90°:

(*a*) sin 105°. (*c*) sec 340°. (*e*) csc 290°.
(*b*) cos 170°. (*d*) cot 242°. (*f*) tan 184°.

5. For each of the following expressions write an equivalent one in terms of an angle no greater than 45°:

(*a*) sin 170°. (*c*) cot 285°. (*e*) sec 100°.
(*b*) cos 195°. (*d*) tan 330°. (*f*) csc 265°.

6. Find in radical form the value of each of the following:

(*a*) cot 120°. (*c*) sin 240°. (*e*) sec 225°.
(*b*) cos 210°. (*d*) csc 135°. (*f*) tan 600°.

7. Evaluate:

$$\frac{\sin 330° \cos 135°}{\tan 225° \cos 180°} + \frac{\cot 240° \cos 150°}{\sec 300° \sin 270°}.$$

8. Evaluate:

$$\csc^2 300° \sin 60° \tan 150° + \sec^2 210° \cot 240° \cos^2 30°.$$

9. Simplify:

$$\cos 255° \sec 75° \sin 100° \cos 260°.$$

10. Prove that

$$\sin 420° \cos 390° + \cos(-300°) \sin(-330°) = 1.$$

11. Prove that

$$\cos 570° \sin 510° - \sin 330° \cos 390° = 0.$$

12. Prove that

$$\tan y + \tan(-x) - \tan(180° - x) = \tan y.$$

13. Prove that

$$\frac{\sin(180° - y)}{\sin(270° - y)} \tan(90° + y) + \csc^2(270° - y) = 1 + \sec^2 y.$$

14. Evaluate $4\sqrt{3} \tan 330° + 3 \sin 270° \cos 90° - 6 \sin(-30°)$.

15. Find in simple radical form the value of

$$\frac{\csc 225° \sec 330° \cos 690° + \tan 240° \sin 600°}{\cot 330° \sin 240° - \cos 210° \cot 120° \sin 270°}.$$

16. Show that

$$\sin 240° = \sin(-90°) \sin 120° - \cos 270° \cos(-60°).$$

17. Verify that $\sin 240° = 2 \sin 120° \cos 840°$.

18. Verify that

$$\cos 255° = \sin 45° \sin 30° - \cos 45° \cos 30°.$$

19. Verify that $\sin 195° = \sin 135° \cos 60° + \cos 135° \sin 60°$.

20. Verify that $\sin(A + B) = \sin A \cos B + \cos A \sin B$ for (*a*) $A = 330°$, $B = 60°$; (*b*) $A = 135°$, $B = 315°$.

21. Verify that $\cos(A + B) = \cos A \cos B - \sin A \sin B$ for (*a*) $A = 30°$, $B = 60°$; (*b*) $A = 240°$, $B = 330°$.

22. Verify that

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

for (*a*) $A = 240°$, $B = 120°$; (*b*) $A = 315°$, $B = 225°$.

23. Verify that

$$\cos 3A = \cos 2A \cos A - \sin 2A \sin A,$$
$$\sin 3A = \sin 2A \cos A + \cos 2A \sin A,$$

for (*a*) $A = 60°$; (*b*) $A = 135°$; (*c*) $A = 600°$.

24. Verify that

$$\tan^2 x \csc^2 x \cot^2 x \sin^2 x = 1$$

for (*a*) $x = 240°$, (*b*) $x = 300°$, (*c*) $x = 480°$.

25. Verify that

$$\frac{\sin x + 1 - \cos x}{\sin x - 1 + \cos x} = \tan x + \sec x$$

for (*a*) $x = 210°$, (*b*) $x = 225°$, (*c*) $x = 315°$, (*d*) $x = 330°$.

26. Verify that

$$\csc 2A = \cot A - \cot 2A$$

for (*a*) $A = 120°$, (*b*) $A = 210°$, (*c*) $A = 225°$.

27. Verify that

$$\frac{\sin (2x + y) + \sin (2x - y)}{\sin x} = 4 \cos x \cos y$$

for (*a*) $x = 120°$, $y = 60°$; (*b*) $x = 150°$, $y = 120°$.

CHAPTER IV

THE RIGHT TRIANGLE

27. Introduction. In the study of the first chapter we solved a number of right triangles. Although the process in this chapter will be essentially the same as that used before, the treatment given here will be more thorough and complete. All cases will be considered, more complicated figures will be solved, and in some of the problems the computation will be carried out by means of logarithms. For this purpose tables that are more complete and accurate will be used. In practice, logarithms are employed when considerable accuracy is desired; but when three-figure accuracy is sufficient the slide rule may be used. Triangles and rectilinear figures can be solved by means of the slide rule in a small fraction of the time required by logarithmic computation; and even when extreme accuracy is desired, the slide-rule results serve as a rough check.

28. Accuracy. Suppose a man knows that his house is longer than 31.5 ft. but shorter than 32.5 ft. How can he express the length of his house on the basis of this meager knowledge? If he should tell an engineer that his house was 32 ft. long, the engineer would be justified in thinking that the length was correct to the nearest foot. Hence he might argue as follows: The house is more than 31.5 ft. long; otherwise 31 ft. would be a closer approximation than 32 ft. Also, the house is shorter than 32.5 ft.; otherwise 33 ft. would be a better approximation. Similarly, if a man gave 32.3 ft. as the length of his house, an engineer would conclude that it was longer than 32.25 ft. but shorter than 32.35 ft. Evidently the error in this case would not be greater than $\frac{5}{100}$ $(= \frac{1}{20})$ ft., or 0.6 in. The first length, 32 ft., would be spoken of as accurate to two significant figures, the second length, 32.3 ft., as accurate to three significant figures. A number is rounded off (or is accurate) to k significant figures when it is expressed, as nearly as possible, by means of a first

digit different from zero, $k - 1$ digits immediately following the first, and enough zeros to place the decimal point. Thus 0.000512 ft., 318000 in., 0.308 mile, all represent data accurate to three significant figures. Note that neither the four zeros in 0.000512 nor the three zeros in 318000 are significant, since they serve merely to place the decimal point. The numbers 27862, 0.3996, and 38.85 when rounded off to three figures would be 27900, 0.400, 38.8, respectively. 38.85 might have been rounded off to 38.9; we chose 38.8 because many computers take the even digit when there is a choice.

Results got by using a 10-in. slide rule are generally considered accurate to three significant figures, although one cannot always be sure of the last figure. With data accurate to four figures four-place logarithm tables are used, with data accurate to five figures, five-place tables are used, etc. The result of computing $0.0038761\sqrt{4.8724}$ would be written 0.00856 if computed with a 10-in. slide rule, 0.008556 if computed with a four-place logarithm table, and 0.0085560 if computed with a five-place table or a more accurate one.

EXERCISES

1. Round off each of the following numbers to three figures.

(*a*) 6.7245, (*b*) 984.55, (*c*) 69349, (*d*) 4935.

2. A careless engineer gave the height of a flagpole as 48.672 ft. However, the measurements were made so poorly that his result might have been 2 in. in error. What height should have been given?

29. Tables of natural trigonometric functions. By means of advanced mathematics the values of the trigonometric functions have been computed for a large number of angles. On page 69 is listed the values, accurate to three figures, of the trigonometric functions for each degree from 0° to 90°.

The value of a function of an angle between 0° and 45° will be found in the row with the number of degrees in the angle and in the column headed by the name of the function. If the angle lies between 45° and 90°, its value will be found in the row with the number of degrees in the angle and in the column having the name of the function at its foot.

If the angle is not an exact number of degrees, the value of a function of the angle may be found by interpolation. For

NUMERICAL VALUES OF THE TRIGONOMETRIC FUNCTIONS

Degrees	**sin**	**csc**	**tan**	**cot**	**cos**	**sec**	
0	0 000	∞	0.000	∞	1.000	1.000	**90**
1	0.017	57.299	0.017	57.290	1 000	1 000	89
2	0 035	28 654	0.035	28.636	0.999	1 001	88
3	0 052	19 107	0 052	19.081	0.999	1 001	87
4	0 070	14.336	0.070	14.301	0.998	1.002	86
5	0 087	11.474	0.087	11.430	0.996	1.004	85
6	0 105	9 567	0 105	9.514	0.995	1.006	84
7	0.122	8 206	0 123	8 144	0.993	1 008	83
8	0 139	7 185	0.141	7.115	0.990	1.010	82
9	0.156	6.392	0.158	6.314	0.988	1.012	81
10	0 174	5 759	0.176	5.671	0.985	1.015	**80**
11	0 191	5.241	0.194	5 145	0 982	1 019	79
12	0 208	4.810	0 213	4 705	0 978	1 022	78
13	0 225	4 445	0 231	4 331	0 974	1 026	77
14	0.242	4 134	0 249	4.011	0 970	1 031	76
15	0 259	3 864	0 268	3.732	0.966	1 035	75
16	0 276	3 628	0 287	3 487	0 961	1 040	74
17	0 292	3 420	0 306	3 271	0 956	1 046	73
18	0 309	3 236	0 325	3 078	0 951	1 051	72
19	0 326	3.072	0.344	2.904	0.946	1 058	71
20	0 342	2 924	0.364	2 747	0 940	1 064	**70**
21	0 358	2 790	0 384	2 605	0 934	1 071	69
22	0 375	2 669	0 404	2 475	0 927	1 079	68
23	0 391	2 559	0 424	2 356	0 921	1 086	67
24	0 407	2 459	0 445	2.246	0 914	1 095	66
25	0 423	2.366	0 466	2 145	0 906	1 103	65
26	0 438	2 281	0 488	2 050	0 899	1.113	64
27	0 454	2 203	0 510	1 963	0 891	1.122	63
28	0 469	2 130	0 532	1 881	0.883	1.133	62
29	0 485	2.063	0.554	1.804	0.875	1.143	61
30	0 500	2 000	0 577	1.732	0 866	1 155	**60**
31	0 515	1.942	0 601	1.664	0 857	1 167	59
32	0 530	1 887	0 625	1 600	0 848	1 179	58
33	0 545	1 836	0 649	1.540	0 839	1 192	57
34	0.559	1.788	0.675	1.483	0.829	1.206	56
35	0 574	1 743	0 700	1.428	0.819	1 221	55
36	0 588	1 701	0 727	1 376	0 809	1 236	54
37	0 602	1 662	0.754	1 327	0 799	1 252	53
38	0 616	1 624	0 781	1.280	0 788	1 269	52
39	0.629	1 589	0 810	1.235	0 777	1.287	51
40	0 643	1 556	0 839	1 192	0 766	1.305	**50**
41	0 656	1.524	0 869	1 150	0 755	1.325	49
42	0 669	1 494	0 900	1.111	0 743	1.346	48
43	0 682	1 466	0 933	1.072	0.731	1.367	47
44	0 695	1 440	0.966	1.036	0 719	1.390	46
45	0 707	1 414	1 000	1.000	0 707	1.414	45
	cos	**sec**	**cot**	**tan**	**sin**	**csc**	**Degrees**

example, to find sin 57°24′, take from the table the values of sin 57° and sin 58°, and make the following form:

$$60' \left\{ \begin{array}{l} 24' \left\{ \begin{array}{l} \sin 57°00'' = 0.839 \\ \sin 57°24' = ? \end{array} \right\} d \\ \qquad \sin 58°00'' = 0.848 \end{array} \right\} 9.$$

For small changes in an angle, the increment of angle is nearly proportional to the increment of its sine. Therefore

$$\tfrac{24}{60} = \frac{d}{9} \text{ (nearly)}, \qquad \text{or} \qquad d = (\tfrac{24}{60})(9) = 4 \text{ (nearly)}.$$

Adding 0.004 to 0.839, we obtain

$$\sin 57°24' = \mathbf{0.843}.$$

When the value of the function is given, a similar process enables us to find the angle. For example, if $\tan\theta = 0.734$, to find θ we use the table to get $\tan 36° = 0.727$, $\tan 37° = 0.754$, and then make the following form:

$$60' \left\{ \begin{array}{l} x' \left\{ \begin{array}{l} \tan 36° = 0.727 \\ \tan \quad \theta = 0.734 \end{array} \right\} 7 \\ \qquad \tan 37° = 0.754 \end{array} \right\} 27.$$

As before, we write $\dfrac{x'}{60} = \tfrac{7}{27}$, or $x' = (\tfrac{7}{27})(60') = 16'$ (nearly).

Therefore $x = \mathbf{36°16'}$.

EXERCISES

Find the value of each of the expressions numbered 1 to 6:

1. sin 42°40′.

2. cos 54°23′.

3. tan 22°10′.

4. cot 20°35′.

5. sec 62°20′.

6. csc 16°18′.

For each of the following equations, find an acute angle satisfying it:

7. $\sin\theta = 0.672$.

8. $\cos\theta = 0.908$.

9. $\tan\theta = 1.630$.

10. $\cot\theta = 0.518$.

30. Solving right triangles. The sides and the angles of a rectilinear figure are called its parts. It is convenient, when no misunderstanding will result, to refer to a part of a figure or to its magnitude by the same name. When, for example, we speak

of the hypotenuse of a right triangle we shall sometimes mean its longest side and sometimes the length of the longest side. The context will always indicate which meaning is intended.

The conventional way of lettering a triangle is to assign, as was done in Fig. 1, the letters a, b, c to the sides and the letters A, B, C, respectively, to the angles opposite.

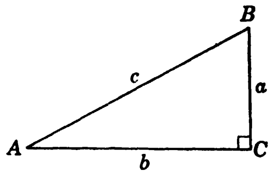


Fig. 1.

When enough parts of a rectilinear figure are given to determine it, the process of finding the remaining parts is called "solving the figure." A right triangle is determined when a side and another part are given. The following italicized rule states the method to be used in solving a right triangle.

Rule. *To find an unknown part of a right triangle when a side and another part are given,* (*a*) *draw a representative figure, and write on each known part its value and on the unknown part a letter;* (*b*) *read from the figure a formula connecting the unknown part and the known parts;* (*c*) *solve for the unknown part, and compute its value.*

When all unknown parts of a triangle have been computed, the work may be checked by reading from the triangle an equation involving the computed parts, finding the value of each member, and observing that these values differ very little if any.

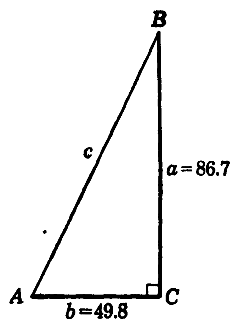


Fig. 2.

Example. Solve the right triangle in which $a = 86.7$ and $b = 49.8$.

Solution. Construct Fig. 2 and from it obtain

$$\tan A = \frac{86.7}{49.8} = 1.741. \qquad (a)$$

From the table of §29 and (*a*) find $A = \mathbf{60°8'}$. To get c, use Fig. 2 to obtain

$$\frac{c}{86.7} = \csc A, \qquad \text{or} \qquad c = 86.7 \csc 60°8'. \qquad (b)$$

Now replace csc 60°8′ by 1.153, its value from the table of §29,

to obtain

$$c = 86.7 \times 1.153 = \mathbf{100.0}.$$

To check, write $\dfrac{49.8}{c} = \cos 60°8'$, or $49.8 = c \cos 60°8'$; replace c by 100.0 and $\cos 60°8'$ by 0.498 to obtain

$$49.8 = 100.0 \times 0.498 = 49.8.$$

EXERCISES

Solve the following right triangles:

1. $a = 32$, $A = 48°25'$.
2. $c = 46.1$, $B = 29°14'$.
3. $c = 46.3$, $a = 25.1$.
4. $a = 3.04$, $b = 2.51$.
5. $b = 67$, $B = 32°15'$.
6. $c = 47.6$, $A = 62°12'$.
7. $a = 41$, $b = 20$.
8. $c = 37$, $A = 69°50'$.

31. Definitions. The terms defined below will be used in the following list of problems and elsewhere in this book.

The **line of sight** is a straight line connecting the eye of an observer with the object viewed.

The **angle of elevation** at a point O of an observed point B higher than O is the angle that the straight line OB makes with the horizontal.

The **angle of depression** at a point C of an observed point O lower than C is the angle that the straight line CO makes with the horizontal.

The **angle subtended by a line** BC at a point O is the angle formed by the rays OB and OC.

For example, in the vertical plane OBC represented in Fig. 3, OB is the line of sight for an observer at O viewing the point B, angle x is the angle of elevation of B at O, angle y is the angle of depression of C at O, and angle BOC is the angle subtended at O by the line BC.

The **compass bearing** of an object is the angle, measured clockwise, that is, from north around toward or through east, between a horizontal line running north from an observer and a horizontal line connecting the observer with the object. The angle measured clockwise in a horizontal plane from north to the direction of motion of an observer is known as his **compass course.**

Thus the bearing of point A for an observer at O in Fig. 4 is 130°; the bearing of B is 330°. A ship sailing from O toward A

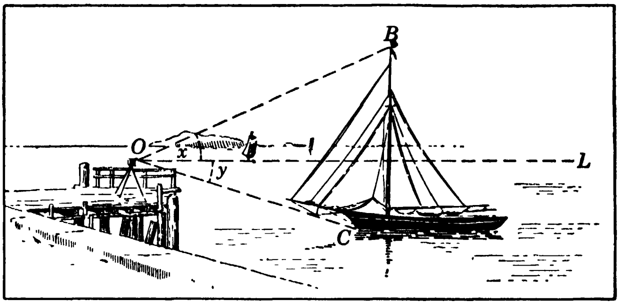


FIG. 3.

would have a compass course of 130°. The direction to an object is often indicated by stating an initial direction, north (*N.*) or south (*S.*), then the angle in degrees, minutes, and seconds, and finally a letter indicating whether the object is east (*E.*) or west (*W.*) of the observer. Thus the bearing of A in Fig. 4 might be given as S. 50° E. and that of B as N. 30° W.

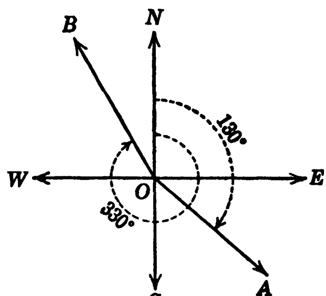


FIG. 4.

When a ship sails a comparatively short distance from a point P to a point P' so as to cut at a constant angle α all meridians crossed by it, we use the words *departure* (*Dep*), *difference in latitude* (*DL*), *distance*, and *course* in speaking of its trip. To understand the meaning of these words, consider the triangular figure $PP'N$ (see Fig. 5) in which PP' represents the path of the ship, PN represents an arc of a meridian, and NP' represents a "small" circle, all points of which have the same lattitude. For practical purposes we consider this triangle as a plane right triangle and call distance NP' **the departure,** PN the **difference in latitude,** PP' the **distance,** and angle α the **course.** The course angle α is measured from the north around through the east from 0° to 360°.

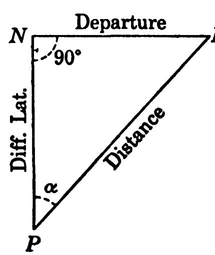


FIG. 5.

EXERCISES

1. The master of a whaling vessel orders his mate to take a position 500 yd. from his ship in a small boat, as shown in Fig. 6. The top of the

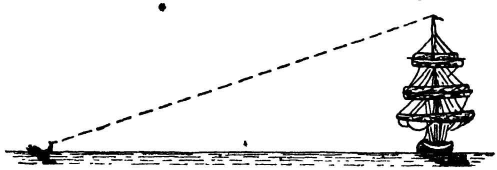


FIG. 6.

whaling vessel's mast above the water line is 213 ft. Find what angle this height will subtend on the mate's sextant when he reaches his position.

2. A ship moving due west at 15 mile per hour passes due north of a given point A, and 20 min. later it bears N. 38°26′ W. from the given point. Find the distance of the ship from A at both times.

3. A surveyor in a barn distant 1 mile from a railroad track observes that a train of cars on the track subtends 35°40′ at his position when one

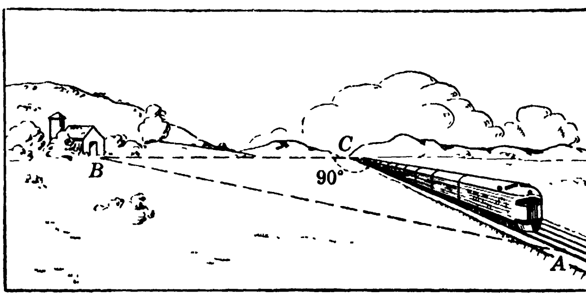


FIG. 7.

end of the train is directly opposite him. How long is the train (see Fig. 7)?

4. From the top of a rock that rises vertically 80 ft. out of the water the angle of depression of a boat is found to be 35°; find the distance of the boat from the foot of the rock.

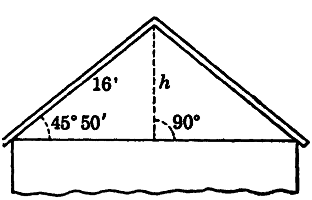


FIG. 8.

5. The shadow of a vertical cliff 113 ft. high just reaches a boat on the sea 93 ft. from its base; find the altitude of the sun.

6. The rafters of a house make an angle of 45°50′ with the horizontal and are 16 ft. long from the top of the wall to the highest point of the roof. Find the height h of the roof above the wall (see Fig. 8).

7. The two stations A and B shown in Fig. 9 are 5200 ft. apart. When an airplane D was directly above A an observer at B found the

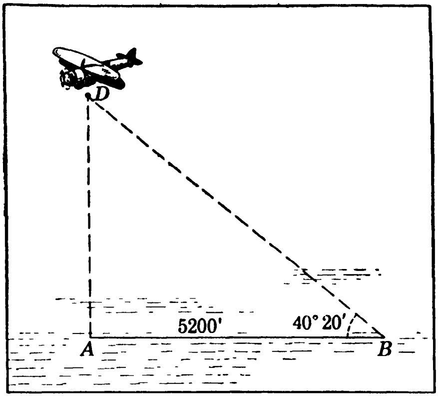


FIG. 9.

angle of elevation of the plane to be 40°20′. Find the distance from the plane to station B.

8. From a point 1420 ft. above a trench, an observer in an airplane finds that the angle of depression of an enemy fort is 23°50′. How far is the trench from the fort?

9. If a ship sails on a course of 42° for 190 miles, what are the departure and difference in latitude?

10. A ship asks bearings from two radio stations A and B. A reports the ship's bearing 82° (Navy Compass) and B reports 127°. Station B is known to be 127 nautical miles from A on bearing 58° from A. Find the difference in latitude and departure of the ship from A.

11. From a point A 175 ft. from the base of a lighthouse a yachtsman finds the angle of elevation of the top to be 29°30′, as shown in Fig. 10. Find the height of the lighthouse.

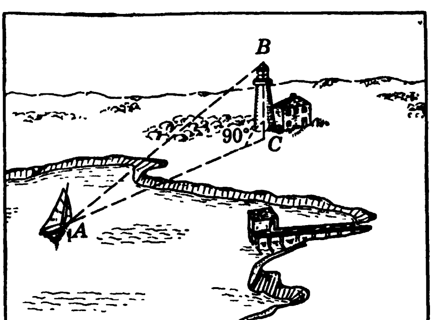


FIG. 10.

12. From an observer's position O, 8.5 ft. above the water (see Fig. 11), the angle of elevation of the top B of the sail was found to be

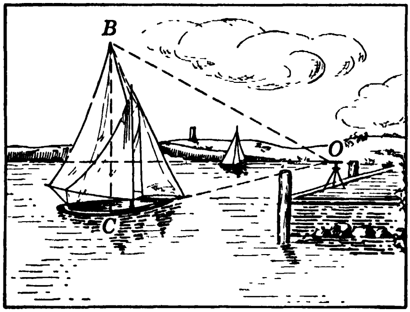


FIG. 11.

28°30′, and the angle of depression of the lowest point C was 20°25′. Find the total height BC of the sailboat.

13. From the top of a hill the angles of depression of two successive milestones on a straight level road leading to the hill are observed to be 5° and 15°. How high is the hill?

32. Solution of the right triangle by slide rule.* A fundamental law of trigonometry, called *the law of sines*, is especially adapted to slide-rule computation. It states that the ratio of the sine of any angle of a triangle to the opposite side is equal to the ratio of the sine of any second angle to its opposite side; or, in symbols,

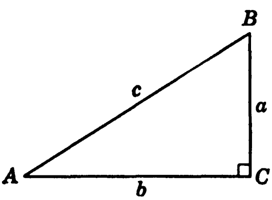


FIG. 12.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}. \tag{1}$$

To prove this for a right triangle, use Fig. 12 to obtain

$$\frac{a}{c} = \sin A, \quad \text{or} \quad \frac{1}{c} = \frac{\sin A}{a}, \tag{2}$$

$$\frac{b}{c} = \sin B, \quad \text{or} \quad \frac{1}{c} = \frac{\sin B}{b}. \tag{3}$$

* A good preparation for making the computations of this article and the next one may be obtained by studying §§127, 128.

Equating the values of $1/c$ in (2) and (3), we get $(\sin A)/a = (\sin B)/b = 1/c$, or replacing 1 by its equal, sin 90°,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin 90^\circ}{c}. \tag{4}$$

To solve the triangle of Fig. 13, substitute 35° for A, 387 for a, and 55° for B in (4) to obtain

$$\frac{S}{D}: \frac{\sin 35^\circ}{387} = \frac{\sin 55^\circ}{b} = \frac{\sin 90^\circ}{c}, \tag{5}$$

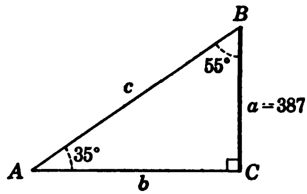


Fig. 13.

where the symbol S/D indicates that the angles in the numerator are to be set on the S scale of the slide rule, and the denominators on the D scale. Hence, in accordance with the proportion principle,

push hairline to 387 on D,
draw 35° of S under the hairline,
push hairline to 55° on S,
at the hairline read b = **552** on D;
push hairline to 90° on S,
at hairline read c = **675** on D.

The student should note that it is unnecessary to write the law of sines to solve a right triangle. Observing that, in accordance with the law of sines, each side and the angle opposite must be set opposite each other on the slide rule, he uses the following rule:

Rule. To solve a right triangle, except when the given parts are two legs, draw the triangle and write on each known part, including the 90° angle, its value, and then

push the hairline to known side on *D*,
draw angle opposite on *S* under hairline,
push hairline to any other known side on *D*;
at the hairline read angle opposite on *S*,
push hairline to any known angle on *S*,
at the hairline read side opposite on *D*.

EXERCISES

Solve the following right triangles by means of the slide rule.

1. $a = 60$, $c = 100$.	**4.** $b = 200$, $A = 64°$.	**7.** $b = 47.7$, $B = 62°56'$.
2. $a = 50.6$, $A = 38°40'$.	**5.** $c = 37.2$, $B = 6°12'$.	**8.** $a = 0.624$, $c = 0.910$.
3. $a = 729$, $B = 68°50'$.	**6.** $c = 11.2$, $A = 43°30'$.	**9.** $a = 83.4$, $A = 72°7'$.

33. Slide-rule solution of a right triangle when two legs are known. When the two legs of a right triangle are known, the smaller acute angle may be found from its tangent, the other acute angle by subtracting the smaller one from 90°, and then the hypotenuse by using the law of sines. Thus, to solve the right triangle shown in Fig. 14, write

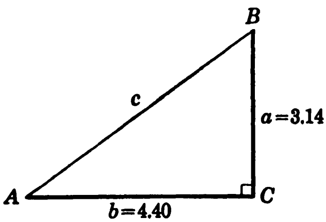


FIG. 14.

$$\tan A = \frac{3.14}{4.40}, \qquad \text{or} \qquad \frac{\tan A}{3.14} = \frac{1}{4.40}.$$

Hence, in accordance with the proportion principle,

set the index of C to 440 on D,
push hairline to 3.14 on D,
at the hairline read $A =$ **35°31′** on T.

Evidently angle $B = 90° - A =$ **54°29′**. To find the hypotenuse c, apply the setting based on the law of sines explained in §32; this leads us to:

push hairline to 3.14 on D,
draw 35°31′ on S under the hairline,
at the index of C read $c =$ **5.40** on D.

If one observes that the first of the three steps just indicated is unnecessary, since the hairline was already set to 3.14 on D

when the angle A was found, he will see that the following rule applies:

Rule. *To solve a right triangle when two legs are known:*

To greater leg on D set proper index of slide,
push hairline to smaller leg on D,
at the hairline read smaller acute angle on T,
draw this angle on S under the hairline,
at index of slide read hypotenuse on D.

EXERCISES

Solve the following right triangles by means of the slide rule:

1. $a = 12.3$, $b = 20.2$.	**4.** $a = 273$, $b = 418$.	**7.** $a = 13.2$, $b = 13.2$.
2. $a = 101$, $b = 116$.	**5.** $a = 28$, $b = 34$.	**8.** $a = 42$, $b = 71$.
3. $a = 50$, $b = 23.3$.	**6.** $a = 12$, $b = 5$.	**9.** $a = 0.31$, $b = 4.8$.

34. Table of logarithms of trigonometric functions. When a high degree of accuracy is desired for the solution of a problem involving trigonometry, the computation should be done by means of logarithms. To facilitate the process, tables of logarithms of the trigonometric functions have been prepared. The sample page printed in the next article is a page from such a table. The complete table gives, accurate to five decimal places, the logarithms of the six trigonometric functions for angles from 0° to 45° at intervals of 1 min. It may be applied directly for all positive angles less than 180°. Tabular differences of successive logarithms are given in the columns headed $d\ 1'$; they are used in the process of interpolation that is designed to take account of seconds of angle.

35. To find the logarithms of a trigonometric function of an angle. The solution of the following example illustrates the method of finding the logarithm of a trigonometric function of a given angle.

Example. Find log sin 35°42′17″.

35° **144°**

′	l sin 9.	d 1′	l csc 10.	l tan 9.	d 1′	l cot 10.	l sec 10.	d 1′	l cos 9.	′
0	75859	18	24141	84523	27	15477	08664	8	91336	60
1	877	18	123	550	26	450	672	9	328	59
2	895	18	105	576	27	424	681	9	319	58
3	913	18	087	603	27	397	690	9	310	57
4	931	18	069	630	27	370	699	9	301	56
5	949	18	051	657	27	343	708	9	292	55
6	967	18	033	684	27	316	717	9	283	54
7	985	18	015	711	27	289	726	8	274	53
8	76003	18	23997	738	26	262	734	9	266	52
9	021	18	979	764	27	236	743	9	257	51
10	039	18	961	791	27	209	752	9	248	50
11	057	18	943	818	27	182	761	9	239	49
12	075	18	925	845	27	155	770	9	230	48
13	093	18	907	872	27	128	779	9	221	47
14	111	18	889	899	26	101	788	9	212	46
15	129	17	871	925	27	075	797	9	203	45
16	146	18	854	952	27	048	806	9	194	44
17	164	18	836	979	27	021	815	9	185	43
18	182	18	818	85006	27	14994	824	9	176	42
19	200	18	800	033	26	967	833	9	167	41
20	218	18	782	059	27	941	842	9	158	40
21	236	17	764	086	27	914	851	8	149	39
22	253	18	747	113	27	887	859	9	141	38
23	271	18	729	140	26	860	868	9	132	37
24	289	18	711	166	27	834	877	9	123	36
25	307	17	693	193	27	807	886	9	114	35
26	324	18	676	220	27	780	895	9	105	34
27	342	18	658	247	26	753	904	9	096	33
28	360	18	640	273	27	727	913	9	087	32
29	378	17	622	300	27	700	922	9	078	31
30	76395	18	23605	85327	27	14673	08931	9	91069	30
31	413	18	587	354	26	646	940	9	060	29
32	431	17	569	380	27	620	949	9	051	28
33	448	18	552	407	27	593	958	9	042	27
34	466	18	534	434	26	566	967	10	033	26
35	484	17	516	460	27	540	977	9	023	25
36	501	18	499	487	27	513	986	9	014	24
37	519	18	481	514	26	486	995	9	005	23
38	537	17	463	540	27	460	09004	9	90996	22
39	554	18	446	567	27	433	013	9	987	21
40	572	18	428	594	26	406	022	9	978	20
41	590	17	410	620	27	380	031	9	969	19
42	607	18	393	647	27	353	040	9	960	18
43	625	17	375	674	26	326	049	9	951	17
44	642	18	358	700	27	300	058	9	942	16
45	660	17	340	727	27	273	067	9	933	15
46	677	18	323	754	26	246	076	9	924	14
47	695	17	305	780	27	220	085	9	915	13
48	712	18	288	807	27	193	094	10	906	12
49	730	17	270	834	26	166	104	9	896	11
50	747	18	253	860	27	140	113	9	887	10
51	765	17	235	887	26	113	122	9	878	9
52	782	18	218	913	27	087	131	9	869	8
53	800	17	200	940	27	060	140	9	860	7
54	817	18	183	967	26	033	149	9	851	6
55	835	17	165	993	27	007	158	10	842	5
56	852	18	148	86020	26	13980	168	9	832	4
57	870	17	130	046	27	954	177	9	823	3
58	887	17	113	073	27	927	186	9	814	2
59	904	18	096	100	26	900	195	9	805	1
60	76922		23078	86126		13874	09204		90796	0
′	9. l cos	d 1′	10. l sec	9. l cot	d 1′	10. l tan	10. l csc	d 1′	9. l sin	′

125° **54°**

Proportional Parts							
″	27	26	18	17	10	9	8
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	1	1	1	1	0	0	0
3	1	1	1	1	0	0	0
4	2	2	1	1	1	1	1
5	2	2	2	1	1	1	1
6	3	3	2	2	1	1	1
7	3	3	2	2	1	1	1
8	4	3	2	2	1	1	1
9	4	4	3	3	2	1	1
10	4	4	3	3	2	2	1
11	5	5	3	3	2	2	1
12	5	5	4	3	2	2	2
13	6	6	4	4	2	2	2
14	6	6	4	4	2	2	2
15	7	6	4	4	2	2	2
16	7	7	5	5	3	2	2
17	8	7	5	5	3	3	2
18	8	8	5	5	3	3	2
19	9	8	6	5	3	3	3
20	9	9	6	6	3	3	3
21	9	9	6	6	4	3	3
22	10	10	7	6	4	3	3
23	10	10	7	7	4	3	3
24	11	10	7	7	4	4	3
25	11	11	8	7	4	4	3
26	12	11	8	7	4	4	3
27	12	12	8	8	4	4	4
28	13	12	8	8	5	4	4
29	13	13	9	8	5	4	4
30	14	13	9	8	5	4	4
31	14	13	9	9	5	5	4
32	14	14	10	9	5	5	4
33	15	14	10	9	6	5	4
34	15	15	10	10	6	5	5
35	16	15	10	10	6	5	5
36	16	16	11	10	6	5	5
37	17	16	11	10	6	6	5
38	17	16	11	11	6	6	5
39	18	17	12	11	6	6	5
40	18	17	12	11	7	6	5
41	18	18	12	12	7	6	5
42	19	18	13	12	7	6	6
43	19	19	13	12	7	6	6
44	20	19	13	12	7	7	6
45	20	20	14	13	8	7	6
46	21	20	14	13	8	7	6
47	21	20	14	13	8	7	6
48	22	21	14	14	8	7	6
49	22	21	15	14	8	7	7
50	22	22	15	14	8	8	7
51	23	22	15	14	8	8	7
52	23	23	16	15	9	8	7
53	24	23	16	15	9	8	7
54	24	23	16	15	9	8	7
55	25	24	16	16	9	8	7
56	25	24	17	16	9	8	7
57	26	25	17	16	10	9	8
58	26	25	17	16	10	9	8
59	27	26	18	17	10	9	8
60	27	26	18	17	10	9	8
″	27	26	18	17	10	9	8
Proportional Parts							

Solution. From the table we find the logarithms in the following form and then compute the differences exhibited.

$$\left.\begin{array}{l}\left.\begin{array}{l}\log\sin 35^\circ 42'00''\\ \log\sin 35^\circ 42'17''\end{array}\right\} 17''\\ \log\sin 35^\circ 43'00''\end{array}\right\} 60'' \quad \left.\begin{array}{l}\left.\begin{array}{l}= 9.76607-10\\ = x\end{array}\right\} y\\ = 9.76625-10\end{array}\right\} d = 18$$

The small changes in angle are nearly proportional to the corresponding changes in logarithm. Therefore

$$\frac{y}{18} = \frac{17}{60}, \qquad \text{or} \qquad y = (18)\,\frac{17}{60} = 5 \text{ (nearly).}$$

and $\log\sin 35^\circ 42'17'' = 9.76607 - 10 + 0.00005 = \mathbf{9.76612 - 10}$.

To perform the interpolation by means of the proportional-parts column, read $9.76607 - 10$ as the $\log\sin 35^\circ 42'$; near this entry in the column headed d 1′ note the number 18, in the proportional parts column headed 18 and in the row with 17 of the column headed ″ read 5, and add 0.00005 to $9.76607 - 10$ to obtain **9.76612 − 10.**

EXERCISES

Find the value of the following:

1. $\log\sin 39^\circ 46'17''$.
2. $\log\sin 59^\circ 31'26''$.
3. $\log\cos 81^\circ 21'43''$.
4. $\log\tan 28^\circ 29'49''$.
5. $\log\cot 49^\circ 16'21''$.
6. $\log\sin 64^\circ 47'51''$.
7. $\log\tan 20^\circ 11'11''$.
8. $\log\csc 16^\circ 17'18''$.
9. $\log\sec 81^\circ 19'31''$.
10. $\log\cos 12^\circ 19'14''$.

36. To find the angle when the logarithm is given. The solution of the following example illustrates the method of finding an angle when the logarithm of a trigonometric function of the angle is given.

Example. Find the acute angle B when $\log\tan B = 0.14920$.

Solution. Observe that 0.14920 lies between the two entries 0.14914 and 0.14941 on the sample page in the column with l tan written at its foot. Therefore write the logarithms in the following form and compute the differences exhibited:

$$\left.\begin{array}{l}\left.\begin{array}{l}\log\tan 54°39' \\ \log\tan B\end{array}\right\}y \\ \log\tan 54°40'\end{array}\right\}60'' \quad \left.\begin{array}{l}\left.\begin{array}{l}= 0.14914 \\ = 0.14920\end{array}\right\}6 \\ = 0.14941\end{array}\right\}d = 27.$$

The small changes in angle are nearly proportional to the small changes in the logarithm. Therefore

$$\frac{y}{60} = \frac{6}{27}, \qquad \text{or} \qquad y = (60)\,\frac{6}{27} = 13'',$$

and

$$B = \mathbf{54°39'13''} \text{ (nearly).}$$

To get the correction y by the proportional parts table: find the tabular difference 27 between the entries 14914 and 14941 of the tangent column; find the difference 14920 − 14914 = 6; opposite the bold-faced 6 in the proportional parts column headed 27 read 13 in the seconds column. Whenever there is a choice between two or more entries, one of which is printed in bold face, *always give preference to the bold-faced entry.*

EXERCISES

Find the value of A in the following:

1. $\log\sin A = 9.31461 - 10.$

2. $\log\tan A = 9.03141 - 10.$

3. $\log\cot A = 0.01210.$

4. $\log\sin A = 9.12867 - 10.$

5. $\log\cos A = 9.92112 - 10.$

6. $\log\cos A = 9.21611 - 10.$

7. $\log\tan A = 0.11161.$

8. $\log\cot A = 9.86192 - 10.$

9. $\log\sin A = 9.02218 - 10.$

10. $\log\sec A = 0.21210.$

37. Solution of the right triangle by means of logarithms. To solve a right triangle by means of logarithms, proceed as indicated in §30, but do the computation with a table of logarithms. The solution of the following example will indicate a very convenient form for the computation as well as the method of procedure.

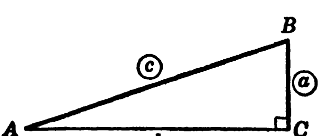


FIG. 15.

Example. Solve the right triangle in which $c = 796.47$, $a = 267.53$.

Solution. Fig. 15 shows the given parts encircled. From it we obtain

$$\sin A = \frac{a}{c}, \tag{a}$$

$$B = 90° - A, \tag{b}$$

$$\frac{b}{c} = \cos A, \qquad \text{or} \qquad b = c \cos A, \tag{c}$$

$$\frac{b}{a} = \cot A, \quad \text{or} \quad b = a \cot A. \quad \text{(Check formula)} \tag{d}$$

From (*a*),

$$\log \sin A = \log a + \operatorname{colog} c.$$

From (*c*),

$$\log b = \log c + \log \cos A.$$

From (*d*),

$$\log b = \log a + \log \cot A.$$

The following forms contains all numbers used in the computation, including the results. Note that every expression on any line refers to the first number in the line

	(*a*)	(*c*)	(*d*)
$a = 267\ 53$	$\log a = 2\ 42737$		$\log a = 2\ 42737$
$c = 796\ 47$	$\operatorname{colog} c = 7\ 09883 - 10$	$\log c = 2\ 90117$	
$A =$ **19°37′37″**	$\log \sin A = 9\ 52620 - 10$	$\log \cos A = 9\ 97401 - 10$	$\log \cot A = 0\ 44780$
$b =$ **750 20**		$\log b = 2.87518$	$\log b - 2\ 87517$
$B = 90°\quad A =$ **70°22′23″**.			

EXERCISES

Solve the following right triangles:

1. $b = 14$, $A = 35°$.

2. $c = 6.275$, $B = 18°47'$.

3. $c = 1200.7$, $a = 885.6$.

4. $a = 8.67892$, $b = 2.4639$.

5. $c = 672.34$, $A = 35°16'25''$.

6. $a = 645.32$, $b = 396.25$.

7. $c = 98.245$, $a = 95.573$.

8. $B = 27°9'14''$, $a = 35.421$.

9. $A = 44°10'38''$, $c = 24.896$.

10. $a = 3.2914$, $b = 5.7842$.

11. $a = 72.131$, $A = 76°17'32''$.

12. $c = 1672.1$, $B = 83°21'11''$.

13. A stay wire for a telephone pole is to be attached to the pole 18 ft. 6 in. above the ground and to make an angle of 42°10′ with the horizontal. Find the length of the stay wire, allowing 3 ft. to make attachment.

14. If a ship sails a course of 19° for 201.85 miles, what is the departure?

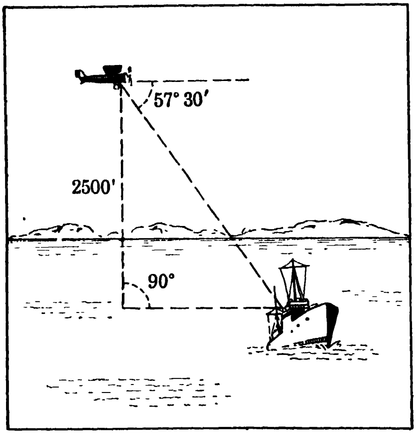


FIG. 16.

15. An observer in an airplane 2500 ft. above the sea sights a destroyer at an angle of depression of 57°30′, as shown in Fig. 16. Find the distance between the plane and the destroyer.

16. If a railroad track rises 30 ft. 4 in. in a horizontal distance of 5280.7 ft., what is its angle of inclination with the horizontal?

17. The area of a right triangle is 23.577 sq. ft., and one angle is 52°24′29″. Find the length of the hypotenuse.

18. A diagonal of a cube intersects a diagonal of one of its faces. Find the angle between these diagonals.

19. A marble $\frac{3}{4}$ in. in diameter subtends an angle of 2°15′30″ at the eye of an observer. How far is it from the observer?

20. If two straight stretches of railway were extended they would meet at a point making an angle of 46°18′ with each other. These two stretches are to be connected by means of a circular arc of radius 4500 ft. Find the distance from the point of tangency to the point of intersection of the straight stretches.

21. A rectangular bin is 42 in. long and 30 in. wide. What angles does a vertical, diagonal partition make with the sides of the bin?

22. In building a suspension bridge a straight cable is run from the top of a pier to a point 852 ft. 7 in. from its foot. If from this point the angle of elevation of the top of the pier is 27°6′, what length of cable is required?

23. In a level field a tunnel was dug into the earth at an angle of 19°20′ with the horizontal. At a point in the field 285 ft. from the entrance of the tunnel an engineer dug a vertical shaft to meet the tunnel. Find the depth of this shaft.

24. Assuming that the earth is a sphere of radius 3958.5 miles, how far is a point in latitude 41°40′ from the earth's axis?

25. On a 2 per cent railroad grade, that is, a rise of 2 ft. in each 100 ft. measured horizontally, what is the angle at which the rails are

inclined to the horizontal? How far must one move along the rails to be 162 ft. higher than at the starting point?

26. Find the radii of the inscribed and circumscribed circles of a regular octagon whose side is 6.2538.

27. At a point A due west of the Washington Monument, which is 555 ft. high, the angle of elevation of its top was observed to be 51°22.9′. Find the angle of elevation of the monument at another point B, 200 ft. west of A, assuming that the points A and B and the base of the monument are in the same horizontal plane.

38. Solution of rectilinear figures. The process of expressing line segments in terms of specified parts of a rectilinear figure was employed in Chap. II. To compute the length of a line segment or the magnitude of an angle forming part of a rectilinear figure, use the process of Chap. II to find an expression for the desired part, and then evaluate this expression.

An expression is convenient for logarithmic computation if its evaluation involves only multiplications and divisions. To obtain such an expression for an unknown length in a rectilinear figure, one generally drops perpendiculars in such a way as to form a chain of right triangles, each of which has a side in common with the next one in the chain. The first triangle has a side of known length, and the last one has as a side the length to be found. The following example will illustrate the procedure.

Example. A surveyor on a mountain peak observes below him two ships lying at anchor 1 mile apart and in the same

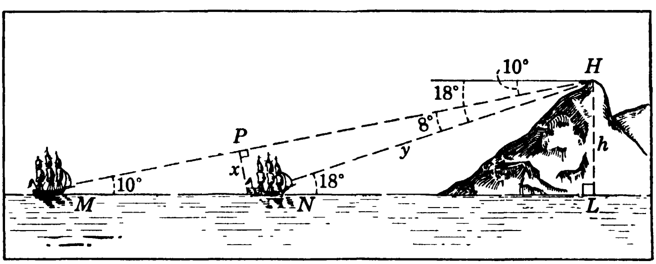


FIG. 17.

vertical plane with his position. He finds the angles of depression of the ships to be 18° and 10°, respectively. How high does the peak rise above the water?

Solution. In Fig. 17, H represents the position of the surveyor, M and N represent the respective positions of the ships,

and the angles marked 10° and 18° represent the angles of depression. Draw NP perpendicular to MH, and denote the length of NP by x and that of NH by y. From triangle MNP,

$$\frac{x}{5280} = \sin 10°, \qquad \text{or} \qquad x = 5280 \sin 10°. \qquad (a)$$

From triangle NPH,

$$\frac{y}{x} = \csc 8°, \qquad \text{or} \qquad y = x \csc 8°. \qquad (b)$$

From triangle LNH,

$$\frac{h}{y} = \sin 18°, \qquad \text{or} \qquad h = y \sin 18°. \qquad (c)$$

Substituting the value of y from (b) and x from (a) in (c), we obtain

$$h = y \sin 18° = x \csc 8° \sin 18° = 5280 \sin 10° \csc 8° \sin 18°.$$

The following form shows the computation:

$$\begin{array}{rll}
 & \log 5280 & = 3.72263 \\
 & \log \sin 10° & = 9.23967 - 10 \\
\log \csc 8° = & \operatorname{colog} \sin 8° & = 0.85644 \\
 & \log \sin 18° & = 9.48998 \\
\hline
h = \mathbf{2035.7} & \log h & = 3.30872
\end{array}$$

Too much accuracy is indicated by this answer for ordinary measurements. The surveyor might be justified in writing 2.0×10^3 ft. or even 2040 ft. as the height of the peak.

EXERCISES

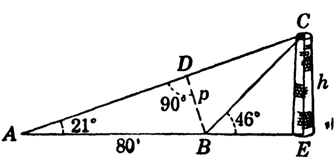


FIG. 18.

1. Two points A and B 80 ft. apart lie on the same side of a tower and in a horizontal line through its foot. If the angle of elevation of the top of the tower at A is 21° and at B is 46°, find the height of the tower (see Fig. 18).

2. Two points A and B 180 ft. apart lie on the same side of a tower on a hill and in a horizontal line passing directly under the tower. The angles of elevation of the top and bottom of the tower viewed from B are 42° and 34°, respectively, and at A the angle of elevation of the bottom is 10°. Find the height of the tower.

Hint. Draw Fig. 19, compute angle $ACB = 24°$, angle $EBC = 8°$, and note that angle $ECF = 42°$. Find in order p_1, BC, p_2, and h.

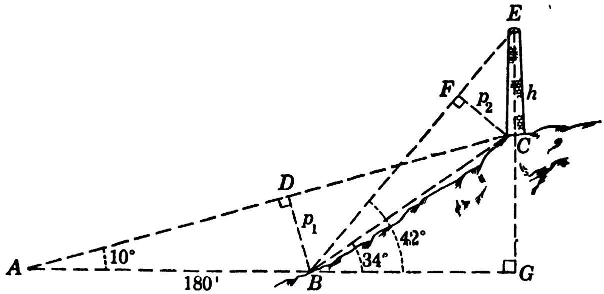


Fig. 19.

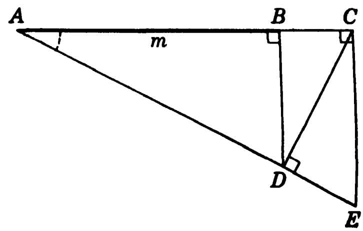


Fig. 20.

3. (*a*) Express BC, DE, and CE in terms of m and A (see Fig. 20).

(*b*) Given $m = 1.96$ in. and $\tan A = 0.482$, find BC, DE, and CE.

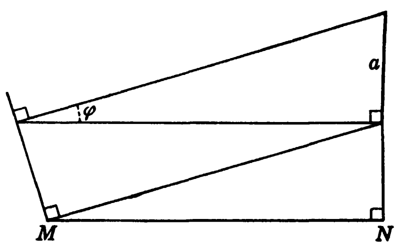


Fig. 21.

4. (*a*) Express all line segments of Fig. 21 in terms of a and φ.

(*b*) Given $a = 34.368$, $\tan \varphi = 0.30517$; use logs to find the length of MN.

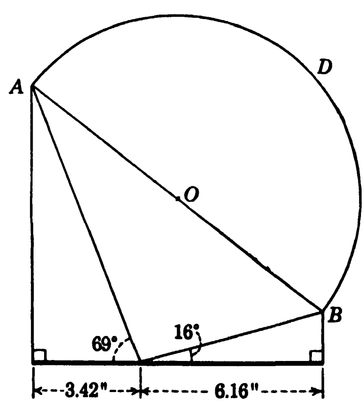


Fig. 22.

5. Find the length of diameter AOB, the length of arc ADB, and the area of the semicircle shown in Fig. 22.

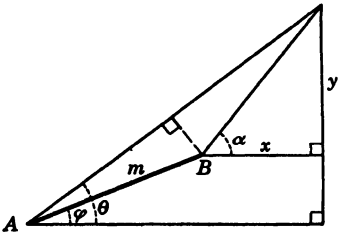


FIG. 23.

6. Given the angles α, φ, θ, and the distance $AB = m$ in Fig. 23; find formulas for x and y.

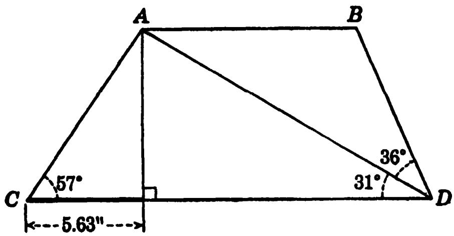


FIG. 24.

7. Given AB parallel to CD, in Fig. 24, find the area of the figure $ABDC$.

8. A mountain peak C is 4135 ft. above sea level, and from C the angle of elevation of a second peak B is 5°. An aviator at A directly

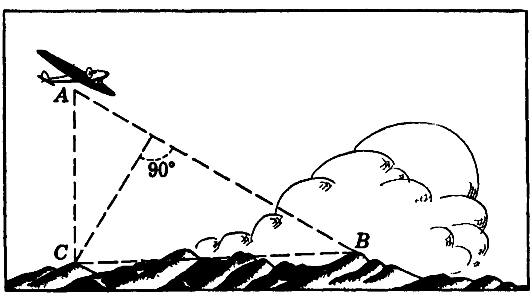


FIG. 25.

over peak C finds that angle CAB is 43°50′ when his altimeter shows that he is 8460 ft. above sea level. Find the height of peak B (see Fig. 25).

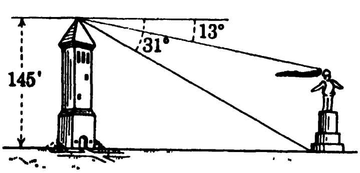


FIG. 26.

9. A tower and a monument stand on a level plain (see Fig. 26). The angles of depression of the top and bottom of the monument viewed from the top of the tower are 13° and 31°, respectively; the height of the tower is 145 ft. Find the height of the monument.

10. As the altitude of the sun decreased from 63°46′ to 50°35′, the length of the shadow of a tower increased 89.65 ft. Find the height of the tower.

11. Figure 27 represents a 600-ft. radio tower. AC and AD are two cables in the same vertical plane anchored at two points C and D on a level with the base of the tower. The angles made by the cables with the horizontal are 44° and 58° as indicated. Find the lengths of the cables and the distance between their anchor points.

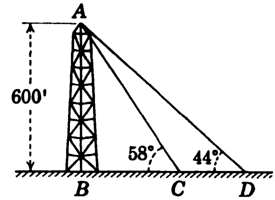


Fig. 27.

12. A building and a tower stand on the same horizontal plane, the tower being 120 ft. high. From the top of the tower the angles of depression of the top and bottom of the building are 22°13.8′ and 44°18.9′, respectively. Find the height of the building.

13. A line AB along one bank of a stream is 315 ft. long, and C is a point on the opposite bank. The angle BAC is 66°30′, and the angle ABC is 54°45′. Find the width of the stream.

14. From a ship two lighthouses bear N. 40° E. After the ship sails at 15 knots on a course of 135° for 1 hr. 20 min., the lighthouses bear 10° and 345°.

(*a*) Find the distance between the lighthouses.

(*b*) Find the distance from the ship in the latter position to the further lighthouse.

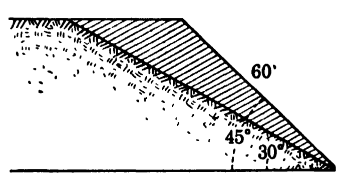
39. MISCELLANEOUS EXERCISES

Solve the following right triangles:

1. $a = 104$, $c = 185$.	**3.** $b = 47.78$, $B = 39°22'$.	**5.** $c = 5.8902$, $B = 67°8'20''$.
2. $c = 625$, $A = 44°$.	**4.** $a = 49967$, $B = 62°43'34''$.	**6.** $a = 4.0007$, $b = 7.9234$.

7. Two straight roads cross at an angle of 52°36′, and there is a town on one road 6520 yd. from the crossing. How far is this town from a point on the other road 2528 yd. from the crossing? (Give two answers.)

8. The Pennsylvania Railroad found it necessary, owing to land slides upon the roadbed, to reduce the angle of inclination of one bank of a certain railway cut near Pittsburgh, Pa., from an original angle of 45° to a new angle of 30°, as shown in Fig. 28. The bank as it originally stood was 200



Cross section
Fig. 28.

ft. long and had a slant length of 60 ft. Find the amount of the earth removed if the top level of the bank remained unchanged.

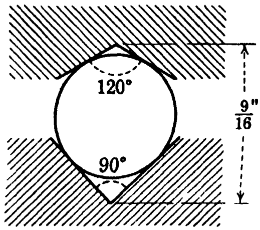


FIG. 29.

9. A slide in a machine is to run on rolling balls. The balls run in grooves with straight sides as shown in Fig. 29. The angle of the upper (moving) groove is 120°, and that of the lower (fixed) groove is 90°. What size of balls should be used?

10. A searchlight situated on a straight coast has a range of 43 miles. A ship sails on a line parallel to the coast and 15 miles from it. What is the distance covered by the ship while it remains within range of the light? What angle is subtended at the light by a line connecting the extreme positions of the ship?

11. A man in a balloon observes that the straight line connecting the bases of two towers, which are 1 mile apart on a horizontal plane, subtends an angle of 70°. If he is exactly above the middle point of this line, find the height of the balloon.

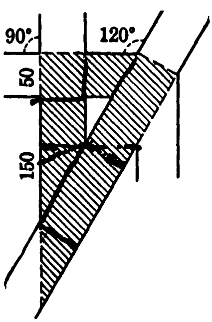


FIG. 30.

12. Find the number of square feet of pavement required for the shaded portion of the streets shown in Fig. 30, all the streets being 50 ft. wide.

13. A flagstaff 25 ft. high stands on the top of a house. From a point on the plain on which the house stands, the angles of elevation of the top and the bottom of the flagstaff are observed to be 60° and 45°, respectively. Find the height of the house.

14. From a point A 10 ft. above the water, the angle of elevation of the top of a lighthouse is 46°, and the angle of depression of its image in the water is 50°. Find the height h of the lighthouse and its horizontal distance from the observer (see Fig. 31).

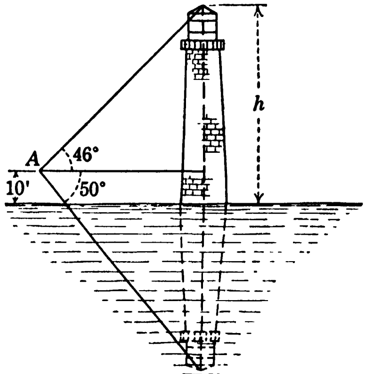


FIG. 31.

15. The pilot in an airplane observes the angle of depression of a light directly below his line of flight to be 30°. A minute later its angle of depression is 45°. If he is flying horizontally in a straight course at the rate of 150 miles per hour, find (*a*) the altitude at which he is flying; (*b*) his distance from the light at the first point of observation.

16. From the top of a building the angle of depression of a point in the same horizontal plane with the base of the building is observed to be 47°13′. What will be the angle of depression of the same point when viewed from a position half way up the building?

17. The captive balloon *C* shown in Fig. 32 is connected to a ground station *A* by a cable of length 842 ft. inclined 65° to the horizontal. In a vertical plane with the balloon and its station and on the opposite side of the balloon from *A* a target *B* was sighted from the balloon on a level with *A*. If the angle of depression of the target from the balloon is 4°, find the distance from the target to a point *C* directly under the balloon.

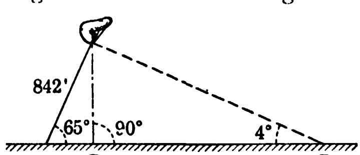


FIG. 32.

18. A straight line *AB* on the side of a hill is inclined at 15° to the horizontal. The axis of a tunnel 486 ft. long is inclined 28°25′ below the horizontal and lies in a vertical plane with *AB*. How long is a vertical hole from the bottom of the tunnel to the surface of the hill?

19. A lighthouse standing on the top of the cliff shown in Fig. 33 is observed from two boats *A* and *B* in a vertical plane through the lighthouse. The angle of elevation of the top of the lighthouse viewed from *B* is 16°, and the angles of elevation of the top and bottom viewed from *A* are 40° and 23°, respectively. If the boats are 1320 ft. apart, find the height of the lighthouse and the height of the cliff.

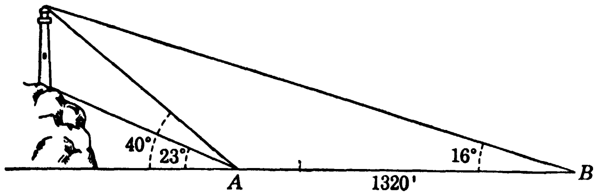


FIG. 33.

20. The church A and the lighthouse B represented in Fig. 34 were observed from a ship at point S to be on a straight line passing through S

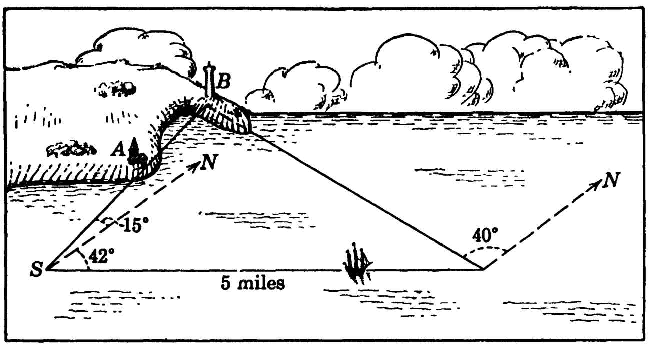


FIG. 34.

and bearing N. 15° W. After sailing 5 miles on a course N. 42° E., the captain of the ship found that A bore due west and B bore N. 40° W. Find the distance from the church to the lighthouse.

21. A tower (Fig. 35) of height h stands on level ground and is due north of point A and due east of point B. At A and B the angles of

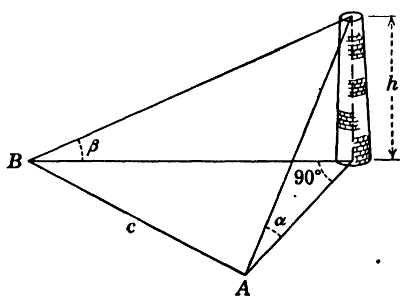


FIG. 35.

elevation of the top of the tower are α and β, respectively. If the distance AB is c, show that

$$h = \frac{c}{\sqrt{\cot^2 \alpha + \cot^2 \beta}}.$$

22. Given the oblique triangle ABC of Fig. 36 in which A, B, and a are known. Show that $b = \frac{a}{\sin A} \sin B$.

Hint. Drop a perpendicular p from the vertex C to the side AB. Find two values of p and equate them.

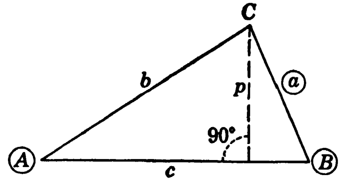


FIG. 36.

23. In the oblique triangle ABC (Fig. 37) show that $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$.

Hint. $AD = b \cos A$, and $DB = c - b \cos A$. Equate two values of p.

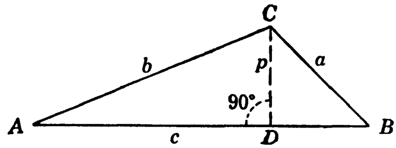


FIG. 37.

24. If R is the radius of a circle, show that the area of a regular circumscribed polygon of n sides is $A = nR^2 \tan \frac{180°}{n}$.

25. Show that the area of a regular polygon of n sides each of length a is given by $A = \frac{na^2}{4} \cot \frac{180°}{n}$.

CHAPTER V

FORMULAS AND GRAPHS

40. Introduction. In Chap. III definitions of the trigonometric functions applicable to an angle of any magnitude were given. In this chapter formulas based on these definitions are deduced, and the graphs of the trigonometric functions are discussed and drawn. A new unit of angular measure, *the radian*, is introduced at this point. It will be used in connection with the graphs and in various places throughout the text.

41. The radian. There is a unit of angular measurement used so frequently in higher mathematics that it is understood to be the unit of measurement when no other is specified. Its importance is due to the fact that various mathematical expressions take simpler forms in terms of this unit than in terms of any other. For this reason we consider it in trigonometry. This unit is called the *radian*.

The angle subtended at the center of a circle by an arc of the circle equal in length to its radius is called a radian. A chord of a circle equal in length to its radius subtends an angle of 60° at its center; an arc on the same circle equal in length to its radius would subtend at its center an angle slightly less. Therefore *an angle of* 1 *radian is slightly less than* 60°. In fact, since the circumference of a circle is $2\pi R$, the length of the radius is contained in the length of the circumference 2π times. Hence, since the complete circumference subtends 360°, 2π radians (= 6.2832 radians) are equivalent to 360°. Accordingly we write

$$2\pi \text{ radians} = 360°, \qquad \text{or} \qquad \pi \text{ radians} = 180°. \tag{1}$$

Since π radians are equivalent to 180°, 1 radian is $1/\pi$ times as much; that is,

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^\circ = 57.2958° = 57°17'45''. \tag{2}$$

Also, from (1), 180° is equivalent to π radians; hence 1° is equivalent to 1/180 times π radians. Accordingly, we write

$$1^\circ = \frac{\pi}{180} \text{ radian} = 0.017453 \text{ radian.} \tag{3}$$

From formulas (2) and (3) it appears that to find the number of degrees in a given number a of radians multiply a by $180/\pi$, and to find the number of radians in a given number b of degrees multiply b by $\pi/180$.

By way of illustration, we write

$$10^\circ = 10\left(\frac{\pi}{180}\right) \text{ radian} = \frac{\pi}{18} \text{ radian;}$$

$$5' = \left(\frac{5}{60}\right)^\circ = \frac{5}{60}\,\frac{\pi}{180} \text{ radian} = \frac{\pi}{2160} \text{ radian;}$$

$$0.75 \text{ radian} = 0.75\left(\frac{180}{\pi}\right)^\circ = 42.9719^\circ = 42^\circ 58' 19''.$$

EXERCISES

1. Express the following angles in radians:

(*a*) 45°.	(*d*) 180°.	(*g*) 22°30′.
(*b*) 60°.	(*e*) 120°.	(*h*) 200°.
(*c*) 90°.	(*f*) 135°.	(*i*) 480°.

2. Express the following angles in degrees:

(*a*) $\pi/3$ radians.	(*c*) $\pi/72$ radian.	(*e*) $20\pi/3$ radians.
(*b*) $3\pi/4$ radians.	(*d*) $7\pi/6$ radians.	(*f*) 0.98π radians.

3. Express in radians the following angles accurate to four significant figures:

(*a*) 1°.	(*c*) 1″.	(*e*) 180°34′20″.
(*b*) 1′.	(*d*) 10°11′25″.	(*f*) 300°25′43″.

4. Find, accurate to the nearest minute, the following angles in degrees and minutes: (*a*) $\frac{1}{10}$ radian; (*b*) $2\frac{1}{2}$ radians; (*c*) 1.6 radians; (*d*) 6 radians.

5. Evaluate the following (without tables):

(*a*) $\tan \frac{1}{6}\pi$.	(*d*) $\tan \frac{1}{3}\pi$.	(*g*) $\cot \frac{4}{3}\pi$.
(*b*) $\sin \frac{1}{3}\pi$.	(*e*) $\sin \frac{1}{2}\pi$.	(*h*) $\sec \frac{2}{3}\pi$.
(*c*) $\cos \frac{1}{4}\pi$.	(*f*) $\cos \pi$.	(*i*) $\tan (-\pi)$.

6. Find the number of radians through which each of the hands of a clock turns in (*a*) 5 min., (*b*) 15 min., (*c*) 45 min., (*d*) 2 hr., (*e*) 6 hr. 30 min.

7. Find the values of x and y in $x = 2(\theta - \sin\theta)$ and $y = 2(1 - \cos\theta)$ when (*a*) $\theta = 0$, (*b*) $\theta = \frac{1}{3}\pi$, (*c*) $\theta = \frac{1}{4}\pi$, (*d*) $\theta = \frac{3}{4}\pi$, (*e*) $\theta = \frac{5}{6}\pi$, (*f*) $\theta = \frac{7}{6}\pi$, (*g*) $\theta = \frac{1}{2}\pi$, (*h*) $\theta = \pi$, (*i*) $\theta = \frac{3}{2}\pi$, (*j*) $\theta = 2\pi$, (*k*) $\theta = 7\pi$.

8. If $x = 5(\cos\theta + \theta\sin\theta)$ and $y = 5(\sin\theta - \theta\cos\theta)$, find the value of x and y when (*a*) $\theta = 0$, (*b*) $\theta = \frac{1}{3}\pi$, (*c*) $\theta = \frac{7}{6}\pi$.

9. Two angles of a triangle are $\frac{1}{3}\pi$ and $\frac{1}{2}$. Find the third angle in sexagesimal units.

42. Length of circular arc. Figure 1 shows a central angle of 1 radian and a central angle of θ radians in a circle of radius r. Since two central angles in a circle have the same ratio as their intercepted arcs, we have

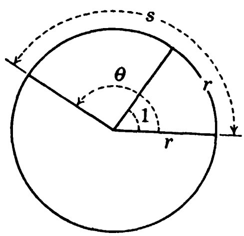


FIG. 1.

$$\frac{\theta}{1} = \frac{s}{r}$$

or

$$\boldsymbol{s = r\theta} \text{ units.} \tag{4}$$

Example 1. A target in the form of a circular arc having its center at a gun is 3000 yd. from the gun and subtends at the gun an angle of 0.015 radian. Find the length of the target.

Solution. Here $r = 3000$ yd., and $\theta = 0.015$ radian. Substituting these numbers in (4), we obtain

$$s = r\theta = 3000(0.015) = \textbf{45 yd.}$$

Example 2. The nautical mile, or sea mile, used in the United States is the arc length subtended on a circle of diameter 7917.59 miles by a central angle of 1′ (7917 miles is approximately the diameter of a sphere having a volume equal to that of the earth). Find the length of the nautical mile accurate to five figures.

Solution. Using formula (4) with

$$r = \tfrac{1}{2}(7917.6)(5280) \qquad \text{and} \qquad \theta = \tfrac{1}{60} \times \frac{\pi}{180},$$

we obtain

$$S = \tfrac{1}{2}(7917.6)(5280)\frac{\pi}{60 \times 180} = \textbf{6080.4 ft.}$$

This is approximately the length of the nautical mile. A more accurate value is 6080.27 ft.

EXERCISES

1. For a circle of radius 720 ft., find the length of arc subtended by a central angle of (*a*) 18°; (*b*) 28°30′; (*c*) 17°20′30″; (*d*) 20′30″; (*e*) 38″; (*f*) $(a/\pi)°$.

2. For a circle having a circumference 3000 ft. in length, find in degrees, minutes, and seconds the central angle subtended by an arc of length (*a*) 300 ft.; (*b*) 10 ft.; (*c*) 1 ft.; (*d*) 12 ft.; (*e*) 2807 ft.

3. Show that a central angle of θ degrees subtends on the circumference of a circle of radius r a length s given by

$$\frac{\theta}{180} = \frac{s}{\pi r}.$$

4. If a circular arc of 30 ft. subtends 4 radians at the center of its circle, find the radius of the circle.

5. If two angles of a plane triangle are respectively equal to 1 radian and $\frac{1}{2}$ radian, express the third angle in degrees.

6. An enemy battery 6000 yd. distant from an observation post subtends at the post an angle of $\frac{1}{80}$ radian. How many yards of front does the battery occupy if the post is directly in front of it?

7. Find approximately the angle in radians subtended by a church spire 160 ft. high at a point in the horizontal plane through the base of the spire and distant 1 mile from it.

8. An automobile whose wheels are 34 in. in diameter travels at the rate of 25 miles per hour. How many revolutions per minute does a wheel make? What is its angular velocity in radians per second?

9. A mil* is $\frac{1}{1600}$ of a right angle. Find the fraction of a radian in 1 mil and the number of mils in 1 radian.

10. A mil is approximately the angle subtended at the center of a circle having a radius of 1000 yd. by an arc length of 1 yd. on the circle. If for a circle r and s are expressed in yards and θ in mils, prove that

$$s = \frac{r\theta}{1000} \text{ (approx.)}.$$

11. An enemy battery, range 6000 yd., subtends an angle of 12 mils. How many yards of front does it occupy (see Exercise 10)?

12. A grade is the hundredth part of a right angle. Express an angle of 1 grade in radians. Also show that a mil is $\frac{1}{16}$ of a grade.

* For a discussion of the mil, see Appendix *A*.

13. Assuming the earth to be a perfect sphere 7917 miles in diameter, find the length of an arc on the equator that subtends an angle of 1° at the center of the earth. Also find the distance between two points on the same meridian if one is 8° north of the equator and the other 5°30′ south of the equator.

14. When the moon is 239,000 miles from the earth, its diameter subtends about 31′ of angle at a point on the earth. Using this fact, compute the diameter of the moon by assuming that the diameter is the arc of a circle having its center at a point on the earth.

15. The larger of two wheels about which a belt is drawn taut has a 3-ft. radius. If the centers of the wheels are 6 ft. apart, and if the arc of the larger wheel in contact with the belt subtends at its center an angle of 3.4 radians, find the radius of the smaller wheel.

16. An automobile has tires 28 in. in diameter. Find the angular velocity in radians per second of the wheel of the automobile when going 50 miles per hour.

17. The drive wheel of a locomotive is 6 ft. in diameter. Find its angular velocity in radians per minute when the train is moving 60 miles per hour.

18. The drive wheel of a locomotive is 6 ft. in diameter. If it makes 500 radians per minute, find the speed of the train in miles per hour.

19. Find the average speed of a man who runs two laps in 30 sec. on a circular track that is 35 ft. in diameter.

In exercises 20 *to* 25, *give approximate answers based on formula* (4).

20. On approaching the shore, the captain of the ship shown in Fig. 2 measured the angle of elevation of the top of a flagstaff and

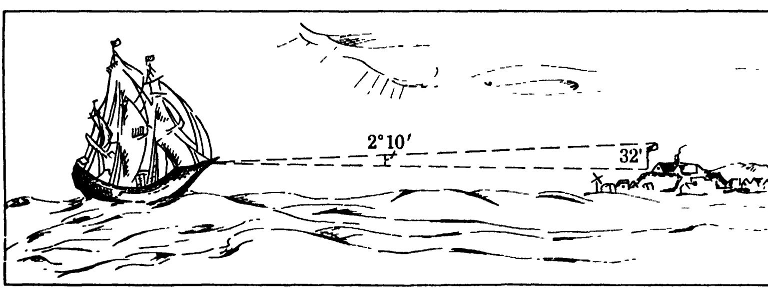


FIG. 2.

found it to be 2°10′. If he knew the height of the staff was 32 ft. and if the foot of the staff was on the same level with the captain's eye, find his distance from the flagstaff.

21. A lighthouse 100 ft. high stands on a rock. From the bottom of the lighthouse the angle of depression of a ship is 2°47′, and from the

top of the lighthouse its angle of depression is 4°2′. What is the height of the rock? What is the horizontal distance from the lighthouse to the ship?

22. The signal-corps man shown in Fig. 3 subtends an angle of 35′ at station S. If he is 6 ft. tall, find his distance from the station.

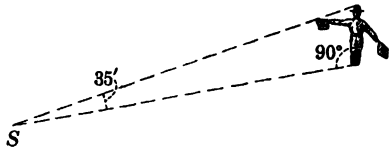


Fig. 3.

23. In approaching a fort situated on a plain, a reconnoitering party finds at one place that the fort subtends an angle of 3° and at a place

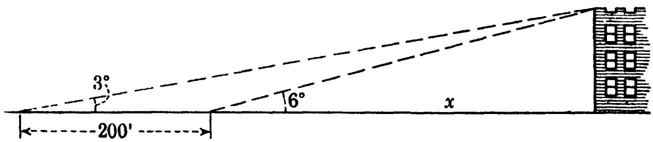


Fig. 4.

200 ft. nearer the fort that it subtends an angle of 6°. How high is the fort, and what is the distance to it from the second place of observation (see Fig. 4)?

24. The line of sight of a gun passes through a target 10,000 yd. away. Through an error in the sighting mechanism of the gun the plane of fire makes an angle of 10 mils with the vertical plane through the line of sight. How far from the target will the shell burst occur if the gun is correctly elevated?

25. Statistics show that when a shell bursts within 50 ft. of an airplane it registers an effective hit. Find, for effective shooting, the maximum deviation from the direction that would give a central hit on an airplane distant 10,000 yd. Assume the airplane extends through a circle of diameter 75 ft.

43. Functions of $90° - \theta$. The trigonometric functions of $90° - \theta$ have been expressed in terms of θ when θ is acute. We shall now show that these same expressions hold true when θ is any angle.

In Fig. 5, OX and OY represent rectangular coordinate axes and angle C_1OB_1 represents an acute angle θ. From B_1 on the terminal side of angle XOB_1, B_1C_1 is drawn perpendicular to the x-axis. Angle XOB is drawn equal to angle $90° - \theta$, OB is taken equal to OB_1, and BC is drawn perpendicular to the x-axis. In Fig. 6 angle θ represents an obtuse angle; in Fig. 7, angle θ is

greater than 180° but less than 270°; and, in Fig. 8, angle θ is greater than 270° but less than 360°. The description of Fig. 5 given above applies also to Figs. 6, 7, and 8 except in the statements of the magnitude of the angle θ.

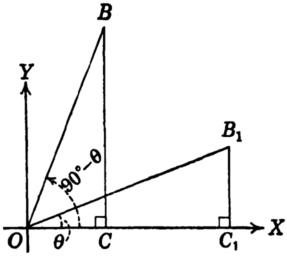


FIG. 5.

The two triangles OC_1B_1 and OCB in each of Figs. 5, 6, 7, and 8 are equal since in each case they have the hypotenuse and an acute angle of one equal, respectively, to the hypotenuse and an acute angle of the other; hence, in each figure, $OB = OB_1$, $OC = C_1B_1$, $CB = OC_1$.

Now let us agree that *a line segment MN parallel to the y-axis is positive when a point moving on this line from M to N is moving in the positive direction of the y-axis,*

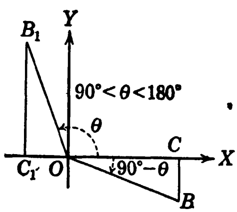


FIG. 6.

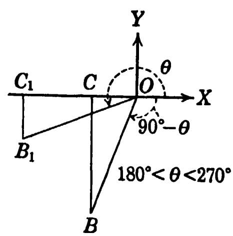


FIG. 7.

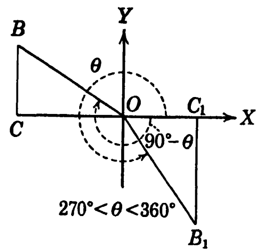


FIG. 8.

and negative when a point moving from M to N is moving in the negative direction of the y-axis. Thus in Fig. 5 the positive direction of the y-axis is toward the top of the page; hence segments C_1B_1 and CB are positive,

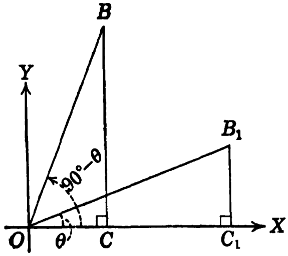


FIG. 5.

but the same segments when read B_1C_1 and BC are considered negative. Let us agree that *a line segment MN parallel to the x-axis is positive when a point moving on this line from M to N is moving in the positive direction of the x-axis, and negative when a point moving from M to N is moving in the negative direction of the x-axis.* Thus in Fig. 5 the positive direction of the x-axis is to the right; hence segments OC_1 and CC_1 are positive but the same segments when read C_1O and C_1C are considered negative. Referring to

Fig. 5, we should write $C_1O = -OC_1$, $C_1C = -CC_1$, $BC = -CB$, and $C_1B_1 = -B_1C_1$. *A line segment forming a hypotenuse will be considered positive in all cases.*

From Fig. 5 we read in accordance with the definitions of the trigonometric functions:

$$\left.\begin{aligned}
\sin(90° - \theta) &= \frac{CB}{OB} = \frac{OC_1}{OB_1} = \cos\theta,\\
\cos(90° - \theta) &= \frac{OC}{OB} = \frac{C_1B_1}{OB_1} = \sin\theta,\\
\tan(90° - \theta) &= \frac{CB}{OC} = \frac{OC_1}{C_1B_1} = \cot\theta,\\
\cot(90° - \theta) &= \frac{OC}{CB} = \frac{C_1B_1}{OC_1} = \tan\theta,\\
\sec(90° - \theta) &= \frac{OB}{OC} = \frac{OB_1}{C_1B_1} = \csc\theta,\\
\csc(90° - \theta) &= \frac{OB}{CB} = \frac{OB_1}{OC_1} = \sec\theta.
\end{aligned}\right\} \quad (5)$$

If, while reading any equation of the group (5), we consider the line segments involved as applying to Fig. 6, Fig. 7, or Fig. 8, we find that the argument holds good in each case. Moreover, the argument will still hold good in the case of each figure if angle θ represents the indicated angle increased or decreased by any number of revolutions; this is true because changing the angle θ by any number of revolutions will not change the line segments of the figure in any way. Hence equations (5) are true for all values of θ.

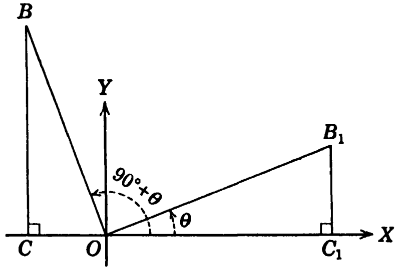


Fig. 9.

44. Functions of $90° + \theta$, $270° + \theta$, $180° \pm \theta$, $-\theta$. In the remaining cases we shall make the argument only for θ an acute angle. However, the directions for drawing the figures and the statements made will apply for all angles θ. For each case considered below, the student may construct figures for angle θ in different quadrants, use the same letters for corresponding positions as are used in the given figure, and note that the statements made apply to his figures as well as to the given one.

In Fig. 9, OX and OY represent rectangular axes of coordinates, angle XOB_1 represents angle θ, and angle XOB represents $90° + \theta$. B_1 is any point on the terminal side of angle θ, and B is taken on the terminal side of $90° + \theta$ so that $OB = OB_1$. The lines B_1C_1 and BC are drawn perpendicular to the x-axis and meet it in points C_1 and C, respectively. Since the triangles OB_1C_1 and OBC are equal, $OC_1 = CB$ and $CO = C_1B_1$. Hence from Fig. 9, we obtain

$$\left.\begin{aligned}
\sin(90° + \theta) &= \frac{CB}{OB} = \frac{OC_1}{OB_1} = \cos\theta,\\
\cos(90° + \theta) &= \frac{OC}{OB} = \frac{-C_1B_1}{OB_1} = -\sin\theta,\\
\tan(90° + \theta) &= \frac{CB}{OC} = \frac{OC_1}{-C_1B_1} = -\cot\theta,\\
\cot(90° + \theta) &= \frac{OC}{CB} = \frac{-C_1B_1}{OC_1} = -\tan\theta,\\
\sec(90° + \theta) &= \frac{OB}{OC} = \frac{OB_1}{-C_1B_1} = -\csc\theta,\\
\csc(90° + \theta) &= \frac{OB}{CB} = \frac{OB_1}{OC_1} = \sec\theta.
\end{aligned}\right\} \qquad (6)$$

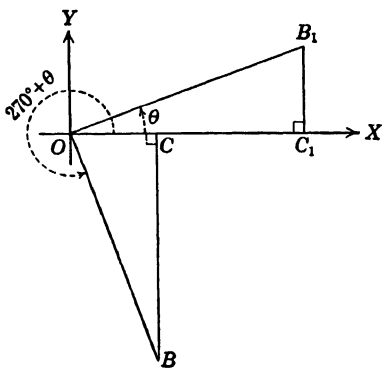


FIG. 10.

Since the construction of the figures for the remaining cases is similar to the constructions already explained, their description will be omitted.

From Fig. 10 we obtain

$$\left.\begin{aligned}
\sin(270° + \theta) &= \frac{CB}{OB} = \frac{-OC_1}{OB_1} = -\cos\theta,\\
\cos(270° + \theta) &= \frac{OC}{OB} = \frac{C_1B_1}{OB_1} = \sin\theta,\\
\tan(270° + \theta) &= \frac{CB}{OC} = \frac{-OC_1}{C_1B_1} = -\cot\theta,
\end{aligned}\right\} \qquad (7)$$

and the other three formulas may be obtained from these by using the reciprocal relations (1) of §11.

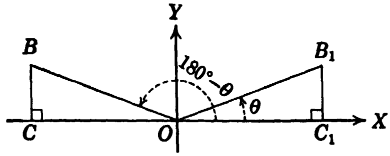


Fig. 11.

From Fig. 11 we obtain

$$\left.\begin{aligned}\sin(180° - \theta) &= \frac{CB}{OB} = \frac{C_1B_1}{OB_1} = \sin\theta,\\ \cos(180° - \theta) &= \frac{OC}{OB} = \frac{-OC_1}{OB_1} = -\cos\theta,\\ \tan(180° - \theta) &= \frac{CB}{OC} = \frac{C_1B_1}{-OC_1} = -\tan\theta,\end{aligned}\right\} \qquad (8)$$

and the other three formulas may be obtained from these by using the reciprocal relations.

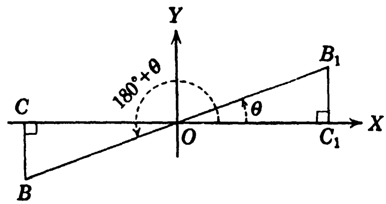


Fig. 12.

From Fig. 12 we obtain

$$\left.\begin{aligned}\sin(180° + \theta) &= \frac{CB}{OB} = \frac{-C_1B_1}{OB_1} = -\sin\theta,\\ \cos(180° + \theta) &= \frac{OC}{OB} = \frac{-OC_1}{OB_1} = -\cos\theta,\\ \tan(180° + \theta) &= \frac{CB}{OC} = \frac{-C_1B_1}{-OC_1} = \tan\theta,\end{aligned}\right\} \qquad (9)$$

and the other three formulas may be obtained from these by using the reciprocal relations.

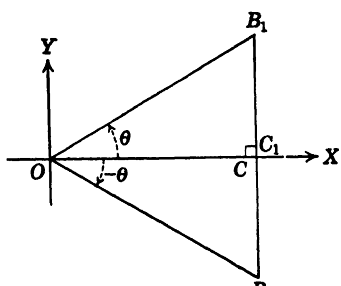


Fig. 13.

From Fig. 13 we obtain

$$\left.\begin{aligned} \sin(-\theta) &= \frac{CB}{OB} = \frac{-C_1B_1}{OB_1} = -\sin\theta, \\ \cos(-\theta) &= \frac{OC}{OB} = \frac{OC_1}{OB_1} = \cos\theta, \\ \tan(-\theta) &= \frac{CB}{OC} = \frac{-C_1B_1}{OC_1} = -\tan\theta, \end{aligned}\right\} \qquad (10)$$

and the other three formulas may be obtained from these by using the reciprocal relations.

45. Functions of $(k\ 90° \pm \theta)$. Observing the formulas (5), (6), and (7) and afterwards the formulas (8), (9), and (10), we perceive the truth of the following statements: (*a*) *each of the six trigonometric functions of k 90° $\pm$ θ, k odd, is numerically equal to the co-function of θ;* (*b*) *each function of k 90° $\pm$ θ, k even, is numerically equal to the same function of θ;* (*c*) *the sign to be placed before the resulting function of θ is the same as the sign of the original function in the quadrant of k 90° $\pm$ θ, where θ is thought of as an acute angle.*

While these rules are convenient, *the student will find that he can draw a rough figure and easily deduce from it the required results.*

EXERCISES

1. Draw the four figures relating to the formulas connected with $90° + \theta$; Fig. 9 is the first figure, in the second one θ should represent an obtuse angle, in the third one θ should represent an angle greater than 180° but less than 270°, and in the fourth one θ should represent an angle greater than 270° but less than 360°. Letter your figures to correspond with Fig. 9 and note that the statements made in group (6) apply to each of your figures.

2. Prove formulas like those in group (6) for $270° + \theta$.

3. If the angles of a triangle are A, B, and C, express each trigonometric function of $A + B$ in terms of a function of C. Do your formulas hold true in each of the cases:

$$0° < A + B < 90°, \qquad A + B = 90°, \qquad 90° < A + B < 180°?$$

4. Derive formulas expressing vers $(180° + \theta)$, vers $(270° - \theta)$, hav $(360° - \theta)$, hav $(-\theta)$, covers $(90° + \theta)$, covers $(180° - \theta)$ in terms of trigonometric functions of θ.*

* For definitions of ***vers*** θ, ***hav*** θ, and ***covers*** θ, see (8), §4.

5. Express as functions of a positive angle less than 90°:

(*a*) $\cos 170°$.
(*b*) $\tan 110°$.
(*c*) $\cot 160°$.
(*d*) $\cos(-20°)$.
(*e*) $\tan(-80°)$.
(*f*) $\sin(-120°)$.

6. Express as functions of θ:

(*a*) $\sin(810° - \theta)$.
(*b*) $\tan(360° - \theta)$.
(*c*) $\cot(270° + \theta)$.
(*d*) $\sin(\theta - 90°)$.
(*e*) $\tan(\theta - 180°)$.
(*f*) $\sec(-180° - \theta)$.
(*g*) $\csc(-630° + \theta)$.
(*h*) $\cos(990° - \theta)$.

7. From the table of natural functions on page 69 find sine, cosine, tangent, and cotangent of

(*a*) $100°15'$.
(*b*) $-395°36'$.
(*c*) $1097°10'$.
(*d*) $-370°10'$.
(*e*) $750°53'$.
(*f*) $-100°18'$.

8. Simplify

(*a*) $\dfrac{\cos(90° + A)}{\sin(-A)} + \dfrac{\sin(90° + A)}{\cos(-A)} + \dfrac{\cot(90° + A)}{\tan(-A)}$.

(*b*) $\cos(270° - \theta)\sin(180° - \theta) - \cos(180° + \theta)\sin(270° + \theta)$.

(*c*) $\dfrac{\cos^2(180° + \theta)}{\sin^2(-\theta)} - \dfrac{\cos(270° - \theta)}{\sin(180° - \theta)}$.

(*d*) $\dfrac{\cos(180° + \theta)}{\sin(270° - \theta)} + \dfrac{\sin^3(-\theta)}{\cos(270° + \theta)}$.

(*e*) $\dfrac{\cot(270° + \theta)}{\cot(270° - \theta)} \times \dfrac{\tan(180° - \theta)}{\tan(180° + \theta)} \times \dfrac{\csc(360° - \theta)}{\sec(360° + \theta)}$.

9. Find the value of

(*a*) $\sin 480° \sin 690° + \cos(-420°)\cos 600°$.

(*b*) $\tan\dfrac{17\pi}{6}\tan\dfrac{14\pi}{3} + \cot\left(-\dfrac{11\pi}{6}\right)\cot\left(-\dfrac{4\pi}{3}\right)$.

(*c*) $\sin\dfrac{19\pi}{6}\cos\left(-\dfrac{11\pi}{6}\right) - \sin\dfrac{7\pi}{3}\cos\left(-\dfrac{4\pi}{3}\right)$.

10. Prove each of the following:

(*a*) $\cos 230° \cos 310° - \sin(-50°)\sin(-130°) = -1$.
(*b*) $\tan 110° \cot 340° - \sin 160° \sec 250° = \csc^2 20°$.

11. Find the numerical value of

$$\tan\frac{11\pi}{6} - 2\sin\frac{4\pi}{3} - \frac{3}{4}\csc^2\frac{3\pi}{4} - 4\cos^2\frac{5\pi}{6}.$$

12. Find the numerical value of

$$\text{vers}\,\frac{11\pi}{6} - \text{covers}\,\frac{23\pi}{3} + \text{hav}\,\frac{7\pi}{6}.$$

13. Simplify

$$\cos\left(\tfrac{1}{2}\pi + x\right)\sin\left(\tfrac{1}{2}\pi - x\right)\tan\left(\tfrac{3}{2}\pi - x\right) - \cos\left(\tfrac{3}{2}\pi + x\right)\cos\left(\tfrac{1}{2}\pi + x\right)\tan\left(\pi - x\right).$$

14. Find the value of each of the following expressions:

(*a*) $\tan^3 660°$. (*c*) $\sin^2 \frac{27}{4}\pi$. (*e*) $\tan\left[(2n + 1)\pi - \frac{1}{3}\pi\right]$.
(*b*) $\cos^3 1020°$. (*d*) $\cot^3 \frac{43}{4}\pi$. (*f*) $\cos\left[(2n - 1)\pi + \frac{1}{6}\pi\right]$.

15. Prove

(*a*) $\cos(\pi - x) + \tan(\pi + x)\sin(-x) = \sec(\pi + x)$.

(*b*) $\sin\left(\dfrac{3\pi}{4} - \theta\right) = -\sin\left(\dfrac{5\pi}{4} + \theta\right)$.

(*c*) $\cos\dfrac{3\pi}{2}\cos\theta + \sin\dfrac{3\pi}{2}\sin\theta = \cos\left(\dfrac{3\pi}{2} - \theta\right)$.

(*d*) $\cos\left(\dfrac{\pi}{2} + x\right)\cos(\pi - x) + \sin\left(\dfrac{\pi}{2} + x\right)\sin(\pi + x) = 0$.

(*e*) $\dfrac{\tan\pi + \tan\theta}{1 - \tan\pi\tan\theta} = \tan(\pi + \theta)$.

(*f*) $\sin(90° + \theta)\sec(270° - \theta) = \tan(270° + \theta)$.

(*g*) $\dfrac{\cos(270° + \theta)}{1 - \cos(180° - \theta)} = \dfrac{1 - \cos(-\theta)}{\cos(90° - \theta)}$.

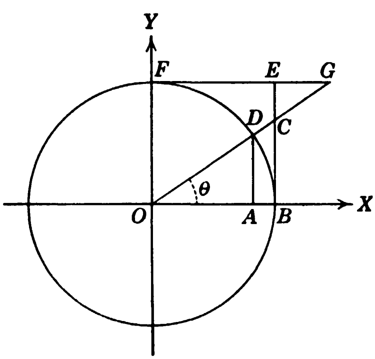


FIG. 14.

16. Express the lengths of the line segments AD, OA, BC, FG, OC, and OG in Fig. 14 in terms of θ if radius OD is 1 unit. Draw figures analogous to Fig. 14 showing θ as (*a*) a second-quadrant angle; (*b*) a third-quadrant angle; (*c*) a fourth-quadrant angle. Do the line values of Fig. 14 apply in the analogous figures?

46. Graph of $y = \sin x$. The graphs of the trigonometric functions are important in that they picture the variations of

these functions and, at the same time, show plainly their periodic nature.

First consider the graph of $y = \sin x$. Using the table of values of trigonometric functions in §29 and using the formulas for expressing the trigonometric functions of any angle in terms of functions of an acute angle, we make Table A:

TABLE A

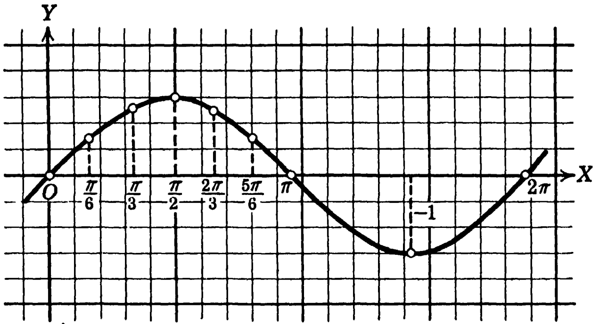
$x°$	x rad.	$y = \sin x$	$x°$	x rad.	$y = \sin x$
0°	0	0	210°	$7\pi/6$	−0.5
30°	$\pi/6$	0.5	240°	$4\pi/3$	−0.866
60°	$\pi/3$	0.866	270°	$3\pi/2$	−1
90°	$\pi/2$	1	300°	$5\pi/3$	−0.866
120°	$2\pi/3$	0 866	330°	$11\pi/6$	−0.5
150°	$5\pi/6$	0 5	360°	2π	0
180°	π	0			

In Fig. 15 are represented the rectangular axes OX and OY. The plotting unit on the x-axis represents $\pi/6$ radian of angle, and three intervals represent the unit of measure to be used in laying off values of $y = \sin x$ along lines parallel to the y-axis.* Plotting points on these axes to correspond with the pairs of values exhibited in Table A and connecting these points with a smooth curve, we obtain the graph shown in Fig. 15. By extending Table A indefinitely for values of x greater than 2π and for negative values of x and by plotting the corresponding points and drawing the curve through them, we should obtain both on the left and on the right of the graph drawn in Fig. 15 curve after curve, each having exactly the same form as the portion shown.

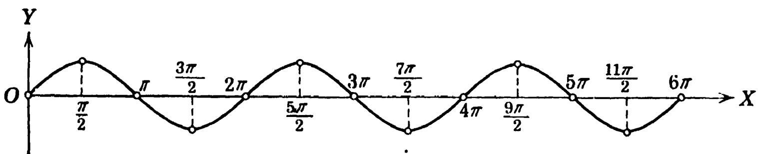
We know that $\sin(2\pi + x) = \sin x$; hence we conclude that when x, starting from any value, varies through 2π radians, $\sin x$

* The unit of measure used for abscissas is not necessarily the same as the unit for ordinates.

varies and takes on all of its possible values once. We express this fact by saying that sin x is periodic and has the period 2π.



$y = \sin x$
FIG. 15.



$y = \sin x$
FIG. 16.

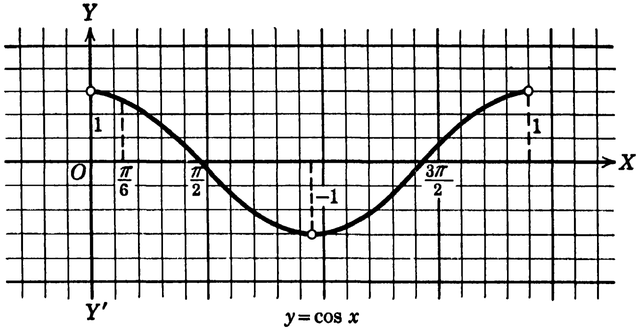
Figure 16 shows the part of the curve $y = \sin x$ corresponding to a change of three periods in x.

47. Graph of $y = \cos x$. Using the table of values of trigonometric functions in §29, and using the formulas for expressing the trigonometric functions of any angle in terms of functions of an acute angle, we make Table B.

Plotting the points to correspond with the pairs of values exhibited in Table B and connecting these points with a smooth curve, we obtain the graph shown in Fig. 17. The complete graph of $y = \cos x$ consists of an endless undulating curve extending both to the right and to the left of the graph drawn in Fig. 17.*

Since $\cos (2\pi + x) = \cos x$, we conclude that $\cos x$ is periodic and has the period 2π.

* Since $\cos x = \sin\left(\frac{\pi}{2} - x\right)$, it appears that the cosine curve has the same form as the sine curve. In fact, if the cosine curve be translated as a whole $\pi/2$ units parallel to the x-axis, it will coincide with the sine curve.



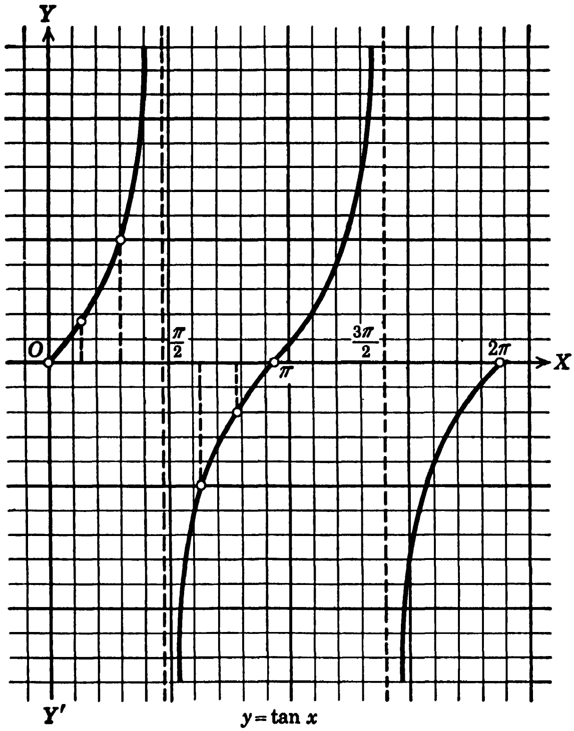
$y = \cos x$
FIG. 17.

TABLE B

$x°$	x rad.	$y = \cos x$
0°	0	1
30°	$\pi/6$	0.866
60°	$\pi/3$	0.5
90°	$\pi/2$	0
120°	$2\pi/3$	−0.5
150°	$5\pi/6$	−0.866
180°	π	−1

$x°$	x rad.	$y = \cos x$
210°	$7\pi/6$	−0.866
240°	$4\pi/3$	−0.5
270°	$3\pi/2$	0
300°	$5\pi/3$	0.5
330°	$11\pi/6$	0.866
360°	2π	1

48. Graph of $y = \tan x$. The Table C of values applies to $y = \tan x$, and Fig. 18 shows the corresponding graph. The straight line perpendicular to the x-axis at $x = \pi/2$ is drawn to indicate that, as the abscissa of a moving point on the curve approaches $\pi/2$ as a limit, the point on the curve approaches indefinitely close to the line, and the length of the ordinate of the point becomes greater and greater without limit. The other line perpendicular to the x-axis where $x = 3\pi/2$ indicates the same kind of situation. Both the table of values and the graph show that the part of the curve from π to 2π has the same form as the part from 0 to π. This follows also from the fact that



$y = \tan x$

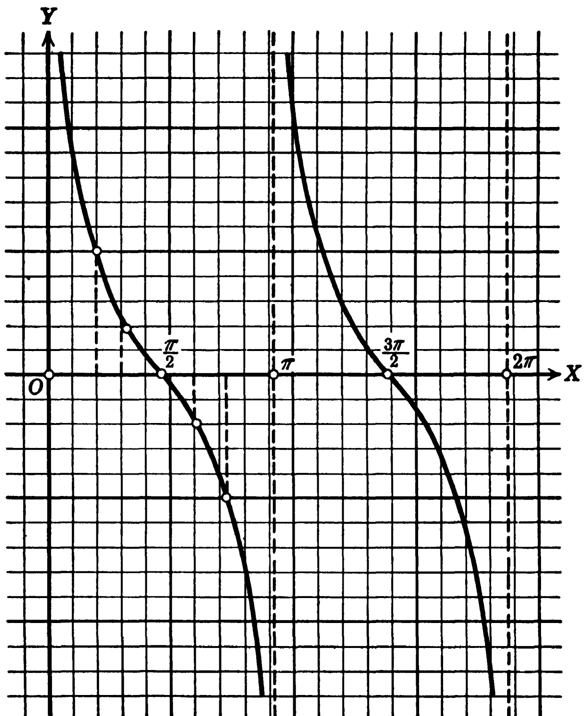
Fig. 18.

Table C

$x°$	x rad.	$y = \tan x$	$x°$	x rad.	$y = \tan x$
0°	0	0	210°	$7\pi/6$	0.577
30°	$\pi/6$	0.577	240°	$4\pi/3$	1 732
60°	$\pi/3$	1.732	270°	$3\pi/2$	∞
90°	$\pi/2$	∞	300°	$5\pi/3$	−1.732
120°	$2\pi/3$	−1.732	330°	$11\pi/6$	−0.577
150°	$5\pi/6$	−0.577	360°	2π	0
180°	π	0			

$\tan x = \tan (\pi + x)$. The complete curve consists of an endless number of branches having the same form as the branch corresponding to the values of x from $\pi/2$ to $3\pi/2$. From this discussion it appears that $\tan x$ is periodic and has the period π.

49. Graphs of $y = \cot x$, $y = \sec x$, $y = \csc x$. The graphs of $y = \cot x$ (see Fig. 19), $y = \sec x$ (see Fig. 20), and $y = \csc x$

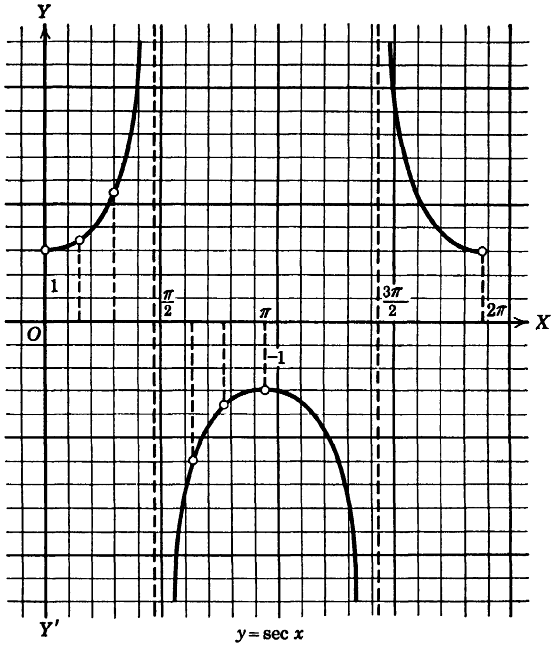


$y = \cot x$
FIG. 19.

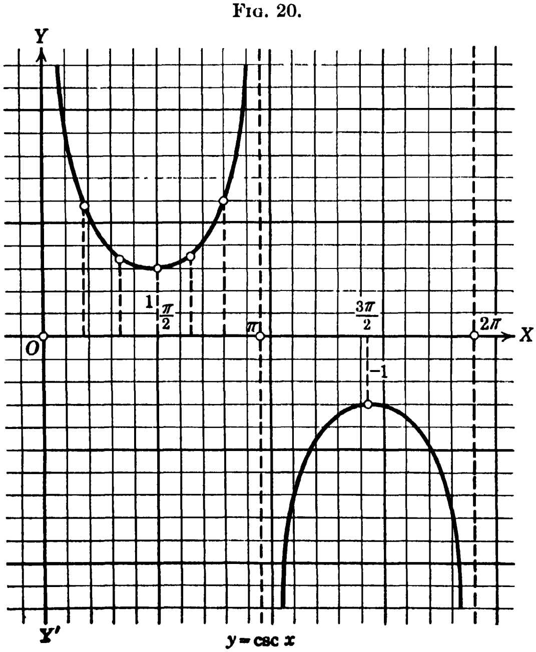
(see Fig. 21) are obtained from the sets of values shown in the following table.

In every case the complete graph consists of an endless number of parts, each congruent with the part shown.

It is easily seen that each of the functions graphed has the same period as its reciprocal function.



$y = \sec x$
FIG. 20.



$y = \csc x$
FIG. 21.

TABLE D

$x°$	x rad.	$y = \cot x$	$y = \sec x$	$y = \csc x$
0°	0	∞	1	∞
30°	$\pi/6$	1.732	1.155	2
60°	$\pi/3$	0.577	2	1.155
90°	$\pi/2$	0	∞	1
120°	$2\pi/3$	−0.577	−2	1 155
150°	$5\pi/6$	−1.732	−1.155	2
180°	π	∞	−1	∞
210°	$7\pi/6$	1.732	−1.155	−2
240°	$4\pi/3$	0.577	−2	−1 155
270°	$3\pi/2$	0	$-\infty$	−1
300°	$5\pi/3$	−0.577	2	−1 155
330°	$11\pi/6$	−1 732	1 155	−2
360°	2π	∞	1	∞

50. Graphs and periods of the trigonometric functions of kθ. First consider the graph of $y = \sin 2x$. The Table E of values is found as in the preceding articles. Plotting the corresponding points and connecting them with a smooth curve, we have Fig. 22. From Table E as well as from Fig. 22 it appears that $y(= \sin 2x)$ has taken its complete set of values twice, once while x passed from 0 to π and once while x passed from π to 2π. Hence we conclude that the period of $\sin 2x$ is $2\pi/2 = \pi$. Since $2x$ passed through 2π radians while x passed through π radians, the period of $\sin 2x$ is one-half the period of $\sin x$. Similarly it appears that kx would pass through 2π radians while x passed through $2\pi/k$ radians; *hence the period of sin kx is $2\pi/k$*. A like argument would show that *the period of cos kx is $2\pi/k$, the period of tan kx is π/k*, and each reciprocal function has the same period as the function of which it is the reciprocal.

In plotting $y = \sin kx$ and $y = \cos kx$, we observe that the greatest value that y may have is unity. Evidently, if we should

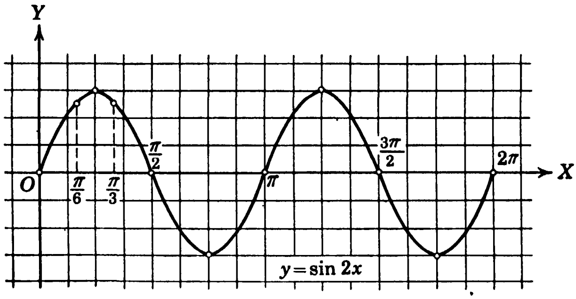


FIG. 22.

TABLE E

x rad.	$x°$	$2x°$	$y = \sin 2x$
0	0°	0°	0
$\pi/6$	30°	60°	0.866
$\pi/3$	60°	120°	0.866
$\pi/2$	90°	180°	0
$2\pi/3$	120°	240°	−0 866
$5\pi/6$	150°	300°	−0.866
π	180°	360°	0
$7\pi/6$	210°	420°	0.866
$4\pi/3$	240°	480°	0.866
$3\pi/2$	270°	540°	0
$5\pi/3$	300°	600°	−0.866
$11\pi/6$	330°	660°	−0.866
2π	360°	720°	0

plot $y = a \sin kx$ or $y = a \cos kx$, the greatest value y could attain in either case would be a. This number a is spoken of as the *amplitude* of y.

EXERCISES

1. Find the period of each of the following functions:

(*a*) $\sin 5\theta$.
(*b*) $3 \cos 8\theta$.
(*c*) $2 \tan \frac{1}{2}\theta$.
(*d*) $\frac{1}{2} \cot 4\theta$.
(*e*) $2 \sec 6\theta$.
(*f*) $242 \csc 2\theta$.
(*g*) $5 \cos (4\theta + 60°)$.
(*h*) $5 \tan \pi\theta$.
(*i*) $3 \cot \frac{1}{3}\varphi$.
(*j*) $7.9 \sec (3\varphi - 45°)$.
(*k*) $2 + \sin 3\varphi$.
(*l*) $6 + \cos 2\varphi$.
(*m*) $-6 \tan \varphi$.
(*n*) $112 \sin (277\theta + 30°)$.

2. Find the amplitude of each of the following functions:

(*a*) $\sin 6\varphi$.
(*b*) $4 \cos 6\varphi$.
(*c*) $\frac{1}{2} \sin \frac{1}{2}\varphi$.
(*d*) $8.6 \cos \varphi$.
(*e*) $334 \cos (\varphi + 60°)$.
(*f*) $\frac{3}{16} \cos (\varphi - \pi)$.
(*g*) $\cos (2 + \theta)$.
(*h*) $8 \sin (241\theta - 45°)$.

3. Plot:

(*a*) $y = \cos x$.
(*b*) $y = 2 \sin x$.
(*c*) $y = 2 \tan x$.
(*d*) $y = 3 \cot x$.
(*e*) $y = 4 \csc x$.
(*f*) $y = 5 \sec x$.
(*g*) $y = 2 \sin 2x$.
(*h*) $y = 4 \tan 2x$.
(*i*) $2y = \cos 2x$.
(*j*) $y = \tan \frac{1}{2}x$.
(*k*) $y = \sin \dfrac{2x}{3}$.
(*l*) $y = \cos \dfrac{x}{4}$.
(*m*) $2y = \cot \dfrac{x}{4}$.
(*n*) $y = \sec (x + \pi)$.
(*o*) $y = \csc \left(\dfrac{\pi}{2} + \theta\right)$.

4. Plot on the same set of axes:

(*a*) $y = \cos x$ and $y = \cos 2x$.
(*b*) $y = \sin x$ and $y = 2 \sin x$.
(*c*) $y = \tan x$ and $y = \cot x$.
(*d*) $y = 2 \sin x$ and $y = 2 \csc x$.
(*e*) $y = \sin 2x$ and $y = \cos \frac{1}{2}x$.
(*f*) $y = 2 \tan 2x$ and $y = \cot \frac{1}{2}x$.

5. Plot the graph of each of the following equations for the indicated range of values of x:

(*a*) $y = \sin x + \cos x$, 0 to 2π.
(*b*) $y = 3 \cos x + 2 \sin x$, $-\pi$ to 2π.
(*c*) $y = \cos x + 3 \sin 2x$, $-\pi$ to π.
(*d*) $y = \sin x - \cos x$, $-\pi$ to π.
(*e*) $y = \sin \frac{1}{2}x - 2 \cos x$, -2π to 2π.

6. By plotting the graph of $y = \sin x$ and using $\csc x = 1/\sin x$, obtain the graph of $y = \csc x$ on the same set of axes and to the same scale.

7. By plotting the graph of $y = \cos x$ and using $\sec x = 1/\cos x$, obtain the graph of $y = \sec x$ on the same set of axes and to the same scale.

8. Plot the curve $y = \sin 3x$. Then construct the curve $y = \csc 3x$ on the same graph by taking account of the fact that $\csc 3x$ and $\sin 3x$ are reciprocal functions.

9. Plot one period of the graph of each of the following equations on the same set of axes and to the same scale:

(*a*) $y = \sin x$, $y = \sin 2x$, and $y = \sin \frac{1}{2}x$.
(*b*) $y = \sin x$, $y = 2 \sin x$, and $y = \frac{1}{2} \sin x$.
(*c*) $y = \cos x$, $y = \cos 2x$, and $y = 2 \cos x$.
(*d*) $y = \cos x$, $y = \frac{1}{2} \cos x$, and $y = \cos \frac{3}{2}x$.

10. If t stands for time in seconds and y for magnitude in volts, then the equation

$$y = 110 \sin 377t$$

represents the voltage causing an alternating current of electricity. Find the period and the maximum magnitude of the voltage.

51. MISCELLANEOUS EXERCISES

1. Express the following angles in radians: 10°, 30°, 45°, 135°, 225°, −270°, −18°, −24°15′.

2. Construct approximately the following angles: 2 radians, $3\frac{1}{2}$ radians, $-\frac{1}{2}$ radian, −4 radians, 9 radians.

3. Construct the following angles:

$$\frac{\pi}{2}, -\frac{\pi}{3}, \frac{\pi}{4}, \pi, -\frac{5\pi}{4}, \frac{5\pi}{2}.$$

4. Express the following angles in degrees: $\frac{\pi}{3}$ radians, π radians, $\frac{2}{3}\pi$ radians, $\frac{7}{4}\pi$ radians, 2 radians, 5 radians, −3 radians.

5. Express the following as functions of an acute angle less than 45°:

(*a*) $\cot \frac{8\pi}{3}$. (*c*) $\tan \frac{17\pi}{10}$.
(*b*) $\sin \frac{37\pi}{14}$. (*d*) $\sec \frac{9\pi}{14}$.

6. In a circle whose radius is 5, the length of an intercepted arc is 12. Find the corresponding central angle (*a*) in radians; (*b*) in degrees.

7. In a circle of radius 12 ft., find the length of the arc intercepted by a central angle of 16°.

8. Find the angle between the tangents to a circle at two points whose distance apart measured on the arc of the circle is 378 ft., the radius of the circle being 900 ft.

9. Assuming the earth's orbit to be a circle of radius 92,000,000 miles, what is the velocity of the earth in its path in miles per second?

10. A belt travels around two pulleys whose diameters are 3 ft. and 10 in., respectively. The larger pulley makes 80 revolutions per minute. Find the angular velocity of the smaller pulley in radians per second; also the speed of the belt in feet per minute.

11. Find the numerical value of:

(*a*) $\cos 30° + \cos 150° + \tan 60° + \tan 120°$.
(*b*) $(\tan 120° - \tan 135°) \times (\tan 120° + \tan 135°)$.
(*c*) $\sin 420° \cdot \cos 390° + \cos (-300°) \cdot \sin (-330°)$.
(*d*) $\cos 570° \cdot \sin 510° - \sin 330° \cdot \cos 390°$.
(*e*) $\tan \frac{2}{3}\pi - \sin \frac{7}{6}\pi + \sec \frac{3}{4}\pi - \csc^2 \frac{5\pi}{3}$.
(*f*) $3 \tan 210° + 2 \tan 120°$.
(*g*) $5 \sec^2 135° - 6 \cot^2 300°$.

12. Simplify each of the following expressions:

(*a*) $\cos \left(\frac{\pi}{2} + x\right) \sin (3\pi - x) - \cos (2\pi + x) \sin \left(\frac{3\pi}{2} - x\right)$.
(*b*) $\sec (180° - \theta) \times \cos \theta \times \tan (180° - \theta) \times \cot \theta$.
(*c*) $\dfrac{\cos (90° - A)}{\sin (180° + A)} + \dfrac{\cos A}{\sin (90° + A)} + \dfrac{\tan (270° + A)}{\tan (-A)}$.
(*d*) $\sec (180° + \theta) \csc (270° + \theta) + \tan (180° - \theta) \cot (270° - \theta)$.
(*e*) $\dfrac{\cos (180° - \theta)}{\sin (90° - \theta)} + \dfrac{\cot (270° + \theta) \cos (270° - \theta)}{\sec (-\theta)}$.
(*f*) $\dfrac{\cos (90° + \alpha)}{\sin (-\alpha)} + \dfrac{\tan (-\alpha)}{\tan (180° + \alpha)}$.
(*g*) $\dfrac{\sin (180° - \theta)}{\cos (90° + \theta)} \times \dfrac{\tan (180° + \theta)}{\cot (90° + \theta)}$.

13. Prove:

(*a*) $\cos (90° + \theta)/\tan (180° + \theta) = 1/\csc (270° - \theta)$.
(*b*) $\dfrac{\tan (180° + \alpha) - \tan (180° - \beta)}{\tan (270° - \alpha) - \cot (-\beta)} = \tan \alpha \tan \beta$.
(*c*) $\dfrac{\tan 3\pi - \tan 2\theta}{1 + \tan 3\pi \tan 2\theta} = \tan (3\pi - 2\theta)$.

(*d*) $(a - b) \tan (90° - x) + (a + b) \cot (90° + x)$
$= (a - b) \cot x - (a + b) \tan x.$

(*e*) $\sin \left(\frac{\pi}{2} + x\right) \sin (\pi + x) + \cos \left(\frac{\pi}{2} + x\right) \cos (\pi - x) = 0.$

(*f*) $\cos (\pi + x) \cos \left(\frac{3\pi}{2} - y\right) - \sin (\pi + x) \sin \left(\frac{3\pi}{2} - y\right) =$
$\cos x \sin y - \sin x \cos y.$

(*g*) $\tan x + \tan (-y) - \tan (\pi - y) = \tan x.$

14. If $\cot 260° = +a$, prove that $\cos 350° = +\frac{1}{\sqrt{1 + a^2}}$.

15. If $\sec 340° = +a$, prove that $\sin 110° = \frac{1}{a}$, and $\tan 110° = -\frac{1}{\sqrt{a^2 - 1}}$.

16. If $\cos 300° = +a$, prove that $\cot 120° = -\frac{a}{\sqrt{1 - a^2}}$.

17. Show that $\cot (270° + x)$ is equal to the negative of the cotangent of the supplementary angle.

18. If $\tan 310° = c$, find $\frac{\sin 320° - \cos 310°}{\tan 140° + \cot 220°}$ in terms of c.

19. If $\sin \theta = -\frac{15}{17}$ and θ is in the third quadrant, find the functions of $(-\theta)$.

20. If $\cot (-\theta) = 2$ and θ is in the second quadrant, find the functions of θ.

21. If $\cos \alpha = -\frac{5}{13}$ and α is in the second quadrant, evaluate:

$$\frac{\sin (180° - \alpha)}{\sec (270° + \alpha)} + \frac{\cos (360° - \alpha)}{\csc (270° - \alpha)}.$$

22. Tan $\beta = \frac{3}{4}$ and β is in the third quadrant, evaluate:

$$\frac{\sin (-\beta) \csc^2 (180° + \beta)}{\sec^2 (90° + \beta)} - \frac{\cot (270° + \beta)}{\tan (180° - \beta)}.$$

23. Plot $y = \sin 2x$.

24. Plot $y = 3 \cos x$.

25. Plot $y = \tan \frac{1}{2}x$.

26. Plot $y = \cos 2x$ and $y = \sec 2x$ on the same set of axes.

27. Express in radians the sum of the angles of a convex polygon of n sides.

28. The rotor of a steam turbine is 2 ft. in diameter and makes 2500 revolutions per minute. The blades of the turbine, situated on the circumference of the rotor, have one-half the velocity of the steam that drives them. What is the velocity of the steam in feet per second?

29. The diameter of the sun is approximately 864,000 miles and at a certain instant it subtends an angle of $32'$ at a point on the earth. Compute the approximate distance from the earth to the sun at this instant.

30. Assuming that the diameter of the smallest sphere clearly visible to the ordinary eye subtends an angle of $1'$ at the eye, find the greatest distance at which a baseball 2.9 in. in diameter can be clearly seen.

31. A horse is tethered to a stake at the corner of a field where the boundaries intersect at an angle of $75°$. How long must the rope be so that the horse can graze over half an acre?

32. Find the length in feet of an arc of $3''$ on the earth's equator.

CHAPTER VI

GENERAL FORMULAS

52. The addition formulas. In many respects, the two formulas

$$\left.\begin{aligned}\sin(A+B) &= \sin A\cos B + \cos A\sin B,\\ \cos(A+B) &= \cos A\cos B - \sin A\sin B,\end{aligned}\right\}\qquad(1)$$

are the most important ones in trigonometry. They are called the addition formulas because they express trigonometric functions of the sum of two angles in terms of the trigonometric functions of the angles. These formulas, holding true as they do for all angles, positive and negative, are the basis of trigonometric analysis. It will appear in what follows that all the formulas of this chapter and many others are derived from them.

53. Proof of the addition formulas. Special case. We shall first prove formulas (1) for the case when both angles A and B are positive acute angles and $A + B < 90°$. In Fig. 1 angles A and B appear as adjacent angles with common vertex O and common side OC. Point D is taken on the terminal side of angle B so that OD is 1 unit long, DC is drawn perpendicular to OC, DG and CE perpendicular to OX, and FC perpendicular to GD.

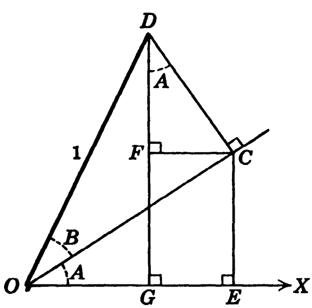


Fig. 1.

The proof of formulas (1) will consist in finding the lengths of the line segments in Fig. 1, writing them on the figure to obtain Fig. 2, and then reading the formulas from Fig. 2. The student may do this for himself without reading the following development.

From Fig. 1 we read

$$\frac{CD}{1} = \sin B, \qquad \frac{OC}{1} = \cos B. \qquad (2)$$

Angle FDC is equal to angle A because its sides are respectively perpendicular to the sides of angle A. Hence, from triangle FCD,

$$\frac{FC}{CD} = \sin A, \qquad \frac{FD}{CD} = \cos A. \tag{3}$$

Replacing CD in (3) by its value $\sin B$ from (2) and multiplying both members of each equation by $\sin B$, we obtain

$$FC = \sin A \sin B, \qquad FD = \cos A \sin B. \tag{4}$$

From triangle OEC,

$$\frac{EC}{OC} = \sin A, \qquad \frac{OE}{OC} = \cos A \tag{5}$$

Replacing OC in (5) by its value $\cos B$ from (2) and multiplying both

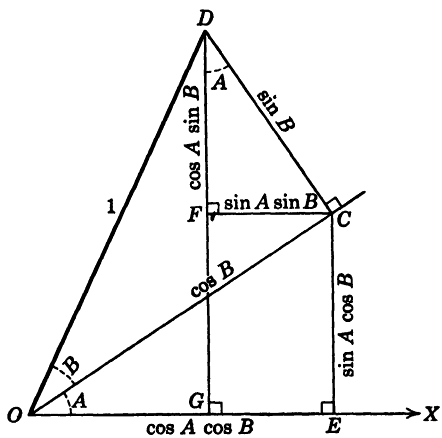


FIG. 2.

members of each equation by $\cos B$, we get

$$EC = \sin A \cos B, \qquad OE = \cos A \cos B. \tag{6}$$

Figure 2 is the result of writing on each line in Fig. 1 its value obtained from one of the equations (2), (4), (5), and (6).

Noting that

$$\sin (A + B) = \frac{GD}{1} = EC + FD$$

and

$$\cos (A + B) = \frac{OG}{1} = OE - FC,$$

we read from Fig. 2

$$\sin(A + B) = \sin A \cos B + \cos A \sin B. \tag{7}$$
$$\cos(A + B) = \cos A \cos B - \sin A \sin B. \tag{8}$$

That the formulas (7) and (8) are true for all values of A and B will be proved in the next article. We shall now assume that they are generally true and use them to obtain two other closely related formulas. Replacing B by $-B$ in (7) and (8), we get

$$\left.\begin{aligned}\sin[A + (-B)] &= \sin A \cos(-B) + \cos A \sin(-B),\\ \cos[A + (-B)] &= \cos A \cos(-B) - \sin A \sin(-B).\end{aligned}\right\} \tag{9}$$

In accordance with §44,

$$\cos(-B) = \cos B \qquad \text{and} \qquad \sin(-B) = -\sin B.$$

Replacing $\cos(-B)$ by $\cos B$ and $\sin(-B)$ by $-\sin B$ in (9), we obtain

$$\sin(A - B) = \sin A \cos B - \cos A \sin B, \tag{10}$$
$$\cos(A - B) = \cos A \cos B + \sin A \sin B. \tag{11}$$

Example. Use (8) to find $\cos 75°$.

Solution. Substituting 45° for A and 30° for B in (8), we obtain

$$\cos 75° = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}.$$

EXERCISES

1. Use (1) to find $\sin(A + B)$ and $\cos(A + B)$ if $\sin A = \frac{1}{3}$ and $\cos B = \frac{2}{3}$, and if A and B are both acute angles.

2. Substitute $A = 30°$, $B = 60°$ in (1) to obtain $\sin 90°$ and $\cos 90°$.

3. Substitute $A = 30°$, $B = 45°$ in (1) to obtain $\sin 75°$ and $\cos 75°$. Then write the values of the trigonometric functions of 75°.

4. By using (1) find $\sin 105°$ and then find the values of the other trigonometric functions of 105° from a right triangle.

5. Given that α and β terminate in the second and in the fourth quadrant, respectively, and that $\sin\alpha = \cos\beta = \frac{3}{5}$, find $\cos(\alpha + \beta)$.

6. Using the table of natural functions, find (*a*) $\sin 31°$ from the functions of 20° and 11°; (*b*) the difference between $\sin(20° + 11°)$ and $\sin 20° + \sin 11°$.

7. Find $\cos(A + B)$ if $\sin A = \frac{3}{5}$ and $\sin B = \frac{5}{13}$, A and B being positive acute angles.

8. If $\tan x = \frac{3}{4}$ and $\tan y = \frac{7}{24}$, find $\sin(x + y)$ and $\cos(x + y)$ when x and y are acute angles.

9. Set $B = A$ in (1) to obtain $\sin 2A$ and $\cos 2A$ in terms of $\sin A$ and $\cos A$.

10. Set $A = 90°$ in (1) and check the result by the methods of Chap. V.

11. Find, by using formulas (7) to (11), the sine and cosine of:

(a) $90° + y$.	(f) $360° - y$.	(k) $-y$.
(b) $180° - y$.	(g) $360° + y$.	(l) $45° - y$.
(c) $180° + y$.	(h) $x - 90°$.	(m) $45° + y$.
(d) $270° - y$.	(i) $x - 180°$.	(n) $30° + y$.
(e) $270° + y$.	(j) $x - 270°$.	(o) $60° - y$.

12. Show that

$$\sin(45° - x) = \frac{\cos x - \sin x}{\sqrt{2}}.$$

13. Show that

$$\cos(210° + x) = \tfrac{1}{2}(\sin x - \sqrt{3}\cos x).$$

14. Show that

$$\cos(60° + \alpha) = \frac{\cos\alpha - \sqrt{3}\sin\alpha}{2}.$$

15. Find $\cos(210° + A)$ if $\sec A = -\sqrt{3}$ and A is a second-quadrant angle.

16. In Fig. 3 let $OB = 1$ unit and express all its line segments in terms of trigonometric functions of θ and φ. Then deduce the formulas

$$\sin(\theta - \varphi) = \sin\theta\cos\varphi - \cos\theta\sin\varphi,$$
$$\cos(\theta - \varphi) = \cos\theta\cos\varphi + \sin\theta\sin\varphi.$$

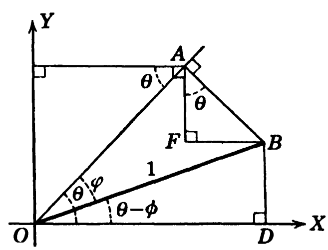


FIG. 3.

17. Show that

$$\sin(\beta - 120°) = -\frac{\sin\beta + \sqrt{3}\cos\beta}{2}.$$

18. Show that

$$\sin(45° + x) = \frac{\cos x + \sin x}{\sqrt{2}}.$$

19. Show that

$$\sin (y + 135^\circ) = \frac{\cos y - \sin y}{\sqrt{2}}.$$

20. Show that

$$\cos (A - B) \cos (A + B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A.$$

21. Show that

$$\sin (x + y) \cos y - \cos (x + y) \sin y = \sin x.$$

22. Show that

$$\sin (x + 60^\circ) - \cos (x + 30^\circ) = \sin x.$$

23. Use (1) to prove that

(a) $\sin 2x = 2 \sin x \cos x$.
(b) $\cos 2x = \cos^2 x - \sin^2 x$.
(c) $\sin 3x = \sin x \cos 2x + \cos x \sin 2x$.
(d) $\sin 3x = \sin 5x \cos 2x - \cos 5x \sin 2x$.

24. Express $\sin 3\theta$ in terms of $\sin \theta$.

25. Express $\cos 3\theta$ in terms of $\cos \theta$.

26. Prove that

$$\frac{\sin (\alpha + \beta) + \sin (\alpha - \beta)}{\cos (\alpha + \beta) + \cos (\alpha - \beta)} = \tan \alpha.$$

54. Removal of restrictions on the addition formulas. In §53 the angles A and B were assumed to be acute angles such that $A + B$ was less than 90°. This article is designed to show that formulas (1) hold true when angles A and B are unrestricted in magnitude and sign.

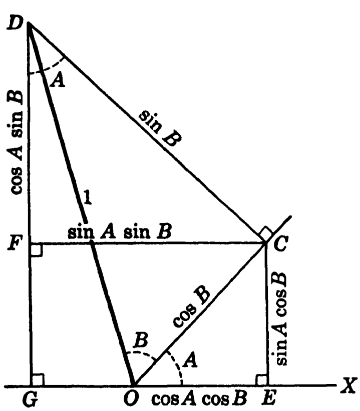


FIG. 4.

The proof given in §53 applies equally well to Fig. 4. Hence formulas (1) are true when A and B are any two acute angles.

Let A be an angle greater than 90° but less than 180°, and let B be a positive acute angle. Let

$$A' = A - 90^\circ. \qquad (12)$$

Since A' and B are acute angles, formulas (1) hold true for them, and we have

$$\left.\begin{aligned}\sin (A' + B) &= \sin A' \cos B + \cos A' \sin B,\\ \cos (A' + B) &= \cos A' \cos B - \sin A' \sin B.\end{aligned}\right\} \tag{13}$$

Replacing A' in (13) by $A - 90°$ from (12) and using the methods of Chap. V, we have

$$\left.\begin{aligned}\sin (A' + B) &= \sin (A + B - 90°) = -\cos (A + B),\\ \cos (A' + B) &= \cos (A + B - 90°) = \sin (A + B),\\ \sin A' &= \sin (A - 90°) = -\cos A,\\ \cos A' &= \cos (A - 90°) = \sin A.\end{aligned}\right\} \tag{14}$$

Substituting the values of $\sin (A' + B)$, $\cos (A' + B)$, $\sin A'$, and $\cos A'$ from (14) in (13), we obtain, after slight simplification,

$$\begin{aligned}\cos (A + B) &= \cos A \cos B - \sin A \sin B,\\ \sin (A + B) &= \sin A \cos B + \cos A \sin B.\end{aligned}$$

Hence it appears that formulas (1) hold true when A is an obtuse angle and B an acute angle.

We next let A be an angle greater than 180° but less than 270° and let B be an acute angle. By letting $A' = A - 90°$ and arguing as above, we prove that formulas (1) hold true for this new case. By continuing this process indefinitely we can show that (1) holds true when A is any positive angle and B is a positive acute angle. Again, letting A be any angle and B an angle greater than 90° but less than 180°, we argue as above and show that (1) holds true in this case. Continuing this process with reference to B, we finally deduce that (1) holds true when A and B are any positive angles.

If (1) holds true for any pair of positive angles A and B, evidently it will still hold true if A and B be decreased by any multiples of 360°. Since any negative angle may be obtained by subtracting some multiple of 360° from a suitable positive angle, and since (1) holds true when A and B are any positive angles, it appears that (1) holds true when A and B represent any negative angles. Hence (1) holds true when A and B represent any angles.

55. Addition and subtraction formulas for the tangent. By using (1), we may deduce addition formulas for the other functions. To express $\tan (A + B)$ in terms of $\tan A$ and $\tan B$ we have

$$\tan (A + B) = \frac{\sin (A + B)}{\cos (A + B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}. \tag{15}$$

Dividing numerator and denominator of the right-hand member of (15) by $\cos A \cos B$, we obtain

$$\tan (A + B) = \frac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}},$$

or

$$\boldsymbol{\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}}. \qquad \textbf{(16)}$$

Since equations (1) hold true for all values of A and B, it follows that (16) holds true for all values of A and B for which $\tan (A + B)$ is defined. Replacing B by $-B$ and therefore $\tan B$ by $\tan (-B) = -\tan B$ in (16), we obtain

$$\boldsymbol{\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}}. \qquad \textbf{(17)}$$

Addition and subtraction formulas for the other functions could be obtained by a similar procedure.

EXERCISES

1. Express the tangent functions in (16) in terms of cotangent functions, and thus deduce that

$$\cot (A + B) = \frac{\cot A \cot B - 1}{\cot A + \cot B}.$$

2. Prove the formula of Exercise 1 by starting from formulas (1).

3. Find $\tan 105°$ in the form of radicals by using (16).

4. Check (16) by substituting in it $A = 4\pi/3$, $B = 3\pi/4$.

5. If $\tan \alpha = \frac{3}{4}$ and $\sin \beta = \frac{12}{13}$, find the functions of $\alpha + \beta$ when α is of the third and β of the second quadrant.

6. If $\cos \alpha = -\frac{40}{41}$ and $\sin \beta = -\frac{5}{13}$, find the functions of $\alpha - \beta$ when α is of the third, and β of the fourth quadrant.

7. If $\tan x = \frac{1}{3}$ and $x - y = 45°$, find $\tan y$.

8. If $\tan y = 2$ and $x + y = 135°$, find $\tan x$.

9. Show that

$$\tan (A - 60°) = \frac{\tan A - \sqrt{3}}{1 + \sqrt{3} \tan A}.$$

10. Show that

$$\tan(x + 45°) + \cot(x - 45°) = 0.$$

11. Show that

$$\cot A - \cot B = \frac{\sin(B - A)}{\sin A \sin B}.$$

12. Show that

$$\frac{\cot(45° - y)}{\cot(45° + y)} = \frac{1 + 2 \sin y \cos y}{1 - 2 \sin y \cos y}.$$

13. In Fig. 1 let $OE = 1$ unit, and express all its line segments in terms of trigonometric functions of A and B. Then deduce formulas (16) and (17).

14. Use (1), (10), and (11) to simplify

(*a*) $\sin 3x \cos 2x + \cos 3x \sin 2x$.
(*b*) $\cos 3x \cos 2x + \sin 3x \sin 2x$.
(*c*) $\sin 3x \cos 2x - \cos 3x \sin 2x$.
(*d*) $\cos(x + 45°) \cos(45° - x) - \sin(x + 45°) \sin(45° - x)$.
(*e*) $\cos^2 x - \sin^2 x$.
(*f*) $\sin x \cos x + \cos x \sin x$.

15. Use (16) to simplify

(*a*) $\dfrac{\tan 3x + \tan 2x}{1 - \tan 2x \tan 3x}$. (*b*) $\dfrac{2 \tan x}{1 - \tan^2 x}$.

16. Express all line segments of Fig. 5 in terms of θ and φ, and from the results deduce a formula for $\sin(\theta + \varphi)$ and a formula for $\cos(\theta + \varphi)$.

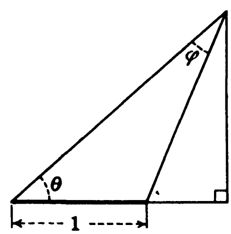


FIG. 5.

17. Taking AC of Fig. 6 equal to 1 unit, express all line segments of the figure in terms of θ and φ, and from your results deduce formula (16).

Hint. Angle $BDC = \varphi$.

18. Taking BC of Fig. 6 equal to 1 unit, deduce from the figure the formula of Exercise 1.

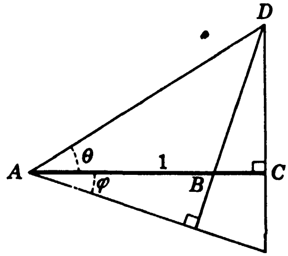


FIG. 6.

19. Prove the following identities:

(*a*) $\tan(45° + \theta) = \dfrac{1 + \tan\theta}{1 - \tan\theta}$.

(*b*) $\tan(45° - x)\tan(135° - x) = -1$.

(*c*) $\cos(60° + x)\cos(30° + x) + \sin(60° + x)\sin(30° + x) = \dfrac{\sqrt{3}}{2}$.

(*d*) $\cos 5x \cos 3x + \sin 5x \sin 3x = 2\cos^2 x - 1$.

(*e*) $\dfrac{\sin(\alpha + \beta)}{\cos(\alpha - \beta)} = \dfrac{\cot\alpha + \cot\beta}{1 + \cot\alpha\cot\beta}$.

(*f*) $\csc 2\theta = \cot\theta - \cot 2\theta$.

20. The expression $a \sin\theta + b\cos\theta$ may be written in the form

$$\sqrt{a^2 + b^2}\left[\frac{a}{\sqrt{a^2 + b^2}}\sin\theta + \frac{b}{\sqrt{a^2 + b^2}}\cos\theta\right].$$

Hence if we let $\tan\alpha = b/a$, we have

$$a\sin\theta + b\cos\theta = \sqrt{a^2 + b^2}(\sin\theta\cos\alpha + \cos\theta\sin\alpha),$$

or

$$a\sin\theta + b\cos\theta = \sqrt{a^2 + b^2}\sin(\theta + \alpha). \tag{A}$$

Write each of the following expressions in the form (A):

(*a*) $2\sqrt{3}\sin\theta + 2\cos\theta$.

(*b*) $a\sin\theta + a\cos\theta$.

(*c*) $\dfrac{1}{\sqrt{2}}\sin\theta + \dfrac{1}{\sqrt{2}}\cos\theta$.

(*d*) $3\sin\theta - \sqrt{3}\cos\theta$.

(*e*) $3\sin\theta + 4\cos\theta$.

(*f*) $\sqrt{2}\cos\theta - \sqrt{2}\sin\theta$.

21. Show that

$$\sin(A + B + C) = \sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C.$$

Hint. $A + B + C = (A + B) + C$.

22. Show that

$$\cos(A + B + C) = \cos A\cos B\cos C - \sin A\cos B\sin C - \cos A\sin B\sin C - \sin A\sin B\cos C.$$

56. The double-angle formulas and the half-angle formulas. To express the trigonometric functions of 2θ in terms of functions of θ replace φ by θ in the addition formulas. Thus, to find $\sin 2\theta$,

substitute θ for ϕ in the formula

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

and obtain

$$\sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\sin\theta$$

or

$$\boldsymbol{\sin 2\theta = 2\sin\theta\cos\theta.} \tag{18}$$

Similarly, from the formula

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

we obtain

$$\boldsymbol{\cos 2\theta = \cos^2\theta - \sin^2\theta,} \tag{19}$$

By using the fact that $\sin^2\theta + \cos^2\theta = 1$, we easily deduce from (19)

$$\boldsymbol{\cos 2\theta = 2\cos^2\theta - 1,} \tag{20}$$

$$\boldsymbol{\cos 2\theta = 1 - 2\sin^2\theta.} \tag{21}$$

From formula (16), we obtain

$$\boldsymbol{\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}.} \tag{22}$$

Solving (20) for $\cos\theta$ and (21) for $\sin\theta$, we obtain

$$\cos\theta = \pm\sqrt{\frac{1 + \cos 2\theta}{2}}, \qquad \sin\theta = \pm\sqrt{\frac{1 - \cos 2\theta}{2}}. \tag{23}$$

To get half-angle formulas, replace θ by $\frac{1}{2}\varphi$ in (23) and obtain

$$\left.\begin{aligned} \boldsymbol{\sin\tfrac{1}{2}\varphi} &= \pm\sqrt{\frac{\mathbf{1 - \cos\varphi}}{\mathbf{2}}},^* \\ \boldsymbol{\cos\tfrac{1}{2}\varphi} &= \pm\sqrt{\frac{\mathbf{1 + \cos\varphi}}{\mathbf{2}}}, \end{aligned}\right\} \tag{24}$$

The plus sign is to be used in the first formula of (24) when $\frac{1}{2}\varphi$ is a first-quadrant*† or a second-quadrant angle, the minus

* Since $\text{hav}\,\varphi = (1 - \cos\varphi)/2$, we have from (24)

$$\sin^2\frac{\varphi}{2} = \frac{1 - \cos\varphi}{2} = \text{hav}\,\varphi.$$

† Occasionally it will be convenient to refer to an angle as belonging to a certain quadrant. If the initial ray of an angle extends from the origin along the positive x-axis, it is called a first-quadrant angle, a second-quad-

sign when $\frac{1}{2}\varphi$ is a third-quadrant or a fourth-quadrant angle. The plus sign is to be used in the second equation of (24) when $\frac{1}{2}\varphi$ is a first-quadrant or a fourth-quadrant angle, the minus sign when $\frac{1}{2}\varphi$ is a second-quadrant or a third-quadrant angle.

To obtain a formula for $\tan \frac{1}{2}\varphi$, divide the first of equations (23) by the second to obtain

$$\tan \tfrac{1}{2}\varphi = \frac{\sin \frac{1}{2}\varphi}{\cos \frac{1}{2}\varphi} = \pm\sqrt{\frac{1 - \cos \varphi}{2}} \times \sqrt{\frac{2}{1 + \cos \varphi}},$$

or

$$\mathbf{\tan \tfrac{1}{2}\varphi = \pm\sqrt{\frac{1 - \cos \varphi}{1 + \cos \varphi}}.} \tag{25}$$

The plus sign is to be used when $\frac{1}{2}\varphi$ is a first-quadrant or a third-quadrant angle, the minus sign when $\frac{1}{2}\varphi$ is a second-quadrant or a fourth-quadrant angle. From (25) we also have

$$\tan \tfrac{1}{2}\varphi = \pm\sqrt{\frac{(1 - \cos \varphi)(1 - \cos \varphi)}{(1 + \cos \varphi)(1 - \cos \varphi)}} = \frac{1 - \cos \varphi}{\sin \varphi}. \tag{26}$$

Since $1 - \cos \varphi$ is never negative and $\sin \varphi$ always has the same sign as $\tan \frac{1}{2}\varphi$, the right-hand member of (26) does not require the $\pm$ signs.

EXERCISES

1. If $\sin \alpha = \frac{3}{5}$, $\cos \alpha = -\frac{4}{5}$, find $\sin 2\alpha$, $\cos 2\alpha$, $\tan 2\alpha$, $\sin \frac{1}{2}\alpha$, $\cos \frac{1}{2}\alpha$, and $\tan \frac{1}{2}\alpha$.

2. Use formulas (24) to find $\sin (22\frac{1}{2})^\circ$ and $\cos (22\frac{1}{2})^\circ$ from the fact that $\cos 45^\circ = 1/\sqrt{2}$.

3. Verify the following identities:

(*a*) $\cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x.$

(*b*) $\dfrac{\sin 2\alpha}{\sin \alpha} - \dfrac{\cos 2\alpha}{\cos \alpha} = \sec \alpha.$

(*c*) $\cos^2 (45^\circ + x) - \sin^2 (45^\circ + x) = -\sin 2x.$

(*d*) $\left(\sin \dfrac{\theta}{2} - \cos \dfrac{\theta}{2}\right)^2 = 1 - \sin \theta.$

(*e*) $\cos^4 \theta - \sin^4 \theta = \cos 2\theta.$

(*f*) $2 \operatorname{hav} \theta = \dfrac{\tan^2 \theta}{1 + \sec \theta + \tan^2 \theta}.$

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rant angle, a third-quadrant angle, or a fourth-quadrant angle according as its terminal side lies in the first, second, third, or fourth quadrant.

(g) $\dfrac{\sin 2\alpha + \sin \alpha}{1 + \cos \alpha + \cos 2\alpha} = \tan \alpha.$

(h) $\tan 2\theta = \dfrac{2}{\cot \theta - \tan \theta}.$

(i) $\tan \frac{1}{2}\varphi = \csc \varphi - \cot \varphi.$

(j) $\operatorname{hav} \varphi = \sin^2 \frac{1}{2}\varphi.$

(k) $\cos^6 \theta - \sin^6 \theta = \cos 2\theta - \frac{1}{8} \sin 4\theta \sin 2\theta.$

4. Substitue $\theta = 2x$, $\varphi = x$ in $\sin (\theta + \varphi) = \sin \theta \cos \varphi + \cos \theta \sin \varphi$ and then use the double-angle formulas to derive

$$\sin 3x = 3 \sin x \cos^2 x - \sin^3 x = 3 \sin x - 4 \sin^3 x.$$

5. Using a method similar to the one suggested in Exercise 4, derive.

(a) $\cos 3x = 4 \cos^3 x - 3 \cos x.$

(b) $\sin 4x = 4 \sin x \cos x(2 \cos^2 x - 1).$

6. Derive a formula expressing $\sin 4x$ in terms of $\sin x$ and a formula expressing $\tan 4x$ in terms of $\tan x$.

7. Prove that, if $z = \tan \dfrac{\theta}{2}$, then

$$\sin \theta = \frac{2z}{1 + z^2}, \qquad \cos \theta = \frac{1 - z^2}{1 + z^2}, \qquad \tan \theta = \frac{2z}{1 - z^2}.$$

8. Find $\sin 18°$ in radical form.

Hint. First write $\cos 3x = \sin 2x$ where $x = 18°$, and express both members in terms of $\sin x$ and $\cos x$. Solve the resulting equation for $\sin x$.

9. If θ is an angle in the second quadrant and $\tan \theta = -\frac{5}{12}$, find

$\cot 2\theta.$	$\cos (270° - 2\theta).$
$\sin (180° - \theta).$	$\csc (180° + 2\theta).$

10. Show that

(a) $\cot \dfrac{x}{4} = \dfrac{\sin \dfrac{x}{2}}{1 - \cos \dfrac{x}{2}}.$

(b) $\cot \dfrac{x}{2} + \tan \dfrac{x}{2} = 2 \csc x.$

(c) $\dfrac{1 - \tan^2 \dfrac{x}{2}}{1 + \tan^2 \dfrac{x}{2}} = \cos x.$

(d) $\tan \frac{1}{2}x = \dfrac{\sin x}{1 + \cos x}.$

(e) $\cot \frac{1}{2}x = \dfrac{\sin x}{1 - \cos x}.$

(f) $\sin 2x = \dfrac{2 \cot x}{1 + \cot^2 x}.$

11. (*a*) Show that $\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$.

(*b*) Show that $\tan 4x = \dfrac{4 \tan x(1 - \tan^2 x)}{1 - 6 \tan^2 x + \tan^4 x}$.

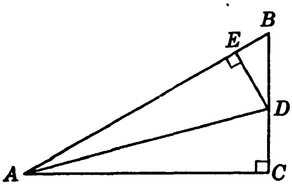

FIG. 7.

12. In Fig. 7, AD bisects the angle A and DE is perpendicular to AB. Hence $DE = CD$. Show from the figure that

$$\tan \tfrac{1}{2}A = \frac{\sin A}{1 + \cos A}.$$

13. Find all line segments of Figs. 8, 9, and 10 in terms of θ, and write several identities from your figures. Verify these identities in the usual way.

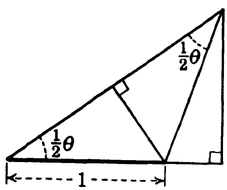

FIG. 8.

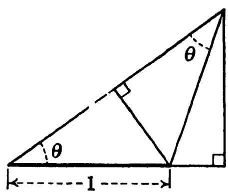

FIG 9.

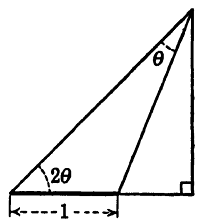

FIG. 10.

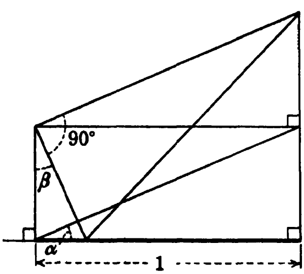

FIG. 11.

14. Prove the formula for $\tan (\alpha + \beta)$ from Fig. 11 by using line values.

15. Prove that in a right triangle, C being the right angle, the following relations are true:

(*a*) $\sin 2A = \sin 2B$.

(*b*) $\tan 2A = \dfrac{2ab}{b^2 - a^2}$.

(*c*) $\cos 2A = \dfrac{b^2 - a^2}{c^2}$.

(*d*) $\cos 2A + \cos 2B = 0$.

(*e*) $\tan B = \cot A + \cos C$.

(*f*) $\sin 3A = \dfrac{3ab^2 - a^3}{c^3}$.

57. Conversion formulas. From (1) and (10), we have

$$\sin(\theta + \varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi,$$
$$\sin(\theta - \varphi) = \sin\theta\cos\varphi - \cos\theta\sin\varphi.$$

Adding these two formulas member by member, we get

$$\boldsymbol{\sin(\theta + \varphi) + \sin(\theta - \varphi) = 2\sin\theta\cos\varphi,} \tag{27}$$

and subtracting the second from the first, we obtain

$$\boldsymbol{\sin(\theta + \varphi) - \sin(\theta - \varphi) = 2\cos\theta\sin\varphi.} \tag{28}$$

From (1) and (11) we get

$$\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi,$$
$$\cos(\theta - \varphi) = \cos\theta\cos\varphi + \sin\theta\sin\varphi.$$

Adding these formulas member by member and afterwards subtracting the second from the first, we obtain

$$\boldsymbol{\cos(\theta + \varphi) + \cos(\theta - \varphi) = 2\cos\theta\cos\varphi,} \tag{29}$$
$$\boldsymbol{\cos(\theta + \varphi) - \cos(\theta - \varphi) = -2\sin\theta\sin\varphi.} \tag{30}$$

Formulas (27) to (30) should not be memorized but should be recalled by mentally carrying out their derivation from the addition formulas. These formulas are important because they enable us to express a product of sines and cosines as a sum of two or more expressions or to express a sum or a difference of two trigonometric functions in the form of a product. The following examples will illustrate the method of doing this.

Example 1. Expand $\cos 2x \cos 3x \sin 4x$ into a sum of sines and cosines of multiple angles.

Solution. Using (29) with $\theta = 2x$, $\varphi = 3x$, we obtain

$$2\cos 2x\cos 3x = \cos(2x + 3x) + \cos(2x - 3x),$$

or

$$2\cos 2x\cos 3x = \cos 5x + \cos x. \tag{a}$$

Multiplying (*a*) through by $\sin 4x$ and dividing by 2, we get

$$\cos 2x\cos 3x\sin 4x = \tfrac{1}{2}(\cos 5x\sin 4x + \cos x\sin 4x). \tag{b}$$

Then using (27) with $\theta = 4x$, $\varphi = 5x$, we obtain

$$2\sin 4x\cos 5x = \sin(4x + 5x) + \sin(4x - 5x),$$

or

$$2\sin 4x\cos 5x = \sin 9x - \sin x. \tag{c}$$

Again using (27) with $\theta = 4x$, $\varphi = x$, we obtain

$$2 \cos x \sin 4x = \sin 5x + \sin 3x. \qquad (d)$$

Substituting $\sin 4x \cos 5x$ from (c) and $\cos x \sin 4x$ from (d) in (b), we obtain, after slight simplification,

$$\cos 2x \cos 3x \sin 4x = \tfrac{1}{4}(\sin 9x - \sin x + \sin 5x + \sin 3x).$$

Example 2. Express $\sin 5x - \sin 3x$ in the form of a product.

Solution. The left-hand member of (28) will be the desired difference if we set

$$\theta + \varphi = 5x, \qquad \theta - \varphi = 3x, \qquad (a)$$

or, solving for θ and φ in terms of x,

$$\theta = 4x, \qquad \varphi = x. \qquad (b)$$

Substituting θ and φ from (b) in (28), we obtain

$$\sin 5x - \sin 3x = 2 \cos 4x \sin x.$$

A process similar to that carried out in (a) and (b) to find θ and φ in terms of the given angles may be used to derive another set of formulas that are convenient for transforming a sum to a product. Let

$$\theta + \varphi = \alpha, \qquad \theta - \varphi = \beta. \qquad (31)$$

Solving (31) simultaneously for θ and φ in terms of α and β, we get

$$\theta = \tfrac{1}{2}(\alpha + \beta), \qquad \varphi = \tfrac{1}{2}(\alpha - \beta). \qquad (32)$$

Replacing θ by $\frac{1}{2}(\alpha + \beta)$ and φ by $\frac{1}{2}(\alpha - \beta)$ in (27), (28), (29), and (30), we obtain

$$\sin \alpha + \sin \beta = 2 \sin \tfrac{1}{2}(\alpha + \beta) \cos \tfrac{1}{2}(\alpha - \beta), \qquad (33)$$

$$\sin \alpha - \sin \beta = 2 \cos \tfrac{1}{2}(\alpha + \beta) \sin \tfrac{1}{2}(\alpha - \beta), \qquad (34)$$

$$\cos \alpha + \cos \beta = 2 \cos \tfrac{1}{2}(\alpha + \beta) \cos \tfrac{1}{2}(\alpha - \beta), \qquad (35)$$

$$\cos \alpha - \cos \beta = -2 \sin \tfrac{1}{2}(\alpha + \beta) \sin \tfrac{1}{2}(\alpha - \beta). \qquad (36)$$

EXERCISES

1. Express in the form of a product

(a) $\sin 35° + \sin 25°$.
(b) $\sin 45° - \sin 30°$.
(c) $\cos 65° + \cos 25°$.
(d) $\cos 75° - \cos 5°$.
(e) $\cos 4x + \cos 2x$.
(f) $\sin 5x - \sin 2x$.
(g) $\sin 3x + \sin x$.
(h) $\cos 5x - \cos 3x$.

2. Expand into a sum of sines and cosines of multiple angles:

(*a*) $\sin 3x \cos 7x$.
(*b*) $\cos 3x \cos 7x$.
(*c*) $\sin x \sin 2x \cos 3x$.
(*d*) $\cos 3x \cos 5x \sin 7x$.

Verify the following identities:

3. $\sin 32° + \sin 28° = \cos 2°$.

4. $\sin 50° - \sin 10° = \sqrt{3} \sin 20°$.

5. $\cos 80° - \cos 20° = - \sin 50°$.

6. $\cos 140° + \cos 100° + \cos 20° = 0$.

7. $\tan 50° + \cot 50° = 2 \sec 10°$.

8. $\cos 60° + \cos 30° = \sqrt{2} \cos 15°$.

9. $\sin 40° - \cos 70° = \sqrt{3} \sin 10°$.

10. $\sin (60° + \alpha) + \sin (60° - \alpha) = \sqrt{3} \cos \alpha$.

11. $\cos 5x + \cos 9x = 2 \cos 7x \cos 2x$.

12. $\dfrac{\sin 7x - \sin 5x}{\cos 7x + \cos 5x} = \tan x$.

13. $\dfrac{\sin 33° + \sin 3°}{\cos 33° + \cos 3°} = \tan 18°$

14. $\dfrac{\sin A - \sin B}{\sin A + \sin B} = \tan \frac{1}{2}(A - B) \cot \frac{1}{2}(A + B)$.

15. $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan \frac{1}{2}(A + B)$.

16. $\cos 20° - \sin 10° - \sin 50° = 0$.

17. $\sin (60° + x) - \sin x = \sin (60° - x)$.

18. $\cos (30° + y) - \cos (30° - y) = - \sin y$.

19. $\cos (x + 45°) + \cos (x - 45°) = \sqrt{2} \cos x$.

20. $\cos (Q + 45°) + \sin (Q - 45°) = 0$.

21. $\dfrac{\sin A + \sin B}{\cos A - \cos B} = - \cot \frac{1}{2}(A - B)$.

22. $\cos 3\alpha - \cos 7\alpha = 2 \sin 5\alpha \sin 2\alpha$.

23. $\dfrac{\sin 5x - \sin 2x}{\cos 2x - \cos 5x} = \cot \dfrac{7x}{2}$.

24. $\sin \theta + \sin 2\theta + \sin 3\theta = \sin 2\theta(1 + 2 \cos \theta)$.

25. $\cos \theta + \cos 2\theta + \cos 3\theta = \cos 2\theta(1 + 2 \cos \theta)$.

26. Express $\sin x + \cos y$ as a product.

27. Express $\sin x - \cos y$ as a product.

28. Show that $\dfrac{\cos 2x - \cos 2y}{\cos 2x + \cos 2y} + \tan (x + y) \tan (x - y) = 0$.

29. Express as a product, $\sin \alpha + \sin 3\alpha + \sin 5\alpha + \sin 7\alpha$.

30. Prove $\dfrac{\cos 5x - \cos 3x}{\sin 5x - \sin 3x} + \dfrac{\cos 2x - \cos 4x}{\sin 4x - \sin 2x} + \dfrac{\sin x}{\cos 4x \cos 3x} = 0.$

31. Prove $\sin \alpha + \sin 2\alpha + \sin 3\alpha + \sin 4\alpha = 4 \cos \frac{1}{2}\alpha \cos \alpha \sin \frac{5}{2}\alpha.$

32. Prove $\sin \alpha + \sin 3\alpha + \sin 5\alpha = \dfrac{\sin^2 3\alpha}{\sin \alpha}.$

33. Prove $\dfrac{\sin (\alpha + \beta) - 2 \sin \alpha + \sin (\alpha - \beta)}{\cos (\alpha + \beta) - 2 \cos \alpha + \cos (\alpha - \beta)} = \tan \alpha.$

34. If $A + B + C = 180°$, prove that

(*a*) $\cos (A + B - C) = -\cos 2C.$

(*b*) $\sin A + \sin B - \sin C = 4 \sin \dfrac{A}{2} \sin \dfrac{B}{2} \cos \dfrac{C}{2}.$

(*c*) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$

(*d*) $\tan A - \cot B = \sec A \csc B \cos C.$

35. Prove $(\cos \alpha + \cos \beta)^2 + (\sin \alpha + \sin \beta)^2 = 4 \cos^2 \frac{1}{2}(\alpha - \beta).$

58. MISCELLANEOUS EXERCISES

1. (*a*) Show that the value of $\sin 2\theta$ is less than the value of $2 \sin \theta$ for all values of θ between 0° and 90°.

(*b*) Show that the value of the fraction $\dfrac{\sin 2\theta}{2 \sin \theta}$ decreases from 1 to 0 as θ increases from 0° to 90°.

2. Given $\cot \alpha = \frac{4}{3}$ and $\cos \beta = -\frac{5}{13}$, find the value of each of the following if α and β each terminate in the third quadrant:

(*a*) $\cos (\alpha - \beta)$.	(*c*) $\sin (\beta - \alpha)$.	(*e*) $\cot (\alpha - \beta)$.
(*b*) $\tan (\alpha + \beta)$.	(*d*) $\cot (\alpha + \beta)$.	(*f*) $\tan (\beta - \alpha)$.

3. If $\cos \alpha = \frac{3}{5}$ and $\sin \beta = -\frac{3}{5}$, and if α is in the fourth and β in the third quadrant show that

(*a*) $\sin (\alpha + \beta) = +\frac{7}{25}$; $\cos (\alpha + \beta) = -\frac{24}{25}$; $\tan (\alpha + \beta) = -\frac{7}{24}$;

(*b*) $\sin (\alpha - \beta) = +1$; $\cos (\alpha - \beta) = 0$; $\tan (\alpha - \beta) = \infty$.

4. Prove that $\sin 180° = 0$ and $\cos 180° = -1$, using the functions of 120° and 60°.

5. Find $\tan (x + y)$ and $\tan (x - y)$, having given $\tan x = \frac{1}{2}$ and $\tan y = \frac{1}{4}$.

Verify each of the following:

6. $\tan (45° + x) = \dfrac{1 + \tan x}{1 - \tan x}.$

7. $\cot(y - 45°) = \dfrac{1 + \cot y}{1 - \cot y}$.

8. $\cot(B + 210°) = \dfrac{\sqrt{3}\cot B - 1}{\cot B + \sqrt{3}}$.

9. $\dfrac{\sin(x + y)}{\sin(x - y)} = \dfrac{\tan x + \tan y}{\tan x - \tan y}$.

10. $\tan x + \tan y = \dfrac{\sin(x + y)}{\cos x \cos y}$.

11. $\dfrac{\tan(\theta - \phi) + \tan\phi}{1 - \tan(\theta - \phi)\tan\phi} = \tan\theta$.

12. $\tan(45° + x) - \tan(45° - x) = 2\tan 2x$.

13. $\tan(45° + C) + \tan(45° - C) = 2\sec 2C$.

14. $\sin 2x = \dfrac{2\tan x}{1 + \tan^2 x}$.

15. $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$.

16. $\dfrac{1 + \sin 2x}{1 - \sin 2x} = \left(\dfrac{\tan x + 1}{\tan x - 1}\right)^2$.

17. $\tan x = \dfrac{\sin 2x}{1 + \cos 2x}$.

18. $\dfrac{\cos(x - y)}{\cos(x + y)} = \dfrac{1 + \tan x \tan y}{1 - \tan x \tan y}$.

19. $\tan A - \tan B = \dfrac{\sin(A - B)}{\cos A \cos B}$.

20. $\cot x + \cot y = \dfrac{\sin(x + y)}{\sin x \sin y}$.

21. $\cos(60° - A) = \dfrac{\cos A + \sqrt{3}\sin A}{2}$.

22. $\cos(x - 315°) = \dfrac{\cos x - \sin x}{\sqrt{2}}$.

23. $\cos 5\alpha \cos 4\alpha + \sin 5\alpha \sin 4\alpha = \cos\alpha$.

24. $\sin(x + 75°)\cos(x - 75°) - \cos(x + 75°)\sin(x - 75°) = \frac{1}{2}$.

25. $\cos(2x + y)\cos(x + 2y) + \sin(2x + y)\sin(x + 2y)$
$= \cos x \cos y + \sin x \sin y$.

26. $\sin(x + y)\sin(x - y) = \sin^2 x - \sin^2 y$.

27. $\cos(x - y + z) = \cos x \cos y \cos z + \cos x \sin y \sin z$
$- \sin x \cos y \sin z + \sin x \sin y \cos z$.

28. $\sin(30° + x)\sin(30° - x) = \frac{1}{4}(\cos 2x - 2\sin^2 x)$.

29. $\sin(A + B)\sin(A - B) = \cos^2 B - \cos^2 A.$

30. $\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2 = 1 + \sin x.$

31. $\dfrac{1 + \sec y}{\sec y} = 2\cos^2\dfrac{y}{2}.$

32. $2\sin\left(45^\circ + \dfrac{x - y}{2}\right)\cos\left(45^\circ - \dfrac{x + y}{2}\right) = \cos y + \sin x.$

33. $1 + \tan x \tan\dfrac{x}{2} = \sec x.$

34. $\tan\dfrac{x}{2} + 2\sin^2\dfrac{x}{2}\cot x = \sin x.$

35. $\dfrac{\cos\theta}{1 - \sin\theta} = \dfrac{1 + \tan\dfrac{\theta}{2}}{1 - \tan\dfrac{\theta}{2}}.$

36. $\dfrac{1 + \sin x + \cos x}{1 + \sin x - \cos x} = \cot\dfrac{x}{2}.$

37. $1 + \cot^2\dfrac{x}{2} = \dfrac{2}{\sin x \tan\dfrac{x}{2}}.$

38. $\dfrac{\tan^2\dfrac{x}{2} + \cot^2\dfrac{x}{2}}{\tan^2\dfrac{x}{2} - \cot^2\dfrac{x}{2}} = -\dfrac{1 + \cos^2 x}{2\cos x}.$

39. Give the behavior of $\tan\dfrac{\theta}{2} + 2\sin^2\dfrac{\theta}{2}\cot\theta$ as θ increases from 0° to 90°.

40. Show that the value of $\tan^2\theta(1 + \cos 2\theta) + 2\cos^2\theta$ is the same for all values of θ.

41. Prove $\dfrac{\sin x + \cos x}{\cos x - \sin x} = \tan 2x + \sec 2x.$

42. Prove $\dfrac{\cot(90^\circ + A)}{\cos 2A - 1} = \csc 2A.$

43. Prove $\dfrac{\cos 3x \sin 2x - \cos 4x \sin x}{\cos 5x \cos 2x - \cos 4x \cos 3x} = -\cot 2x.$

44. Prove $4\sin x \sin(60^\circ - x)\sin(60^\circ + x) = \sin 3x.$

45. Find $\cos 6\alpha$ in terms of $\sin\alpha$.

Verify each of the following:

46. $\sin^6 x + \cos^6 x = \sin^4 x + \cos^4 x - \sin^2 x \cos^2 x.$

47. $\sin(x + y - z) + \sin(x + z - y) + \sin(y + z - x)$
$= \sin(x + y + z) + 4 \sin x \sin y \sin z.$

48. $\cos x \sin(y - z) + \cos y \sin(z - x) + \cos z \sin(x - y) = 0.$

49. $\sin x \cos(y + z) - \sin y \cos(x + z) = \sin(x - y) \cos z.$

50. $1 - 4 \sin^4 x - 2 \sin^2 x \cos 2x = \cos 2x.$

51. If $\alpha + \beta + \gamma = 180°$, prove that

(*a*) $\sin^2 \alpha + \sin^2 \beta - \sin^2 \gamma = 2 \sin \alpha \sin \beta \cos \gamma.$

(*b*) $\tan \frac{\alpha}{2} \tan \frac{\beta}{2} + \tan \frac{\beta}{2} \tan \frac{\gamma}{2} + \tan \frac{\gamma}{2} \tan \frac{\alpha}{2} = 1.$

(*c*) $\cot \frac{\alpha}{2} + \cot \frac{\beta}{2} + \cot \frac{\gamma}{2} = \cot \frac{\alpha}{2} \cot \frac{\beta}{2} \cot \frac{\gamma}{2}.$

52. Prove $\cos(x + y - z) + \cos(y + z - x) + \cos(z + x - y)$
$+ \cos(x + y + z) = 4 \cos x \cos y \cos z.$

53. Prove $\cos(x + y) \cos(x - y) + \sin(y + z) \sin(y - z)$
$- \cos(x + z) \cos(x - z) = 0.$

CHAPTER VII

IMPORTANT FORMULAS RELATING TO TRIANGLES

59. Law of sines. The object of this chapter is to develop important formulas that are useful in solving rectilinear figures and to indicate how they are applied.

In any triangle such as ABC of Fig. 1(*a*), A, B, and C represent the angles, and a, b, and c represent, respectively, the lengths

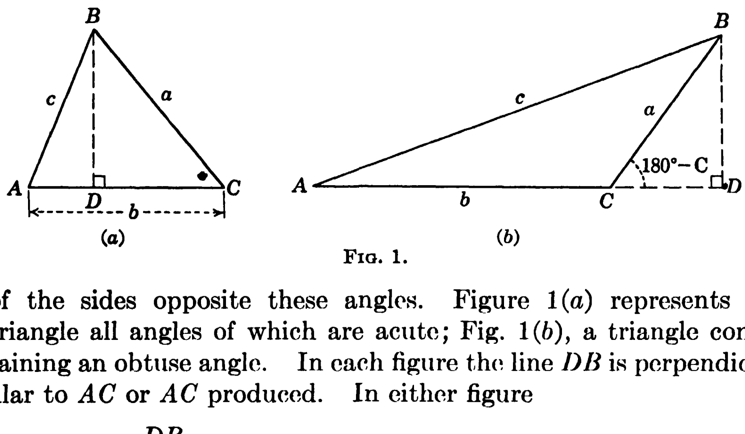

FIG. 1.

of the sides opposite these angles. Figure 1(*a*) represents a triangle all angles of which are acute; Fig. 1(*b*), a triangle containing an obtuse angle. In each figure the line DB is perpendicular to AC or AC produced. In either figure

$$\frac{DB}{c} = \sin A, \qquad \text{or} \qquad DB = c \sin A. \tag{1}$$

In Fig. 1(*a*), $DB/a = \sin C$ and, in Fig. 1(*b*), $DB/a = \sin (180° - C) = \sin C$. In either case

$$DB = a \sin C. \tag{2}$$

Equating the value of DB from (1) to the value of DB from (2) and dividing the result by $\sin A \sin C$, we obtain

$$\frac{a}{\sin A} = \frac{c}{\sin C}. \tag{3}$$

Similarly by drawing a perpendicular from C to the opposite side of the triangle and reasoning as above, we obtain

$$\frac{a}{\sin A} = \frac{b}{\sin B}. \tag{4}$$

Equations (3) and (4) may be combined in the equations

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \tag{5}$$

The equations (5) are referred to as *the law of sines.* This law may be stated as follows: *The sides of a triangle are proportional to the sines of the opposite angles.*

Example. Express all line segments of Fig. 2(*a*) in terms of the given parts.

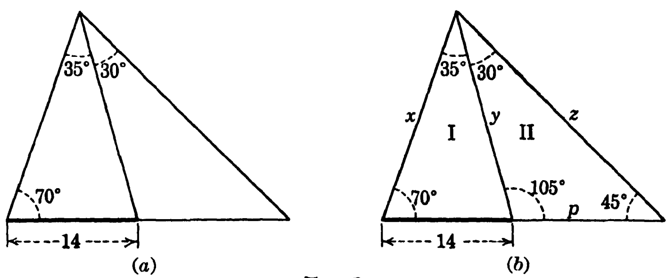

Fig. 2.

Solution. Compute the angles of Fig. 2(*a*) and represent the unknown sides by letters; this gives us Fig. 2(*b*). Attending to triangle I, we think: x over sine of angle opposite (75°) equals 14 over sine of angle opposite (35°), and write

$$\frac{x}{\sin 75^\circ} = \frac{14}{\sin 35^\circ}, \qquad \text{or} \qquad x = \mathbf{14 \sin 75^\circ \csc 35^\circ}. \tag{a}$$

Again from triangle I, we write

$$\frac{y}{\sin 70^\circ} = \frac{14}{\sin 35^\circ}, \qquad \text{or} \qquad y = \mathbf{14 \sin 70^\circ \csc 35^\circ}. \tag{b}$$

From triangle II, we write

$$\frac{p}{\sin 30^\circ} = \frac{y}{\sin 45^\circ}, \qquad \frac{z}{\sin 105^\circ} = \frac{y}{\sin 45^\circ}, \tag{c}$$

or

$$p = y\,\frac{\sin 30^\circ}{\sin 45^\circ}, \qquad z = y\,\frac{\sin 105^\circ}{\sin 45^\circ}. \tag{d}$$

Replacing y in (d) by its value from (b) and simplifying slightly, we obtain

$$p = 14 \sin 70° \csc 35° \sin 30° \csc 45°.$$
$$z = 14 \sin 70° \csc 35° \sin 105° \csc 45°.$$

EXERCISES

1. Find x and y in radical form from Fig. 3 and also from Fig. 4.

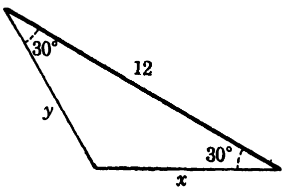

FIG. 3.

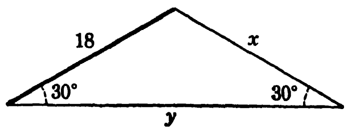

FIG. 4.

2. Express x and y in each of Figs. 5 to 8 in terms of the given parts.

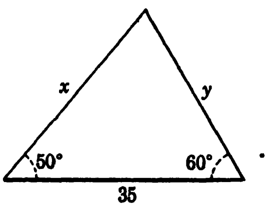

FIG. 5.

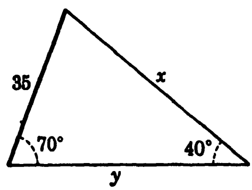

FIG. 6.

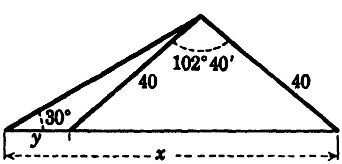

FIG. 7.

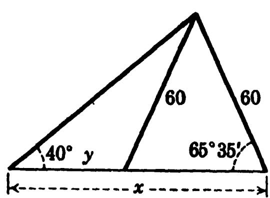

FIG. 8.

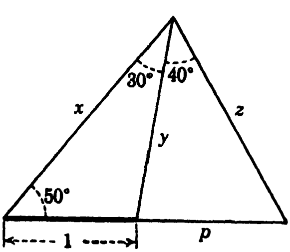

FIG. 9.

3. Find x, y, z, and p of Fig. 9 in terms of the given angles.

4. Find sin B where B is defined by Fig. 10. Also find the value of x in terms of B and the given parts.

5. Find the area of the triangle of Fig. 10 in terms of B and given parts.

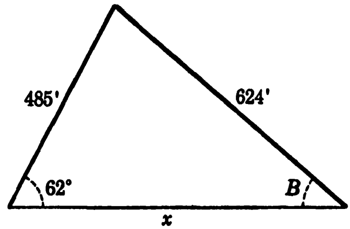

FIG. 10.

6. Express the lines x and y in Figs. 11 and 12 in terms of a and the given angles.

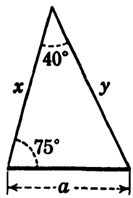

FIG. 11.

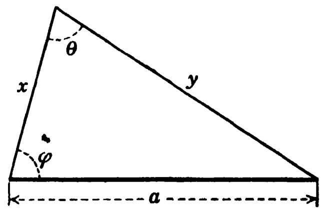

FIG. 12.

7. Express the lengths represented by x, y, z, and w of Fig. 13 in terms of the given parts.

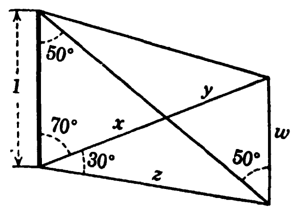

FIG. 13.

8. Use Fig. 14 to prove that

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

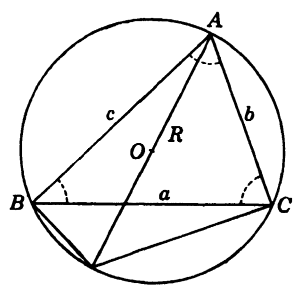

FIG. 14.

9. Show that $\sin (45° - \alpha) = \frac{2}{3} \sin 85°$ where α is defined by Fig. 15.

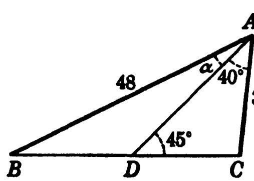

FIG. 15.

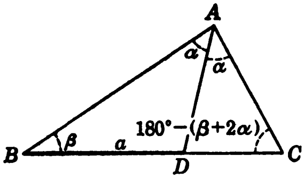

FIG. 16.

10. Express all segments in Fig. 16 in terms of a, α, and β and then show that $\frac{BD}{DC} = \frac{BA}{AC}$.

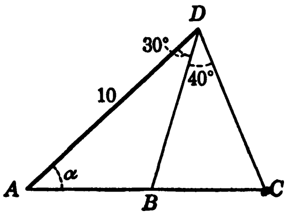

FIG. 17.

11. If $AB = BC$ in Fig. 17, prove that

$$\cot \alpha = \frac{\sin 40° - \sin 30° \cos 70°}{\sin 30° \sin 70°}.$$

60. The law of tangents. Mollweide's equations. The equations referred to in the title of this article are easily deduced from the law of sines. The law of tangents, the proof of which follows directly, is used to solve a triangle when two sides and the included angle are given. Mollweide's equations are excellent equations for checking purposes.

From the law of sines, we have

$$\frac{a}{b} = \frac{\sin A}{\sin B}. \tag{6}$$

Subtracting 1 from each side of (6), we have

$$\frac{a}{b} - 1 = \frac{\sin A}{\sin B} - 1, \quad \text{or} \quad \frac{a - b}{b} = \frac{\sin A - \sin B}{\sin B}. \tag{7}$$

Adding 1 to each side of (6), we have

$$\frac{a}{b} + 1 = \frac{\sin A}{\sin B} + 1, \quad \text{or} \quad \frac{a + b}{b} = \frac{\sin A + \sin B}{\sin B} \tag{8}$$

By dividing (7) and (8) member by member, we obtain

$$\frac{a - b}{a + b} = \frac{\sin A - \sin B}{\sin A + \sin B}.$$

Transforming the right-hand member of this equation by means of the formulas of §57, we obtain

$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{2 \cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)}{2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)}.$$

The right-hand member reduces to

$$\tan \tfrac{1}{2}(A - B) \div \tan \tfrac{1}{2}(A + B).$$

$$\therefore \frac{a - b}{a + b} = \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)}. \tag{9}$$

Another formula may be obtained by replacing a by c and A by C in (9) and a third, by replacing b by c and B by C in (9).

When $b > a$, both sides of (9) are negative. In this case it is convenient to write the formula in the form

$$\frac{b - a}{b + a} = \frac{\tan \frac{1}{2}(B - A)}{\tan \frac{1}{2}(B + A)}, \tag{10}$$

so that both members are positive.

The formulas often called Mollweide's equations are derived as follows:

From the law of sines, we have

$$\frac{a}{c} = \frac{\sin A}{\sin C}, \quad \text{and} \quad \frac{b}{c} = \frac{\sin B}{\sin C}, \tag{11}$$

Adding equations (11) member by member, we obtain

$$\frac{a + b}{c} = \frac{\sin A + \sin B}{\sin C}. \tag{12}$$

Transforming the right-hand member of this equation by means of formula (18) of §56 and formula (33) of §57, we obtain

$$\frac{a + b}{c} = \frac{2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)}{2 \sin \frac{1}{2}C \cos \frac{1}{2}C}. \tag{13}$$

Since $A + B = 180° - C$,

$$\sin \tfrac{1}{2}(A + B) = \sin \tfrac{1}{2}(180° - C) = \cos \tfrac{1}{2}C.$$

Hence Mollweide's first equation may be written in the form

$$\frac{a + b}{c} = \frac{\cos \frac{1}{2}(A - B)}{\sin \frac{1}{2}C}. \tag{14}$$

Mollweide's second equation,

$$\frac{a - b}{c} = \frac{\sin \frac{1}{2}(A - B)}{\cos \frac{1}{2}C}, \tag{15}$$

is derived in a similar manner.

61. The law of cosines. In the triangles of Fig. 18 denote the angles by A, B, and C, and the sides opposite these angles by a, b, and c, respectively. Draw the perpendicular p from

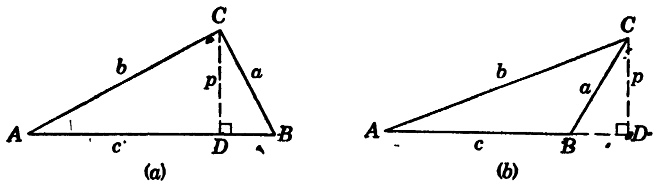

FIG. 18.

one of the vertices C of the triangle to the opposite side c, Fig. 18(a), or c produced, Fig. 18(b). In either figure

$$AD = b \cos A. \tag{16}$$

In Fig. 18(a)

$$DB = c - AD = c - b \cos A,$$

and in Fig. 18(b)

$$BD = AD - AB = b \cos A - c. \tag{17}$$

Since $(c - b \cos A)^2 = (b \cos A - c)^2$, we have for each triangle

$$b^2 - b^2 \cos^2 A = p^2 = a^2 - (c - b \cos A)^2.$$

Simplifying and solving for a^2, we obtain

$$a^2 = b^2 + c^2 - 2bc \cos A. \tag{18}$$

Similarly, by drawing perpendiculars from A and B to the opposite sides or the opposite sides produced, we obtain

$$\left.\begin{aligned} b^2 &= a^2 + c^2 - 2ac \cos B, \\ c^2 &= a^2 + b^2 - 2ab \cos C. \end{aligned}\right\} \tag{19}$$

The law of cosines embodied in equations (18) and (19) may be stated as follows: *The square of any side of a plane triangle is equal to the sum of the squares of the other two sides diminished by twice the product of those two sides and the cosine of their included angle.*

The law of sines, the law of cosines, and the law of tangents will be used in the next chapter to compute parts of rectilinear figures. Here we shall use them to write expressions for lengths of line segments of rectilinear figures and to write identities.

Example 1. Write several equations relating to Fig. 19.

Solution. From the law of sines, we have

$$\frac{x}{\sin 40°} = \frac{y}{\sin 65°} = \frac{20}{\sin 75°}.$$

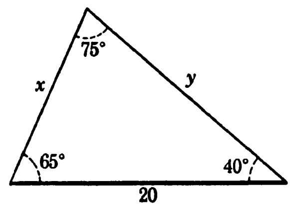

FIG. 19.

Substituting $a = 20$, $A = 75°$, $b = x$, $c = y$ in (18), we obtain

$$20^2 = x^2 + y^2 - 2xy \cos 75°.$$

Substituting $a = 20$, $A = 75°$, $b = x$, $B = 40°$ in (9), we obtain

$$\frac{20 - x}{20 + x} = \frac{\tan \frac{1}{2}(75° - 40°)}{\tan \frac{1}{2}(75° + 40°)} = \frac{\tan (17°30')}{\tan (57°30')}.$$

Substituting $a = 20$, $A = 75°$, $b = x$, $B = 40°$, $c = y$, $C = 65°$ in (14), we obtain

$$\frac{20 + x}{y} = \frac{\cos \frac{1}{2}(75° - 40°)}{\sin \frac{1}{2}(65°)} = \frac{\cos (17°30')}{\sin (32°30')}.$$

Example 2. Express the line segments x, y, z, and w of Fig. 20(a) in terms of the given parts, and write an identity based on these results.

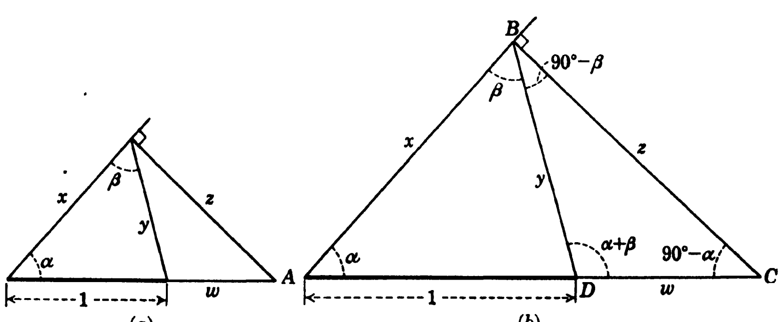

FIG. 20.

Solution. First we devise Fig. 20(*b*). Applying the law of sines to triangle ABD of Fig. 20(*b*) we obtain

$$\frac{x}{\sin(\alpha+\beta)} = \frac{1}{\sin\beta} = \csc\beta, \qquad \frac{y}{\sin\alpha} = \csc\beta, \tag{a}$$

or

$$x = \sin(\alpha+\beta)\csc\beta, \qquad y = \sin\alpha\csc\beta. \tag{b}$$

Applying the law of sines to triangle DBC, we obtain

$$\frac{w}{\sin(90^\circ-\beta)} = \frac{z}{\sin(\alpha+\beta)} = \frac{y}{\sin(90^\circ-\alpha)}. \tag{c}$$

Replacing y by $\sin\alpha\csc\beta$, solving for z and w, and simplifying slightly, we have

$$w = \tan\alpha\cot\beta, \qquad z = \sin(\alpha+\beta)\tan\alpha\csc\beta. \tag{d}$$

Applying the law of cosines to triangle BDC, we obtain

$$z^2 = y^2 + w^2 - 2yw\cos(\alpha+\beta). \tag{e}$$

Replacing y, z, and w by their values from (*b*) and (*d*), we obtain

$$\sin^2(\alpha+\beta)\tan^2\alpha\csc^2\beta = \sin^2\alpha\csc^2\beta + \tan^2\alpha\cot^2\beta - 2\sin\alpha\csc\beta\tan\alpha\cot\beta\cos(\alpha+\beta).$$

EXERCISES

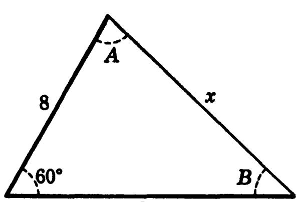

FIG. 21.

1. Use the law of cosines to find x in Fig. 21; then express $\sin A$ and $\sin B$ in terms of x.

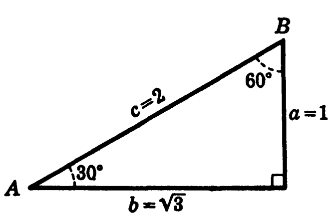

FIG. 22.

2. In Fig. 22 find $\tan\frac{1}{2}(A-B)$ by using formula (9) in §60.

3. In each of Figs. 23 and 24 use the law of cosines to find x. Then express sin A and sin B in terms of x.

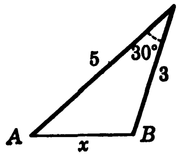

FIG. 23.

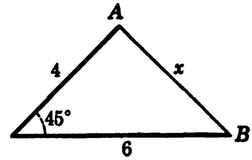

FIG. 24.

4. In each of Figs. 25 and 26 find $\tan \frac{1}{2}(A - B)$ by using formula (9) in §60.

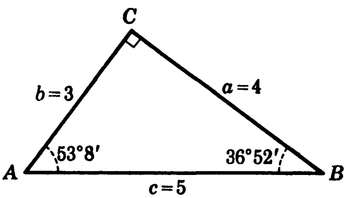

FIG. 25.

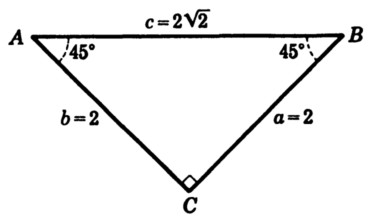

FIG. 26.

62. MISCELLANEOUS EXERCISES

In the following exercises check each identity by substituting one or more of such angles as 0°, 30°, 45°, 120°, 240°, 270°, etc., for the unknown angles involved.

1. Use the law of cosines to find the value of x in Fig. 27.

2. Find the value of $\tan \frac{1}{2}(A - B)$ where A and B are defined by Fig. 27.

3. Find an expression for the area of the triangle in Fig. 27.

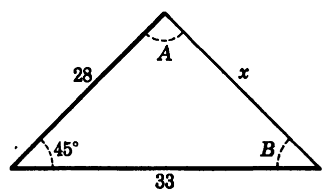

FIG. 27.

4. Write equations applying to Fig. 28 by using each of the following: law of sines, law of cosines, law of tangents, Mollweide's equations.

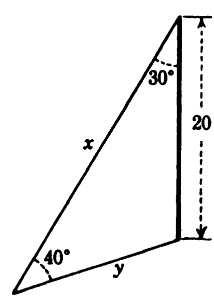

FIG. 28.

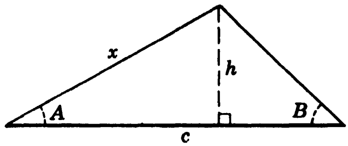

FIG. 29.

5. Find an expression for the area of the triangle in Fig. 29 in terms of c, A, and B.

Hint. First find x and then h.

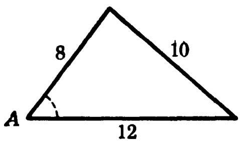

FIG. 30.

6. Find the value of $\cos A$ where A is defined by Fig. 30.

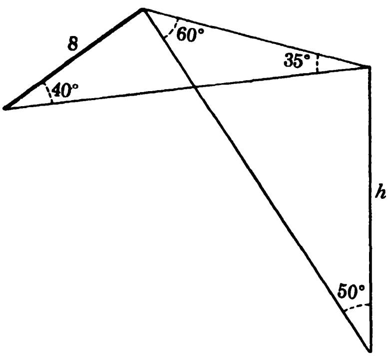

FIG. 31.

7. (*a*) From Fig. 31 find a formula for h in terms of the given parts.

(*b*) Using the formula found in (*a*), compute h.

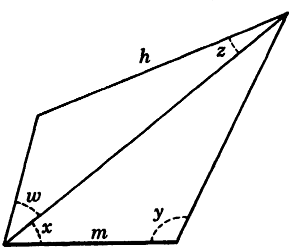

FIG. 32.

8. Using Fig. 32, express h in terms of m, x, y, z, w.

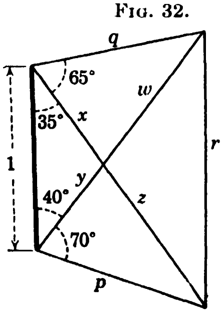

FIG. 33.

9. Find the length of *all* line segments of Fig. 33 in terms of the given parts.

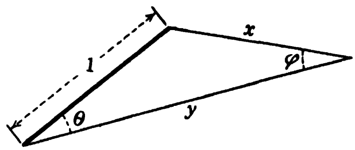

FIG. 34.

10. Draw the altitude to the side lettered x in Fig. 34 and find its length in terms of θ and φ; then write a formula for the area of the triangle. Check this formula by using the values $\theta = 90°$, $\varphi = 45°$.

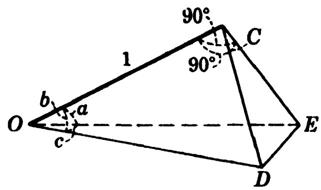

FIG. 35.

11. In Fig. 35 trihedral angle O has the face angles a, b, c, and trihedral angle C has the face angles C, 90°, 90°. Express the length of each line segment in terms of a, b, c, then find and equate two line values of DE, and simplify to obtain $\cos c = \cos a \cos b + \sin a \sin b \cos C$.

12. From the law of cosines derive algebraically the law of sines. *Hint.* Find $\cos A$ in terms of a, b, and c; then find $(\sin^2 A)/a^2 = (1 - \cos^2 A)/a^2$.

13. O-ABC in Fig. 36 represents a pyramid. Find the length of each edge in terms of α, β, γ, θ, and φ.

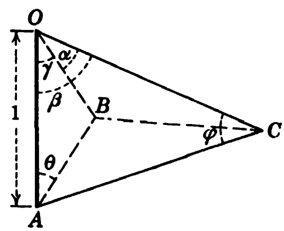

FIG. 36.

CHAPTER VIII

OBLIQUE TRIANGLES

63. Introduction. In this chapter we shall develop formulas and exhibit plans of calculation for the solution of oblique triangles.

When the length of a side and two other parts of a triangle are known, the remaining parts can generally be found. The four cases that arise in the solutions of oblique triangles are referred to as

Case I. *Given one side and two angles.*

Case II. *Given two sides and an angle opposite one of them.*

Case III. *Given two sides and the included angle.*

Case IV. *Given three sides.*

All triangles can be solved by means of the law of sines, the law of cosines, and the law of tangents. However, formulas especially adapted to logarithmic computation will be developed to solve triangles classified under Case IV. Although any formula not used in the solution of a triangle may be used as a check formula, Mollweide's equations are particularly desirable check formulas because they contain all six parts of the triangle and are well adapted to logarithmic computation. A single setting of the slide rule will serve to check, within its range of accuracy, the solution of any triangle.

For convenience of reference we repeat here the slide-rule setting for applying the law of sines to solve a triangle:

Rule A. *To apply the law of sines for solving a triangle,*
push the hairline to any known side on D,
draw under the hairline the opposite known angle on S;
push the hairline to any other side on D,
read at the hairline the angle opposite on S;
push the hairline to any other known angle on S,
read at the hairline the side opposite on D.

64. Form for computation by logarithms to be used in the solution of oblique triangles. The student should now recall the forms and the general method of procedure used in the solution of right triangles by logarithms. When oblique triangles are solved, a similar method will be used. This method may be summarized as follows:

a. Draw a figure of the triangle to be solved, lettering it in the conventional way. Encircle the given parts.

b. Write the formulas to be used in the solution.

c. Make a complete form for the computation before looking up any logarithms.

d. Fill in the form.

65. Case I. Given one side and two angles.

Example. Given $a = 24.31$, $A = 45°18'$, and $B = 22°11'$ (see Fig. 1). Find b, c, and C.

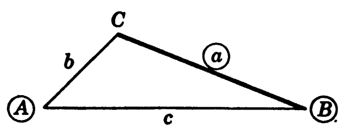

FIG. 1.

Solution. Since $A + B + C = 180°$,

$$C = 180° - (45°18' + 22°11') = 112°31'.$$

To find b, choose the formula from the law of sines which contains b and three known parts. Solve this formula by algebra for b, to obtain

$$b = \frac{a \sin B}{\sin A} = a \sin B \csc A. \qquad (a)$$

Similarly,

$$c = \frac{a \sin C}{\sin A} = a \sin C \csc A. \qquad (b)$$

The solution for the unknown parts in (a) and (b) and the check by Mollweide's equation (14) §60 are displayed below. The letter in parenthesis above each column refers to the formula associated with the column.

	(a)	(b)
$a = 24.31$	$\log a = 1.38578$	$\log a = 1.38578$
$A = 45°18'$	$l \csc^* A = 0.14825$	$l \csc^* A = 0.14825$
$B = 22°11'$	$l \sin B = 9.57700 - 10$	
$b = \mathbf{12.913}$	$\log b = 1.11103$	
$C = \mathbf{112°31'}$		$l \sin C = 9.96556 - 10$
$c = \mathbf{31.593}$		$\log c = 1.49959$

Check. For convenience of computation, we write Mollweide's equation (14) of §60

$$\frac{a+b}{c} = \frac{\cos \frac{1}{2}(A - B)}{\sin \frac{1}{2}C}$$

in the form

$$\frac{a+b}{c} \sin \tfrac{1}{2}C \sec \tfrac{1}{2}(A - B) = 1.$$

$a + b = 37.223$	$\log (a + b) = 1.57081$
$c = 31.593$	$\text{colog } c = 8.50041 - 10$
$\frac{1}{2}C = 56°15'30''$	$l \sin \frac{1}{2}C = 9.91989 - 10$
$\frac{1}{2}(A - B) = 11°33'30''$	$l \sec^* \frac{1}{2}(A - B) = 0.00890$
1	$\log 1 = 0.00001$

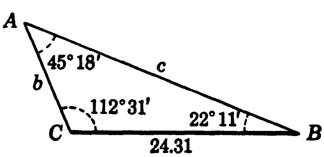

FIG. 2.

To solve the triangle by means of the slide rule, we first find $C = 112°31'$ from the relation $A + B + C = 180°$ and then use Rule A of §63. Hence, construct the triangle shown in Fig. 2, and

push hairline to 24.31 on D,
draw $45°18'$ of S under the hairline,
push hairline to $22°11'$ on S,
at the hairline read b = **12.91,**
push hairline to $67°29'$ ($= 180° - 112°31'$) on S,
at the hairline read c = **31.6.**

* Note that l is used in these forms to abbreviate the word log. If your tables of logarithms of trigonometric functions do not give the values of the logarithms of the secant and cosecant, in the above form write colog cos for l sec and colog sin for l csc.

EXERCISES

Solve the following triangles:

1. $A = 54°28'$, $B = 103°8'$, $a = 3.695$.

2. $B = 38°12'48''$, $C = 60°$, $a = 7012.6$.

3. $A = 64°56'18''$, $B = 47°29'11''$, $c = 913.45$.

4. $A = 47°23'18''$, $C = 70°16'49''$, $c = 227.22$.

5. $A = 71°13'30''$, $B = 40°34'15''$. $c = 236.53$.

6. $A = 25°32'35''$, $B = 133°13'5''$, $a = 411.41$.

7. A line AB along one bank of a stream is 562 ft. long, and C is a point on the opposite bank. The angle BAC is $53°18'$, and the angle ABC is $48°36'$. Find the width of the stream.

8. A vertical plane contains a 132-ft. hillside tunnel sloping downward at 14° with the horizontal and cuts the hillside in a line sloping upwards at 18°. What is the vertical distance from the bottom of the tunnel to the surface of the hill?

9. Prove that the area K of triangle ABC in Fig. 3 is given by

$$K = \frac{b^2 \sin A \sin C}{2 \sin (A + C)}.$$

Hint. First find c in terms of encircled parts; then find h and use the formula $K = \frac{1}{2}ch$.

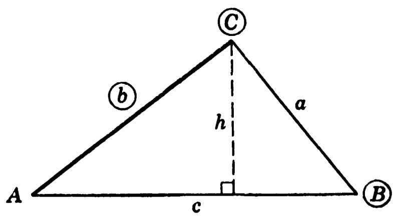

FIG. 3.

10. Use the formula in Exercise 9 to find the area of the triangle in (*a*) Exercise 1; (*b*) Exercise 6.

11. A shore station at point A is 5280 ft. from another at point B. Find the distance from each of the shore stations to an enemy ship at point C if angle ABC is $83°37'$ and angle BAC is $85°1'$.

12. A surveyor running a line due east reached the edge of a swamp. He then ran a line 2000 ft. in a direction S. 47° E., and from the point thus reached he ran a line in the direction N. 52° 20′ E. How far had he continued on this latter line when he reached a point on the original line extended?

13. A building 75.2 ft. high stands at the upper end of a street that slopes down at an angle of $6°52'$ with the horizontal. How far down the street from the building is a point at which the angle of elevation of the top of the building is $13°58'$?

14. From the top of a hill the angles of depression of the top and the bottom of a building 42.5 ft. high are observed to be 36° and 43°,

respectively. Find the height of the hill if the building is at the foot of the hill.

66. Case II. Given two sides and the angle opposite one of them. In this case, as in Case I, the triangle is solved by means of the law of sines and the relation $A + B + C = 180°$. The result may be checked by means of Mollweide's equations. However, this case needs further discussion, for in one instance an ambiguity exists.

Ambiguous case. When the side opposite the given angle is less than the other given side, there are three possibilities: no solution, one solution, or two solutions. Let us investigate the situation in detail.

Let A, a, and b of Figs. 4, 5, 6 be the given parts in which $a < b$. The perpendicular from C to side c is $b \sin A$.

a. If, in Fig. 4, $a < b \sin A$, side a is too short to reach side c. Hence there is no solution.

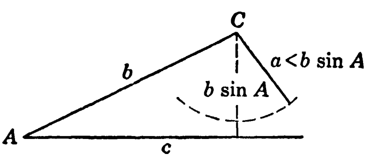

FIG. 4.

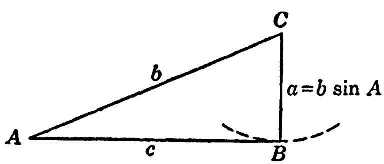

FIG. 5.

b. If, in Fig. 5, $a = b \sin A$, side a just reaches side c. Hence there is one solution, a right triangle.

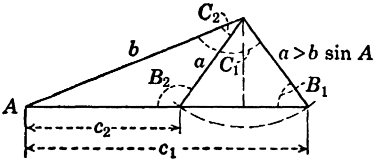

FIG. 6.

c. If, in Fig. 6, $a > b \sin A$, there are two solutions. In practice this is the most probable condition. Notice that B_1 and B_2 are supplementary angles.

These results may be summarized thus: If in triangle ABC, $a < b$, we have no solution when $a < b \sin A$; one solution when $a = b \sin A$; two solutions when $a > b \sin A$.

In the ambiguous case it is not necessary to determine the number of solutions in the foregoing manner before proceeding to solve the triangle, for we shall discover the nature of the situation as soon as we have added the first column of logarithms in the solution. Hence proceed with the computation, and when $\log \sin B$ has been found observe that

(a) if log sin $B > 0$, then sin $B > 1$, and there is no solution;
(b) if log sin $B = 0$, then sin $B = 1$ and there is one solution, a right triangle;
(c) if log sin $B < 0$, then sin $B < 1$, and there are two solutions.

Hence in Case II the procedure is as follows:

a. Determine whether the ambiguous case exists by noting whether the side opposite the given angle is less than the side adjacent to the given angle ($a < b$).

b. Proceed with the computation and if the ambiguous case is involved expect two solutions, but keep in mind that there may be no solution or one solution.

Example 1. Given $a = 67.528$, $b = 56.827$, and $A = 79° 15'20''$ (see Fig. 7). Find c, B, and C.

Solution. By inspection it is observed that $a > b$. Hence this is not the ambiguous case.

To find B, from the law of sines choose the formula containing B and the three known parts. Solve this formula for B to obtain

$$\sin B = \frac{b \sin A}{a}. \qquad (a)$$

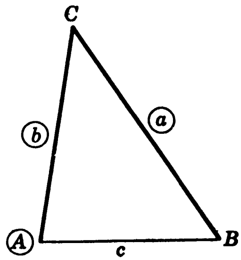

FIG. 7.

After finding B from (a), determine C from the relation

$$A + B + C = 180°.$$

Then write the law of sines involving c, C, and the knowns a and A to obtain

$$c = \frac{a \sin C}{\sin A} = a \sin C \csc A. \qquad (b)$$

The solution is displayed in the following form. The letter in parenthesis above each column refers to the formula associated with the column.

	(a)	(b)
$a = 67.528$	colog $a = 8.17051 - 10$	log $a = 1.82949$
$b = 56.827$	log b = 1 75456	
$A = 79°15'20''$	l sin A = 9 99232 − 10	l csc $A = 0.00768$
B = **55°46′8″**	l sin $B = 9.91739 - 10$	
C = **44°58′32″**		l sin $C = 9.84930 - 10$
c = **48.581**		log $c = 1.68647$

The results should be checked by means of one of Mollweide's equations, as in Case I. One setting of the slide rule serves to check the results.

To solve Example 1 by means of the slide rule, set the proportion

$$\frac{67.5}{\sin 79°15'} = \frac{56.8}{\sin B} = \frac{c}{\sin C}$$

on the rule, and read $B =$ **55°45′**. From the relation $A + B + C = 180°$, get $C =$ **45°**; then on the slide rule read $c =$ **48.6.**

Example 2. Given $a = 9.467$, $b = 14.433$, and $A = 11°14'18''$ (see Fig. 8). Find c, B, and C.

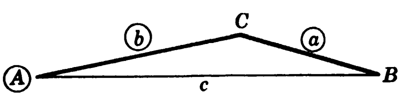

Fig. 8.

Solution. By inspection it is observed that $a < b$. Hence this is the ambiguous case. When log sin B has been computed, we shall determine the number of solutions. The formulas, obtained as in Example 1, are

$$\sin B = \frac{b \sin A}{a}, \tag{a}$$

$$C = 180° - (A + B),$$

$$c = \frac{a \sin C}{\sin A} = a \sin C \csc A. \tag{b}$$

The solution is displayed in the following form:

	(a)	(b)	(b)
$a = 9.467$	$\text{colog}\, a = 9.02379 - 10$	$\log a = 0.97621$	$\log a = 0.97621$
$b = 14.433$	$\log b = 1.15936$		
$A = 11°14'18''$	$l \sin A = 9.28979 - 10$	$l \csc A = 0.71021$	$l \csc A = 0.71021$
$B_1 = 17°17'6''$	$l \sin B_1 = 9.47294 - 10$		
$B_2 = 162°42'54''$			
$C_1 = 151°28'36''$		$l \sin C_1 = 9.67899 - 10$	
$C_2 = 6°2'48''$			$l \sin C_2 = 9.02259 - 10$
$c_1 = 23.196$		$\log c_1 = 1.36541$	
$c_2 = 5.1169$			$\log c_2 = 0.70901$

Since log sin B from the first column was found to be negative, we concluded that there were two solutions. Since sin B is positive in both the first and the second quadrants, we obtained two supplementary angles B_1 and B_2 from log sin B.

One of Mollweide's equations should be employed to check the results. It is interesting to check the results of both solutions by a single setting of the slide rule.

To solve the triangle of Example 2 by means of the slide rule, use the same general line of argument applied in the logarithmic solution, but employ Rule (*A*) of §63 for the computation. Hence draw Fig. 9 and

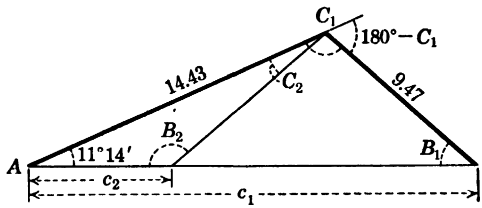

Fig. 9.

push hairline to 947 on D,
draw 11°14′ of S under hairline,
push hairline to 14.43 on D,*
at the hairline read B_1 = **17°17′** on S;
push hairline to 180° − C_1 = 28°31′ on S,
at the hairline read c_1 = **23.2** on D;
compute $C_2 = B_1 - 11°14'$ = **6°3′**,
push hairline to 6°3′ on S,
at the hairline read c_2 = **5.12** on D.

Example 3. Given $a = 96.55$, $b = 124.98$, and $A = 50°34'51''$ (see Fig. 10). Find c, B, and C.

Solution. Upon observing that $a < b$, we know that this is the ambiguous case. The number of solutions will be determined from log sin B. The formulas, obtained as in Example 1, are

$$\sin B = \frac{b \sin A}{a}, \qquad (a)$$

$$C = 180° - (A + B),$$

$$c = \frac{a \sin C}{\sin A} = a \sin C \csc A. \qquad (b)$$

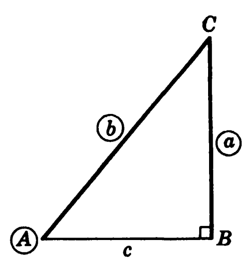

Fig. 10.

* Occasionally it will be necessary to use the following rule: when a number is to be read on the D scale opposite a number on the slide and cannot be read because the slide projects beyond the body of the rule, push

The solution is displayed in the following form:

	(a)	(b)
$a = 96.55$	$\text{colog } a = 8.01525 - 10$	$\log a = 1.98475$
$b = 124.98$	$\log b = 2.09684$	
$A = 50°34'51''$	$l \sin A = 9.88791 - 10$	$l \csc A = 0.11209$
$B = \mathbf{90°00'00''}$	$l \sin B = 0.00000$	
$C = \mathbf{39°25'9''}$		$l \sin C = 9.80276 - 10$
$c = \mathbf{79.360}$		$\log c = 1.89960$

While computing, we found that $\log \sin B = 0$. Therefore $\sin B = 1$, $B = 90°$, and there is one solution.

The computation should be checked by one of Mollweide's equations.

EXERCISES

Solve the following triangles:

1. $a = 309$, $b = 360$, $A = 21°14'25''$.

2. $b = 316$, $c = 360$, $B = 21°16'44''$.

3. $A = 41°13'$, $a = 77.04$, $b = 91.06$.

4. $b = 115.97$, $c = 139.06$, $B = 43°11'32''$.

5. $a = 294$, $b = 189$, $A = 67°32'$.

6. $b = 71.818$, $c = 78.493$, $B = 66°12'10''$.

7. $a = 48.134$, $b = 35.826$, $A = 36°24'0''$.

8. $a = 32.239$, $b = 50.204$, $A = 32°18'30''$.

9. $a = 4.2356$, $b = 5.1234$, $A = 54°18'0''$.

10. $b = 216.45$, $c = 177.01$, $C = 35°36'20''$.

11. $a = 341.91$, $b = 745.91$, $A = 43°35'39''$.

12. $a = 95.21$, $b = 126.4$, $A = 51°40'30''$.

13. It is desired to measure the distance AB between two points on opposite sides of a lake. A point C, easily accessible to both A and B,

---

the hairline to the index of the C scale inside the body and draw the other index of the C scale under the hairline. The desired reading can then be made.

is chosen. It is found that $AC = 8461$ and $BC = 10{,}246$. At A the angle BAC is found to be $26°33'$. Find the distance AB.

14. Two wires are run from the same point on the vertical edge of a building to a level courtyard below. One wire is 42.45 ft. long and makes an angle of 58° with the horizontal. The other wire is 48.60 ft. long and lies in the same vertical plane with the first but on the opposite side of the edge. Find the inclination of the second wire to the yard and the distance between anchor points.

15. The distance from a point A to a point C cannot be measured directly but is estimated to be about $\frac{1}{4}$ mile. From a point B, $BA = 7201.5$ ft., and $BC = 6180.3$ ft. Angle BAC is found to be $41°14'25''$. Find the distance AC.

67. Case III. Given two sides and the included angle. When two sides and the included angle are the given parts, the triangle can be solved by means of the law of tangents and the law of sines. The law of tangents gives the angles opposite the given sides, and the law of sines can then be used to find the third side. The result may be checked by means of Mollweide's equations.

Example 1. Given $c = 1.0398$, $a = 6.7517$, and $B = 127°9'18''$ (see Fig. 11). Find A, C, and b.

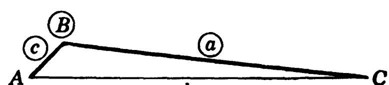

FIG. 11.

Solution. From the relation $A + B + C = 180°$, we have $A + C = 180° - B$, or

$$\tfrac{1}{2}(A + C) = \tfrac{1}{2}(180° - 127°9'18'') = 26°25'21''.$$

From the law of tangents, (see §60) we have

$$\tan \tfrac{1}{2}(A - C) = \frac{(a - c)}{(a + c)} \tan \tfrac{1}{2}(A + C), \qquad (a)$$

and from the law of sines

$$b = \frac{a \sin B}{\sin A} = a \sin B \csc A.^* \qquad (b)$$

* In this case one of Mollweide's equations may be used to find the unknown side and the other as a check.

The solution is displayed in the following form:

	(a)	(b)
$B = 127°9'18''$		$l \sin B = 9.90146 - 10$
$a = 6.7517$		$\log a = 0.82941$
$c = 1.0398$		
$a - c = 5.7119$	$\log (a - c) = 0.75678$	
$a + c = 7.7915$	$\text{colog}\ (a + c) = 9.10838 - 10$	
$\frac{1}{2}(A + C) = 26°25'21''$	$l \tan \frac{1}{2}(A + C) = 9.69626 - 10$	
$\frac{1}{2}(A - C) = 20°0'54''$	$l \tan \frac{1}{2}(A - C) = 9.56142 - 10$	
$A = \mathbf{46°26'15''}$		$l \csc A = 0.13989$
$C = \mathbf{6°24'27''}$		
$b = \mathbf{7.4262}$		$\log b = 0.87076$

The following solution will illustrate the method of using the slide rule to solve a triangle when two of its sides and the included angle are known.

Example 2. Solve the triangle in which $b = 28.7$, $c = 45.2$, $A = 47°$.

Solution. In Fig. 12 draw line CD perpendicular to AB, and solve the right triangle ACD. Knowing x, get $z = 45.2 - x$.

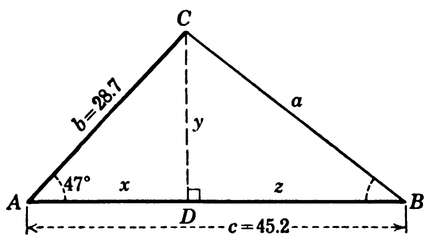

FIG. 12.

Then, knowing the two legs y and z of right triangle DBC, solve it by the method of §128. This leads to the following settings:

set right index of C to 28.7 on D,
opposite 43° on S read $x = 19.6$ on D,
opposite 47° on S read $y = 21$ on D;
compute $z = 45.2 - 19.6 = 25.6$,
set right index of C to 25.6 on D,
push hairline to 21 on D,
at hairline read $B = \mathbf{39°22'}$ on T;

draw $39°22'$ of S under the hairline,
opposite index of C read a = **33.1** on D.
Evidently angle $C = 43° + 90° - 39°22' =$ **93°38′**.

EXERCISES

Solve the following triangles:

1. $a = 17$, $b = 12$, $C = 59°17'$.

2. $a = 748$, $b = 375$, $C = 63°35'30''$.

3. $b = 232.23$, $c = 195.59$, $A = 61°13'0''$.

4. $a = 27.92$, $b = 42.38$, $C = 39°40'$.

5. $b = 85.249$, $c = 105.63$, $A = 50°40'24''$.

6. $a = 0.59312$, $b = 0.22734$, $C = 64°38'0''$.

7. $a = 6.2387$, $b = 2.3475$, $C = 110°32'$.

8. $a = 35.237$, $b = 18.482$, $C = 110°40'30''$.

9. The end A of a boom AB is attached to the platform of a crane and a cable BC connects the end B to a point C on top of the crane (see Fig. 13). If AB = 35 ft., AC = 15 ft., and angle CAB = 95°, find the length of the cable.

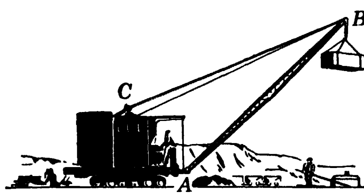

FIG. 13.

10. From a point 5890 ft. from one end of a lake and 6728 ft. from the other end, the lake subtends an angle of 47°18′. Find the length of the lake.

11. A triangular tract of land is to be enclosed by a fence. The side AB = 54.235 ft.; side CB = 29.483 ft.; the included angle B is 95°40′25″. Find the amount of fencing needed to enclose the triangular plot.

12. From the top of a lighthouse 188.6 ft. above sea level, the angle of depression of a ship was 5°30′30″, and its compass bearing was 16°48′0″. One hour later the angle of depression was 4°10′0″ and the compass bearing, 143°4′0″. Find the distance traveled by the ship and its compass course.

13. Two yachts start from the same place at the same time. Yacht A sails at 10 knots on compass course 62°. Yacht B sails at 8 knots on compass course 135°. How far apart are they at the end of 40 min., and what is the bearing of yacht B from yacht A?

14. Prove that the area K of the triangle shown in Fig. 14 is given by

$$K = \tfrac{1}{2}ab \sin C.$$

Use the formula just derived to find the area of the triangle of (*a*) Exercise 1; (*b*) Exercise 7.

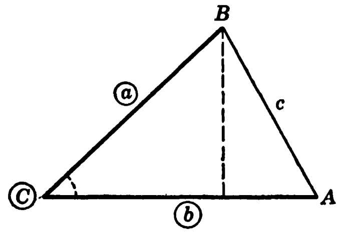

FIG. 14.

15. From a mountain peak in a vertical plane through a straight tunnel, the angles of depression of its ends are 42°41′ and 52°22′, and the corresponding distances from the peak to the ends of the tunnel are 3710 ft. and 4100 ft., respectively. Determine whether the tunnel is horizontal and find its length.

16. From a ship two lighthouses bear N. 40° E. After the ship has sailed 15 miles on a course of 135°, they bear 10° and 345°, respectively. Find the distance between them and the distance from the ship in the latter position to the more distant lighthouse.

17. Two men, A and B, start at the same point on the circumference of a circle of radius 900 ft. and walk at the rate of 350 ft. per minute. If A walks toward the center of the circle and B walks along the circumference, find how far apart the two men are at the end of 1 min.

68. The half-angle formulas. While the law of cosines may be used to solve a triangle when the three sides are given, it is not convenient to use in logarithmic computation. We shall now derive from the law of cosines other formulas that are well adapted to logarithmic computation.

From the first equation of (24) §56, we obtain

$$2 \sin^2 \tfrac{1}{2}A = 1 - \cos A, \tag{1}$$

and from the law of cosines, we have

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}. \tag{2}$$

Substituting the value of cos A from (2) in (1), we get

$$
\begin{aligned}
2\sin^2 \tfrac{1}{2}A &= 1 - \frac{b^2 + c^2 - a^2}{2bc} \\
&= \frac{2bc - b^2 - c^2 + a^2}{2bc} \\
&= \frac{a^2 - (b^2 - 2bc + c^2)}{2bc} \\
&= \frac{a^2 - (b - c)^2}{2bc} \\
&= \frac{(a + b - c)(a - b + c)}{2bc}
\end{aligned} \tag{3}
$$

Let

$$a + b + c = 2s. \tag{4}$$

Subtracting $2a$, $2b$, and $2c$ from each member of (4), we obtain, respectively,

$$
\begin{aligned}
-a + b + c &= 2(s - a), \\
a - b + c &= 2(s - b), \\
a + b - c &= 2(s - c).
\end{aligned}
$$

Substituting from the last two of these equations in (3) and simplifying slightly, we get

$$\sin \tfrac{1}{2}A = \sqrt{\frac{(s - b)(s - c)}{bc}}. \tag{5}$$

Similarly,

$$\sin \tfrac{1}{2}B = \sqrt{\frac{(s - c)(s - a)}{ca}}, \tag{6}$$

and

$$\sin \tfrac{1}{2}C = \sqrt{\frac{(s - a)(s - b)}{ab}}. \tag{7}$$

Using the second definition (8) of §4 together with (1) above, we have

$$\sin^2 \tfrac{1}{2} A = \text{hav } A.$$

From this equation and (5), we easily derive

$$\text{hav } A = \frac{(s - b)(s - c)}{bc}. \tag{8}$$

Similar formulas for hav B and hav C may be obtained from (6) and (7). Formula (8) is often used when haversine tables are available.

From the second equation of (24) §56 and (2), we obtain

$$\begin{aligned} 2\cos^2 \tfrac{1}{2}A &= 1 + \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{2bc + b^2 + c^2 - a^2}{2bc} \\ &= \frac{(b+c)^2 - a^2}{2bc} \\ &= \frac{(a+b+c)(-a+b+c)}{2bc}, \\ &= \frac{(2s)2(s-a)}{2bc}. \end{aligned}$$

Hence

$$\cos \tfrac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}. \tag{9}$$

Similarly,

$$\cos \tfrac{1}{2}B = \sqrt{\frac{s(s-b)}{ca}}, \tag{10}$$

and

$$\cos \tfrac{1}{2}C = \sqrt{\frac{s(s-c)}{ab}}. \tag{11}$$

Since $\tan \tfrac{1}{2}A = \dfrac{\sin \tfrac{1}{2}A}{\cos \tfrac{1}{2}A}$, we get by substitution from (5) and (9)

$$\tan \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}. \tag{12}$$

Similarly,

$$\tan \tfrac{1}{2}B = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}, \tag{13}$$

and

$$\tan \tfrac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}. \tag{14}$$

Formula (12) may be written

$$\tan \tfrac{1}{2}A = \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}. \tag{15}$$

If we let

$$r^* = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

we may write

$$\tan \tfrac{1}{2}A = \frac{r}{s-a}. \tag{16}$$

Similarly

$$\tan \tfrac{1}{2}B = \frac{r}{s-b}, \tag{17}$$

$$\tan \tfrac{1}{2}C = \frac{r}{s-c}. \tag{18}$$

When calculating the angles of a triangle, the tangents of the half angles should be used, since the complete calculation of A, B, C may be performed by taking from the tables only the four logarithms $\log s$, $\log (s - a)$, $\log (s - b)$, and $\log (s - c)$.

69. Case IV. Given three sides. When the three sides of a triangle are given, its solution may be effected by means of the half-angle formulas and the results checked by means of the relation $A + B + C = 180°$.

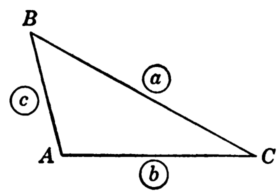

FIG. 15.

Example. Given $a = 6.8235$, $b = 5.2063$, and $c = 3.1628$ (see Fig. 15). Find A, B, and C.

Solution. The half-angle formulas are

$$\tan \frac{A}{2} = \frac{r}{s-a}, \tag{a}$$

$$\tan \frac{B}{2} = \frac{r}{s-b}, \tag{b}$$

$$\tan \frac{C}{2} = \frac{r}{s-c}, \tag{c}$$

where

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}. \tag{d}$$

* r is the radius of the circle inscribed in the triangle.

The solution is displayed in the following form:

	(d)	(a)	(b)	(c)
$a = 6.8235$				
$b = 5.2063$				
$c = 3.1628$				
$2s = 15.1926$				
$s = 7.5963$	$\operatorname{colog} s = 9.11940-10$			
$s-a = 0.7728$	$\log s-a = 9.88807-10$	$\operatorname{colog} s-a = 0.11193$		
$s-b = 2.3900$	$\log s-b = 0.37840$		$\operatorname{colog} s-b = 9.62160-10$	
$s-c = 4.4335$	$\log s-c = 0.64675$			$\operatorname{colog} s-c = 9.35325-10$
$s = 7.5963$				
	$2)\log r^2 = 0.03262$			
r	$\log r = 0.01631$	$\log r = 0.01631$	$\log r = 0.01631$	$\log r = 0.01631$
$A = 106°40'40''$	$\frac{1}{2}A = 53°20'20''$	$l \tan \frac{1}{2}A = 0.12824$		
$B = 46°57'42''$	$\frac{1}{2}B = 23°28'51''$		$l \tan \frac{1}{2}B = 9.63791-10$	
$C = 26°21'38''$	$\frac{1}{2}C = 13°10'49''$			$l \tan \frac{1}{2}C = 9.36956-10$
$A+B+C = 180°0'0''$ (*Check*)				

The arithmetic involved in computing $s - a$, $s - b$, and $s - c$ was checked by verifying that their sum was s.

By means of the law of cosines, we can find by the use of the slide rule one of the angles of the triangle. Then, by applying the law of sines, we read on the slide rule the other two angles.

EXERCISES

Solve the following triangles:

1. $a = 3.41$, $b = 2.60$, $c = 1.58$.

2. $a = 111$, $b = 145$, $c = 40$.

3. $a = 14.493$, $b = 55.436$, $c = 66.913$.

4. $a = 97.862$, $b = 105.98$, $c = 138.72$.

5. $a = 95.321$, $b = 113.72$, $c = 179.84$.

6. $a = 2.2361$, $b = 2.4495$, $c = 2.6458$.

7. $a = 1.4932$, $b = 2.8711$, $c = 1.9005$.

8. $a = 529.37$, $b = 716.49$, $c = 635.21$.

Use the law of cosines to solve the following triangles:

9. $a = 13$, $b = 11$, $c = 9$.

10. $a = 6$. $b = 7$, $c = 8$.

11. $a = 60$, $b = 40$, $c = 35$.

12. $a = 2$. $b = 3$, $c = 4$.

13. Find the largest angle of the triangle whose sides are 13, 14, 16.

14. To find the width of a river, a point A (Fig. 16) is located on one bank and two points B and C on the other. A fourth point D is located in line with AB, and a fifth point E in line with AC. The distances were measured as follows: $BC = 506$ ft., $BD = 453$ ft., $BE = 809$ ft., $CD = 753$ ft., $CE = 392$ ft. Find the width of the river.

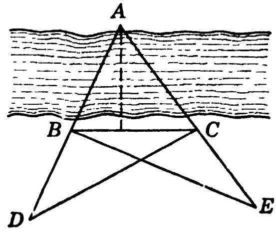

FIG. 16.

15. Three towns, A, B, and C, are situated so that $AB = 23.37$ miles, $BC = 11.84$ miles, and $AC = 16.29$ miles. A road from A to B is met at D by a perpendicular road from C. Find the length of this latter road and the distance DB.

16. Derive Heron's formula for the area K of a triangle in terms of its three sides a, b, c, and $s = \frac{1}{2}(a + b + c)$, namely:

$$K = \sqrt{s(s - a)(s - b)(s - c)}.$$

Hint. The area of the triangle shown in Fig. 17 is $K = \frac{1}{2}bh = \frac{1}{2}cb \sin A$. Replace $\sin A$ by $2 \sin \frac{1}{2}A \cos \frac{1}{2}A$, and then use (5) and (9).

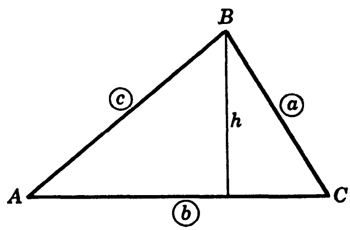

FIG. 17.

17. Use Heron's formula to find the area of the triangle of (*a*) Exercise 1; (*b*) Exercise 7.

18. The sides of a triangular field measure 223.6 ft., 244.9 ft., and 264.6 ft. Find the area of the field.

70. Summary. A summary of the four cases of oblique triangles is given below in tabular form.

Given	One side and two angles	Two sides and the angle opposite one of them	Two sides and the included angle	Three sides
Using logarithms, solve by	Law of sines	Law of sines	Law of tangents and law of sines	Tangent of half-angle formulas
Using slide rule, solve by	Law of sines	Law of sines	Dropping a perpendicular	Law of cosines and law of sines
Check by	Mollweide's equations			$A + B + C = 180°$, and slide rule

71. MISCELLANEOUS EXERCISES

Solve the following triangles:

1. $a = 42.365$,	**3.** $a = 412.67$,	**5.** $a = 6.342$,
$b = 25.863$,	$A = 50°38'50''$,	$b = 7.295$,
$C = 115°39'$	$B = 60°7'25''$.	$c = 8.4177$.
2. $a = 365.74$,	**4.** $a = 0.062387$,	**6.** $a = 31.239$,
$b = 445.84$,	$b = 0.023475$,	$b = 49.001$,
$c = 545.62$.	$C = 110°32'$.	$A = 32°18'$.

7. Two points A and B are inaccessible from C. If $AB = 1308$ ft., angle $CAB = 53°7'$, and angle $CBA = 70°15'$, find the distance from C to each of the other two points.

8. The angles of elevation of a balloon, directly above a straight road, from two points of the road on opposite sides of the balloon, are $78°15'20''$ and $59°47'40''$. If the two points are 5000 ft. apart, what is the height of the balloon?

9. A 52-ft. ladder is set against an inclined buttress and reaches 46 ft. up its face. If the foot of the ladder is 20 ft. from the foot of the inclined face, what is the inclination of the face of the buttress?

10. A and B are separated by an obstruction, but C is accessible from both. If $AC = 161.3$ ft., $CB = 793.6$ ft., and angle $C = 58°22'30''$, what is the distance AB?

11. A ship sails 23 miles on compass course 15°, thence 15 miles on compass course 78°. How far and in what direction is she from her starting point?

12. The area of a triangle whose angles are $61°9'32''$, $34°14'46''$ and $84°35'42''$ is 680.60. What is the length of the longest side?

13. The captain of a ship traveling at 14 knots on compass course 66° sights a lighthouse bearing 39°. After 10 min. the lighthouse bears $17°30'$. How long does it take to get to the point nearest the lighthouse, and how far away is it at that time?

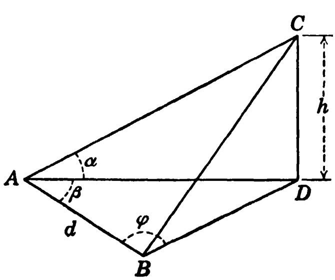

FIG. 18.

14. The magnitude h of an inaccessible vertical height DC is desired. A base line AB of length d in the horizontal plane through the base D of the object is laid off, and the angles DAC, DAB, and DBA are found by measurement to be α, β, and φ, respectively (see Fig. 18).

(*a*) Show that

$$h = d \sin \varphi \tan \alpha \csc (\beta + \varphi).$$

(*b*) If $d = 132.1$ ft., $\alpha = 32°16'$, $\beta = 22°35'$, $\varphi = 20°48'$, find h.

15. From the top of a hill the angles of depression of the top and bottom of a flagstaff 25 ft. high at the foot of the hill are observed to be 45°13′ and 47°12′, respectively. Find the height of the hill.

16. The angle of elevation of a balloon ascending uniformly and vertically at a height of 1 mile is observed to be 35°20′; 20 min. later the elevation is observed to be 55°40′. How fast is the balloon moving?

17. A flagpole 160.43 ft. high is situated at the top of a hill. At a point 600 ft. down the hill the angle between the surface of the hill and a

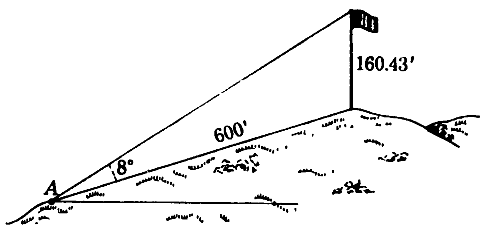

Fig. 19.

line to the top of the flagpole is 8°. Find the distance from the point to the top of the flagpole and the inclination of the ground to a horizontal plane (see Fig. 19).

18. From a point on a horizontal plane the angle of elevation of the top of a mountain peak is 40°28′36″, and 4163.2 ft. farther away in the same vertical plane the angle of elevation is 28°50′24″. Find the height of the peak above the horizontal plane.

19. A tower (Fig. 20) stands on a hill inclined 22° with the horizontal. At a point A some distance down the hill the angle of elevation of the top

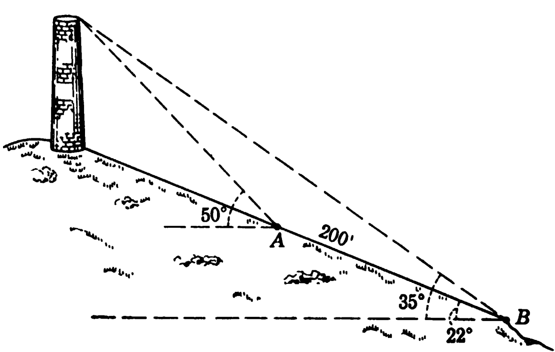

Fig. 20.

of the tower is 50° and at B, 200 ft. farther down the hill, the angle is 35°. Find the height of the tower.

20. A tower stands at the foot of a hill inclined 18° with the horizontal. At a point A some distance up the hill the angle of elevation of the top of the tower is 28°, and at B, 120 ft. farther up the hill, the angle is 15°. Find the height of the tower.

21. From a ship two lighthouses bear N. 45° E. After the ship sails at 11 knots on a course of 130° for 2 hr., the lighthouses bear 6° and 356°, respectively. Find the distance between the lighthouses.

22. A 50-ft. vertical pole casts a shadow 62 ft. 3 in. in length along the ground when the sun's altitude is 41°38′. Find the inclination of the ground in the line of the shadow.

23. The diagonals of a parallelogram are 376.14 ft. and 427.21 ft., and the included angle is 70°12′38″. Find the length of the sides.

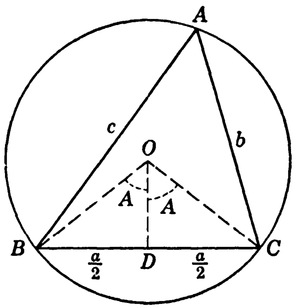

FIG. 21.

24. If R is the radius of a circle circumscribed about the triangle ABC (Fig. 21), show that

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Hint. Angle BAC = angle DOC.

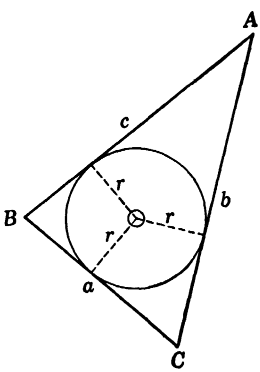

FIG. 22.

25. Find the radius of a circle inscribed in a triangle whose sides are a, b, and c (see Fig. 22).

Hint. The area K of the triangle ABC is $\frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = rs$.

26. Prove that the area K of a triangle is given by the formula

$$K = \frac{abc}{4R},$$

where R is the radius of the circumscribing circle.

27. Show that in any triangle

(a) $a^2 + b^2 + c^2 = 2(ab \cos C + bc \cos A + ac \cos B)$;

(b) $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{a^2 + b^2 + c^2}{2\,abc}$.

28. An observer whose eye is 37 ft. above the surface of the water measures the compass bearing and depression of two buoys as follows: A, compass bearing 103°, depression 3°50′; B, compass bearing 165°, depression 2°45′. Find the length AB and the compass bearing of B from A.

29. Find the value of x in Fig. 23.

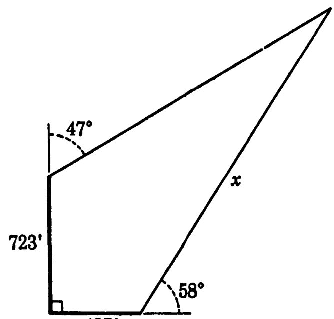

FIG. 23.

30. Two stations, B and C, are situated on a horizontal plane 1200 ft. apart. A balloon is directly above a point A in the same horizontal plane as B and C. At B the angle of elevation of the balloon is 61°30′, and the angle at B subtended by AC is 53°12′, and at C the angle subtended by AB is 71°37′. Find the height of the balloon.

31. A plane through a vertical flagpole on a small hill contains two points A and B lying 130 ft. apart in a horizontal plane, both on the same side of the hill. From A the angles of elevation of the top and bottom of the flagpole are 13° and 6°, respectively, and from B the angle of elevation of its top is 10°. Find the height of the flagpole.

32. A, B, C are three objects at known distances apart; namely, AB = 1056 yd., AC = 924 yd., BC = 1716 yd. An observer places himself at a station P, from which C appears directly in front of A and observes the angle CPB to be 14°24′. Find the distance CP.

33. The foremast on a freighter sailing west bears N. 35° W. for an observer on a submarine 10,000 yd. from the mast. A torpedo fired from the submarine in a direction N. 53° W. travels at the rate of 27 knots and crosses the path of the freighter 235 yd. ahead of its mast. Find the speed of the freighter (see Fig. 24 on page 174). (Take 2000 yd. = 1 nautical mile.)

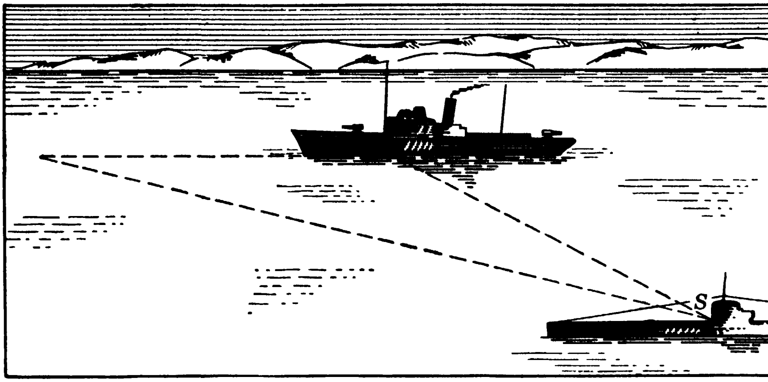

FIG. 24.

34. A vertical plane through the foremast of an anchored freighter cuts a hill on the near-by shore in a line AB inclined 37° to the horizontal. From A the angle of depression of the top T of the mast is 9°, and from B, 98 ft. downhill from A, the angle of elevation of T is 7°. If the mast subtends an angle of 14° at B, find its height.

35. P and Q are two inaccessible objects; a straight line AB, in the same plane with P and Q, is measured and found to be 280 yd. long. If angle $PAB = 95°$, angle $QAB = 47°30'$, angle $QBA = 110°$, and angle $PBA = 52°20'$, find the length of PQ.

36. A and B are two stations 1 mile apart, and B is due east of A. When an airplane is due north of A its angles of elevation at A and B are 37° and 23°, respectively, and when due north of B, its angles of elevation at A and B are 12° and 19°, respectively. Find its altitude at each time of observation and the compass course it is traveling.

37. On the bank of a river there is a column 200 ft. high supporting a statue 30 ft. high. The statue to an observer on the opposite bank subtends the same angle that subtends a man 6 ft. high standing at the base of the column. Find the breadth of the river.

38. From a certain station the angular elevation of a mountain peak in the northeast is observed to be α. A hill $22\frac{1}{2}°$ south of east whose height above the station is known to be h is then ascended, and the mountain peak is now seen in the north at an elevation β. Prove that the height of its summit above the first station is $h \sin \alpha \cos \beta \csc (\alpha - \beta)$.

39. A tower is situated on a horizontal plane at a distance a from the base of a hill whose inclination is α. A person on the hill, looking over the tower, can just see a pond, the distance of which from the tower is b. Show that, if the distance of the observer from the foot of the hill be c, the height of the tower is $\dfrac{bc \sin \alpha}{a + b + c \cos \alpha}$.

40. The angular elevation of a column as viewed from a station due north of it is α, and as viewed from a station due east of the former station and at a distance c from it is β. Prove that the height of the column is

$$\frac{c \sin \alpha \sin \beta}{[\sin (\alpha - \beta) \sin (\alpha + \beta)]^{\frac{1}{2}}}.$$

41. An observer found the angle of elevation of the summits of two spires which appear in a straight line to be α, and the angles of depression of their reflections in still water to be β and γ. If the height of the observer's eye above the level of the water was c, show that the horizontal distance between the spires is

$$\frac{2c \cos^2 \alpha \sin (\beta - \gamma)}{\sin (\beta - \alpha) \sin (\gamma - \alpha)}.$$

42. A, B, C are three objects so situated that $AB = 320$ yd., $AC = 600$ yd., and $BC = 435$ yd. From a station P it is observed that $APC = 15°$, and $BPC = 30°$. Find the distances of P from A, B, and C if the point A is nearest P and the angle APB is the sum of the angles APC and BPC.

Hint. From Fig. 25, $PC = 600 \sin x/\sin 15° = 435 \sin y/\sin 30°$. Solve this equation for $\sin x/\sin y$, apply composition and division, and in the result replace $\sin x - \sin y$ by $2 \cos \frac{1}{2}(x + y) \sin \frac{1}{2}(x - y)$ and $\sin x + \sin y$ by $2 \sin \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y)$, and simplify to obtain

$$\tan \tfrac{1}{2}(x - y) = \frac{435 \sin 15° - 600 \sin 30°}{435 \sin 15° + 600 \sin 30°} \tan \tfrac{1}{2}(x + y). \qquad (A)$$

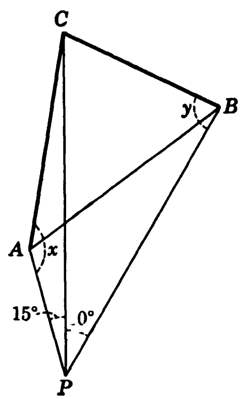

FIG. 25.

Compute angle C, replace $x + y$ in (A) by $360° - (15° + 30° + C)$, and solve the result for $x - y$, etc.

43. Solve a triangle, having given the length of the median to a side, and the angles into which this divides the vertical angle.

44. Three vertical flagstaffs stand on a horizontal plane. At each of the points A, B, and C in the horizontal plane, the tops of two staffs are seen in the same straight line, and these straight lines make angles α, β, γ with the horizon. The plane containing the tops makes an angle θ with the horizon. Prove that their heights are $BC/[\sqrt{\cot^2 \beta - \cot^2 \theta} \pm \sqrt{(\cot^2 \gamma - \cot^2 \theta}]$ and two similar expressions. Explain how the signs of the roots must be taken.

45. A certain gun with a shooting range of 1000 yd. per degree of elevation is pointed 20° above a horizontal plane. If a direct hit is registered on a target at a range of 20,000 yd. when the trunion axis is horizontal, find the variation in range and the variation in deflection to be expected on the second shot if for it the trunion axis is tilted through 5°.

46. Find the answer to the problem resulting when, in Exercise 45, the angle of elevation is replaced by θ, the range by R, and the angle of trunion tilt by ϕ.

CHAPTER IX

INVERSE TRIGONOMETRIC FUNCTIONS

72. Inverse trigonometric functions. To any angle there corresponds one and only one value of each trigonometric function, but to any value of a trigonometric function there correspond many angles. Thus $\sin 30° = \frac{1}{2}$, but 30°, 150°, 390°, and many other angles have a sine whose value is $\frac{1}{2}$.

The problem of finding the value of a trigonometric function of a given angle has already been considered in detail. The inverse problem, namely that of expressing the angles when the value of a trigonometric function is known, is the problem of this chapter. Consider the equation

$$y = \sin x. \tag{1}$$

Evidently x in this equation is an angle whose sine is y. To express this we introduce the symbol $\sin^{-1}$,* write

$$x = \sin^{-1} y, \tag{2}$$

and read the symbol $\sin^{-1} y$ as *the angle whose sine is y.* Since the problem of finding x in equation (1) when y is given is the inverse of finding y when x is given, the symbol $\sin^{-1} y$ is often read as the *inverse sine of y* or the *arc sine of y.*

Similarly, the symbol $\cos^{-1} x$ means the angle whose cosine is x and is read the *angle whose cosine is x*, the *inverse cosine of x*, or the *arc cosine of x.* The symbols $\tan^{-1} x$, $\cot^{-1} x$, $\sec^{-1} x$, and $\csc^{-1} x$ are defined and read in an analogous manner.

Example. Find two positive angles x less than 360° for which (*a*) $x = \tan^{-1} 1$, (*b*) $x = \cos^{-1} (-\frac{1}{2})$.

Solution. Since the tangent of a first-quadrant angle or of a third-quadrant angle is positive, it appears that $x = $ **45°** and

* In the notation $\sin^{-1} x$, -1 is not an algebraic exponent, and $\sin^{-1} x$ does not denote $1/\sin x$. To avoid confusion, when $1/\sin x$ is meant, write $(\sin x)^{-1}$.

$x = \mathbf{225°}$ satisfy $x = \tan^{-1} 1$. The cosine of a second-quadrant angle or of a third-quadrant angle is negative; hence $x = \mathbf{120°}$ and $x = \mathbf{240°}$ satisfy $x = \cos^{-1}(-\frac{1}{2})$.

EXERCISES

For each of the following equations find two positive values of y less than 360° satisfying it:

1. $y = \sin^{-1} \frac{1}{2}$.

2. $y = \sin^{-1} \frac{1}{2}\sqrt{3}$.

3. $y = \sin^{-1}(-\frac{1}{2}\sqrt{2})$.

4. $y = \tan^{-1} \sqrt{3}$.

5. $y = \tan^{-1}(-1)$.

6. $y = \cos^{-1}(-\frac{1}{2})$.

7. $y = \cos^{-1}(-\frac{1}{2}\sqrt{2})$.

8. $y = \sec^{-1} \sqrt{2}$.

9. $y = \sec^{-1} 2$.

10. $y = \csc^{-1}(-2)$.

11. $y = \csc^{-1} \frac{2}{3}\sqrt{3}$.

12. $y = \sin^{-1} 0.432$.

73. Graphs of the inverse trigonometric functions. Since

$$x = \sin y \qquad \text{and} \qquad y = \sin^{-1} x$$

express the same relation between x and y, we may make a table showing corresponding values of x and y for plotting $y = \sin^{-1} x$

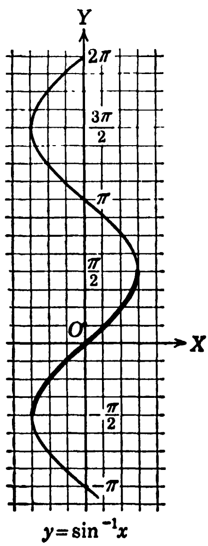

FIG. 1.

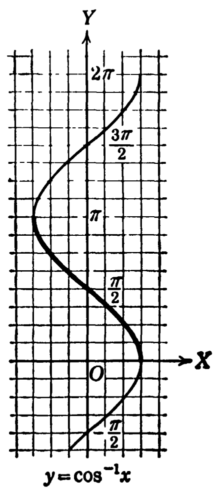

FIG. 2.

by using $x = \sin y$. Since this latter equation is the result of interchanging x and y in $y = \sin x$, we can obtain a table of values

for plotting $y = \sin^{-1} x$ by interchanging x and y in the table of values used in §46 to plot $y = \sin x$. Hence, interchanging x and y in the table of §46, plotting the points represented by the pairs of values in this new table, and connecting them by a smooth curve, we obtain the graph of $y = \sin^{-1} x$ (see Fig. 1).

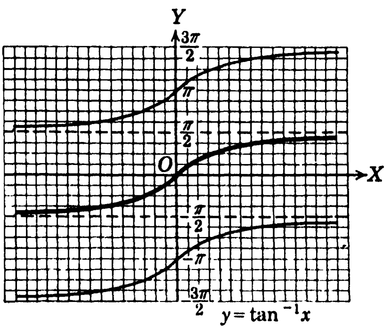

FIG. 3.

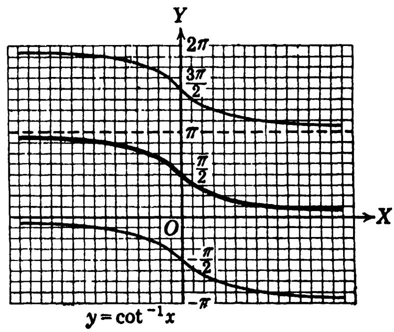

FIG. 4.

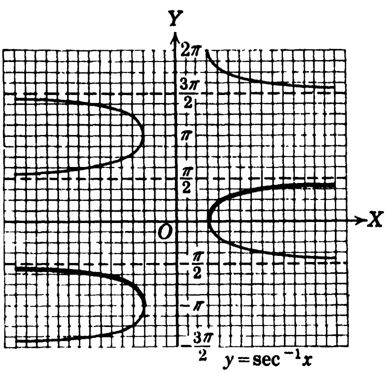

FIG. 5.

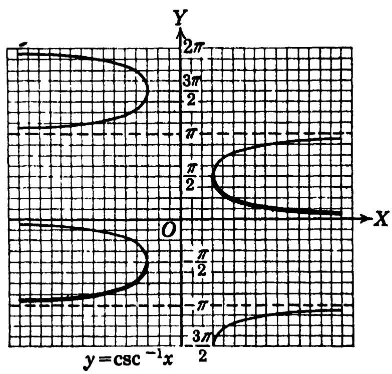

FIG. 6.

By a similar procedure tables of values are prepared for plotting the other inverse trigonometric functions; their graphs are shown in Figs. 2 to 6.

EXERCISES

Construct the graphs of the following equations:

1. $y = \sin^{-1} \frac{x}{2}$.

2. $y = \cos^{-1} \frac{x}{3}$.

3. $y = \tan^{-1} 2x$.

4. $y = \cot^{-1} \frac{x}{2}$.

5. $y = \sec^{-1} 2x$.

6. $y = \csc^{-1} 3x$.

7. $2y = \sin^{-1} 3x$.

8. $y = 4 \cos^{-1} 2x$.

9. $y = 2 \tan^{-1} \frac{x}{3}$.

10. $\frac{1}{3}y = 2 \cot^{-1} \frac{1}{2}x$.

11. $y = \frac{1}{2} \sec^{-1} x$.

12. $y = \frac{2}{3} \csc^{-1} \frac{3}{2}x$.

74. Representation of the general value of the inverse trigonometric functions. In §72, we saw that there are generally two positive values of x less than 360° satisfying an equation of the form

$$x = fn^{-1}(a) \tag{3}$$

where fn stands for sin, cos, tan, cot, sec, or csc. If α_1 and α_2 are two such values satisfying (3), then

$$x = \alpha_1 + n360° \qquad \text{and} \qquad x = \alpha_2 + n360° \tag{4}$$

satisfy (3) if n is an integer; for the six trigonometric functions of an angle are unaffected when the angle is changed by an integral multiple of 360°. When radians are used, the solution (4) is written

$$x = \alpha_1 + 2n\pi, \qquad \text{and} \qquad x = \alpha_2 + 2n\pi. \tag{5}$$

Example. Find the general value of $\sin^{-1}(-\frac{1}{2})$.

Solution. Expressed in degrees, the two positive angles less than 360° each of which has a sine equal to $-\frac{1}{2}$, are 210° and 330°. Hence the general value of $\sin^{-1}(-\frac{1}{2})$ is

$$\mathbf{210° + n360°, 330° + n360°,}$$

or, expressed in radians,

$$\frac{7\pi}{6} + n2\pi, \frac{11\pi}{6} + n2\pi.$$

EXERCISES

1. Find the general value of the angles represented by the following symbols:

(*a*) $\sin^{-1} \frac{1}{2}$.
(*b*) $\sin^{-1} \frac{1}{2}\sqrt{3}$.
(*c*) $\sin^{-1} \frac{1}{2}\sqrt{2}$.
(*d*) $\sin^{-1} (-\frac{1}{2}\sqrt{3})$.
(*e*) $\sin^{-1} 0$.
(*f*) $\sin^{-1} (-1)$.
(*g*) $\sin^{-1} \frac{1}{3}$.
(*h*) $\sin^{-1} 0.4321$.
(*i*) $\sin^{-1} (-\frac{5}{12})$.
(*j*) $\cos^{-1} \frac{1}{2}\sqrt{2}$.
(*k*) $\sec^{-1} (-\sqrt{2})$.
(*l*) $\cos^{-1} (-\frac{1}{2}\sqrt{3})$.
(*m*) $\csc^{-1} (-2)$.
(*n*) $\tan^{-1} (-1)$.
(*o*) $\tan^{-1} \infty$.
(*p*) $\cot^{-1} 1$.
(*q*) $\cot^{-1} \infty$.
(*r*) $\cot^{-1} 0.432$.

2. For each pair of the following equations, find all values of x that satisfy both of them:

(a) $x = \sin^{-1}(-\frac{1}{2})$, $x = \cos^{-1}\frac{1}{2}\sqrt{3}$.
(b) $x = \tan^{-1}\frac{1}{3}\sqrt{3}$, $x = \sin^{-1}(-\frac{1}{2})$.
(c) $x = \sin^{-1}\frac{1}{2}\sqrt{2}$, $x = \tan^{-1}(-1)$.
(d) $x = \sec^{-1}(-\sqrt{2})$, $x = \cot^{-1} 1$.
(e) $x = \csc^{-1} 2$, $x = \cot^{-1}(-\sqrt{3})$.
(f) $x = \cos^{-1}\frac{1}{2}$, $x = \csc^{-1}(-\frac{2}{3}\sqrt{3})$.

3. Find the general value of the angles represented by the following symbols:

(a) $\sin^{-1} 0.36$.
(b) $\cos^{-1} 0.60$.
(c) $\tan^{-1} 0.90$.
(d) $\cot^{-1} 2.1$.
(e) $\sec^{-1} 3.42$.
(f) $\csc^{-1} 1.21$.
(g) $\cos^{-1}\frac{3}{5}$.
(h) $\sin^{-1}\frac{2}{3}$.
(i) $\tan^{-1}\frac{5}{4}$.
(j) $\sec^{-1}\frac{3}{2}$.
(k) $\cot^{-1}\frac{7}{8}$.
(l) $\csc^{-1} 15$.

4. Show that the general values of $\tan^{-1} a$ are $\alpha + k \times 180°$, where α is a particular value. Also show that $\sin^{-1} 0 = k \times 180°$ and $\cos^{-1} 0 = 90° + k \times 180°$ (see Fig. 7).

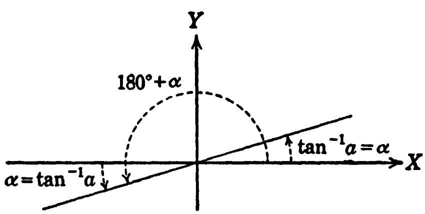

FIG. 7.

5. Using the formulas of Exercise 4, find the general values of θ when

(a) $3\theta = \cos^{-1} 0$,
(b) $5\theta = \sin^{-1} 0$,
(c) $2\theta = \tan^{-1}\sqrt{2}$,
(d) $3\theta = \tan^{-1}\sqrt{3}$.

In each case write all angles less than 360°.

6. Using the formulas given in Exercise 4, find the general values of the angles represented by the following symbols:

(a) $\tan^{-1} 1$.
(b) $\cot^{-1}\sqrt{3}$.
(c) $\tan^{-1}(-1)$.
(d) $\tan^{-1} 0.342$.

75. Principal values. Of the many values of an inverse trigonometric function, a special one is often called the *principal value.* Many ways of choosing a principal value could be devised. The choice dictated by advanced mathematics may be obtained by using the following statements.

Let a represent a *positive* number throughout this article. The *principal value* of $\sin^{-1} a$, $\cos^{-1} a$, $\tan^{-1} a$, etc., (if it exists) is zero or a positive angle no greater than 90°. For example, the principal value of $\sin^{-1} \frac{1}{2}$ is 30°, that of $\cos^{-1} 1$ is zero, and that of $\tan^{-1} 1$ is 45°.

The principal value of $\sin^{-1} (-\mathbf{a})$ (if it exists) or of $\tan^{-1} (-\mathbf{a})$ is a negative angle no greater numerically than 90°. For example, the principal value of $\sin^{-1} (-\frac{1}{2})$ is $-30°$, and that of $\tan^{-1} (-1)$ is $-45°$.

The principal value of $\cos^{-1} (-\mathbf{a})$ (if it exists) or of $\cot^{-1} (-\mathbf{a})$ is either 90°, 180°, or a positive second-quadrant angle. For example, the principal value of $\cos^{-1} (-1/\sqrt{2})$ is 135°, that of $\cot^{-1} (-1)$ is 135°, and that of $\cos^{-1} (-1)$ is 180°.

The principal value (if it exists) of $\sec^{-1} (-\mathbf{a})$ or $\csc^{-1} (-\mathbf{a})$ is a negative angle lying between $-90°$ and $-180°$. For example, the principal value of $\sec^{-1} (-2)$ is $-120°$, that of $\csc^{-1} (-\sqrt{2})$ is $-135°$, and that of $\csc^{-1} (-1)$ is $-90°$.

Figure 8 may help in choosing principal values. In §73, the part of each graph drawn with a heavy line is the graph repre-

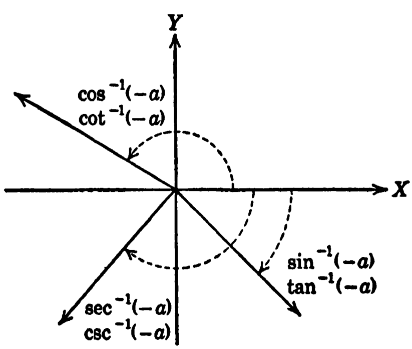

FIG. 8.

senting the principal value of the associated inverse trigonometric function.

EXERCISES

1. Find the principal values of the following:

(*a*) $\sin^{-1} \frac{1}{2}\sqrt{2}$. (*g*) $\cot^{-1} 1$. (*m*) $\csc^{-1} 1$.
(*b*) $\sin^{-1} \frac{1}{2}\sqrt{3}$. (*h*) $\cos^{-1} \frac{1}{2}$. (*n*) $\cot^{-1} \sqrt{3}$.
(*c*) $\sin^{-1} 0$. (*i*) $\cos^{-1} \frac{1}{2}\sqrt{2}$. (*o*) $\sec^{-1} 2$.
(*d*) $\tan^{-1} 1$. (*j*) $\cos^{-1} 0$. (*p*) $\cos^{-1} 1$.
(*e*) $\tan^{-1} \sqrt{3}$. (*k*) $\cos^{-1} \frac{1}{2}\sqrt{3}$. (*q*) $\sec^{-1} \frac{2}{3}\sqrt{3}$.
(*f*) $\tan^{-1} 0$. (*l*) $\csc^{-1} \frac{2}{3}\sqrt{3}$. (*r*) $\cot^{-1} \dfrac{1}{\sqrt{3}}$.

2. Find the principal values of the following:

(*a*) $\sin^{-1} (-\frac{1}{2})$. (*d*) $\tan^{-1} (-1)$.
(*b*) $\sin^{-1} \left(-\dfrac{1}{\sqrt{2}}\right)$. (*e*) $\tan^{-1} (-\sqrt{3})$.
(*c*) $\sin^{-1} \left(-\dfrac{\sqrt{3}}{2}\right)$. (*f*) $\tan^{-1} \left(-\dfrac{1}{\sqrt{3}}\right)$.

3. Find the principal values of the following:

(*a*) $\cos^{-1} \left(-\dfrac{1}{\sqrt{2}}\right)$. (*d*) $\cot^{-1} (-1)$.
(*b*) $\cos^{-1} \left(-\dfrac{\sqrt{3}}{2}\right)$. (*e*) $\cot^{-1} (-\sqrt{3})$.
(*c*) $\cos^{-1} (-\frac{1}{2})$. (*f*) $\cot^{-1} \left(-\dfrac{1}{\sqrt{3}}\right)$.

4. Find the principal values of the following:

(*a*) $\sec^{-1} (-2)$. (*d*) $\sec^{-1} (-\frac{2}{3}\sqrt{3})$. (*g*) $\csc^{-1} (-\frac{2}{3}\sqrt{3})$.
(*b*) $\sec^{-1} (-\sqrt{2})$. (*e*) $\csc^{-1} (-2)$. (*h*) $\csc^{-1} (-1)$.
(*c*) $\sec^{-1} (-1)$. (*f*) $\csc^{-1} (-\sqrt{2})$. (*i*) $\csc^{-1} (\tan 135°)$.

5. Find the principal values of the following:

(*a*) $\sin^{-1} (-\frac{1}{2})$. (*e*) $\csc^{-1} (-\sqrt{2})$. (*i*) $\sin^{-1} \frac{1}{2}\sqrt{3}$.
(*b*) $\tan^{-1} 1$. (*f*) $\sec^{-1} (-1)$. (*j*) $\sec^{-1} -\sqrt{2}$.
(*c*) $\cot^{-1} (-\sqrt{3})$. (*g*) $\tan^{-1} (\sin 270°)$. (*k*) $\cos^{-1} (-1)$.
(*d*) $\cos^{-1} 0$. (*h*) $\cot^{-1} \frac{1}{3}\sqrt{3}$.

6. Find the principal values of the following:

(*a*) $\sin^{-1} (-0.866)$. (*d*) $\sec^{-1} (-2.73)$. (*g*) $\sin^{-1} (-0.074)$.
(*b*) $\cos^{-1} (-0.414)$. (*e*) $\cot^{-1} (-0.472)$. (*h*) $\cos^{-1} (-0.913)$.
(*c*) $\tan^{-1} (-1.414)$. (*f*) $\csc^{-1} (-6.41)$. (*i*) $\tan^{-1} (-13.0)$.

7. Using principal values evaluate the following expressions, giving your answer in radian measure.

(*a*) $\sin^{-1}(\frac{1}{2}) - \sin^{-1}(-\frac{1}{2})$.

(*b*) $\sin^{-1}(-1) - \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$.

(*c*) $\tan^{-1}(\sqrt{3}) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

(*d*) $\cos^{-1}(\frac{1}{2}) - \cos^{-1}(-\frac{1}{2})$.

(*e*) $\sec^{-1}(1) - \sec^{-1}(-1)$.

(*f*) $\csc^{-1}(-2) - \sin^{-1}(-\frac{1}{2})$.

8. Verify for principal values the following equations:

(*a*) $\sin^{-1}\frac{1}{2} + \sin^{-1}\frac{1}{2}\sqrt{3} = -\sin^{-1}(-1)$.

(*b*) $\sin^{-1}\frac{1}{2}\sqrt{2} - 3\sin^{-1}\frac{1}{2}\sqrt{3} = -\frac{3}{4}\pi$.

(*c*) $\sin^{-1}(-\frac{1}{2}) + \sin^{-1}\frac{1}{2}\sqrt{2} = \frac{1}{12}\pi$.

(*d*) $\sin^{-1}\frac{1}{2}\sqrt{2} - \sin^{-1}\frac{1}{2}\sqrt{3} = \sin^{-1}\frac{1}{2} - \frac{1}{4}\pi$.

(*e*) $\sin^{-1}\frac{1}{2} + \cos^{-1}\frac{1}{2} = \sin^{-1} 1$.

(*f*) $\tan^{-1} 1 + \tan^{-1}\frac{1}{3}\sqrt{3} = \frac{9}{12}\pi - \tan^{-1}\sqrt{3}$.

(*g*) $\tan^{-1}\infty - \sin^{-1}\frac{1}{2}\sqrt{2} = \tan^{-1}\sqrt{3} - \frac{1}{12}\pi$.

(*h*) $\cos^{-1}\frac{1}{2} + \sin^{-1}\frac{1}{2} = \tan^{-1} 1 + \cos^{-1}\frac{1}{2}\sqrt{2}$.

(*i*) $\sin^{-1}\frac{1}{2} - \cos^{-1}(-\frac{1}{2}) = \cot^{-1}\sqrt{3} + \sec^{-1}(-2)$.

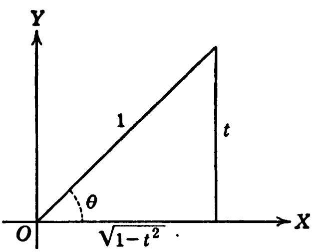

FIG. 9.

76. Relations among the inverse functions. Let t be a positive number less than 1 and θ a positive acute angle such that $\sin\theta = t$. Figure 9 shows a right triangle having an angle equal to θ, the hypotenuse equal to 1, the leg opposite θ equal to t, and the leg adjacent to θ equal to $\sqrt{1-t^2}$. From the figure we read

$$\sin\theta = t, \quad \text{or} \quad \theta = \sin^{-1} t,$$

$$\cos\theta = \sqrt{1-t^2}, \quad \text{or} \quad \theta = \cos^{-1}\sqrt{1-t^2},$$

$$\tan\theta = \frac{t}{\sqrt{1-t^2}}, \quad \text{or} \quad \theta = \tan^{-1}\frac{t}{\sqrt{1-t^2}},$$

$$\cot\theta = \frac{\sqrt{1-t^2}}{t}, \quad \text{or} \quad \theta = \cot^{-1}\frac{\sqrt{1-t^2}}{t},$$

$$\sec\theta = \frac{1}{\sqrt{1-t^2}}, \quad \text{or} \quad \theta = \sec^{-1}\frac{1}{\sqrt{1-t^2}},$$

$$\csc\theta = \frac{1}{t}, \quad \text{or} \quad \theta = \csc^{-1}\frac{1}{t}.$$

Since all these values of θ are equal, we have for principal values

$$\sin^{-1} t = \cos^{-1} \sqrt{1 - t^2} = \tan^{-1} \frac{t}{\sqrt{1 - t^2}} = \csc^{-1} 1/t$$
$$= \sec^{-1} \frac{1}{\sqrt{1 - t^2}} = \cot^{-1} \frac{\sqrt{1 - t^2}}{t}.$$

Hence, for principal values, we have the following relations:

$$\sin^{-1} u = \csc^{-1} \frac{1}{u},$$
$$\cos^{-1} u = \sec^{-1} \frac{1}{u},$$

provided u is a positive number less than 1, and

$$\tan^{-1} u = \cot^{-1} \frac{1}{u},$$

when u is any positive number.

77. Examples involving inverse trigonometric functions. The solutions of many trigonometric equations are effected by employing the relations existing among the inverse trigonometric functions. When solving an equation involving inverse functions, the student will find it advantageous to draw a right triangle for each of the angles involved in the original equation, and designate the lengths of the sides appropriately. From these triangles the value of any desired trigonometric function is taken directly. The following examples will illustrate the method.

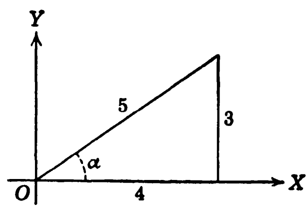

FIG. 10.

Example 1. Find the value of $\cos (\sin^{-1} \frac{3}{5})$ using the principal value of $\sin^{-1} \frac{3}{5}$.

Solution. Let α represent the principal value of $\sin^{-1} \frac{3}{5}$. The right triangle exhibiting α is shown in Fig. 10 with the sides appropriately numbered. From this figure we read directly

$$\cos (\sin^{-1} \tfrac{3}{5}) = \cos \alpha = \frac{4}{5}.$$

Example 2. Using principal values for the inverse functions involved, find

$$\cos\left[\cos^{-1}\left(-\tfrac{1}{3}\right) + \sin^{-1}\left(-\tfrac{1}{4}\right)\right]. \qquad (a)$$

Solution. Let α represent the principal value of $\cos^{-1}\left(-\frac{1}{3}\right)$ and β the principal value of $\sin^{-1}\left(-\frac{1}{4}\right)$. Substitution of these values in (a) gives $\cos(\alpha + \beta)$. Expanding this, we obtain

$$\cos\alpha\cos\beta - \sin\alpha\sin\beta. \qquad (b)$$

Consider the two right triangles in Fig. 11, one exhibiting angle α, the other angle β. In accordance with the definitions of

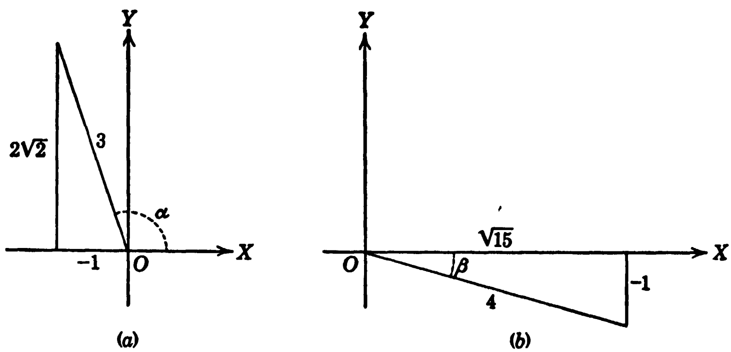

FIG. 11.

principal values we must take α in the second quadrant and β in the fourth quadrant.

Reading the values of $\cos\alpha$, $\cos\beta$, etc., direct from the triangles and substituting them in (b), we obtain

$$\left(-\frac{1}{3}\right)\left(\frac{\sqrt{15}}{4}\right) - \left(\frac{2\sqrt{2}}{3}\right)\left(-\frac{1}{4}\right) = \frac{-\sqrt{15} + 2\sqrt{2}}{12}.$$

Example 3. Show that

$$\tan^{-1}\left(-\frac{2}{9}\right) + \sin^{-1}\left(-\frac{1}{\sqrt{17}}\right) = \tfrac{1}{2}\cos^{-1}(-0.6) - 90° \qquad (a)$$

provided principle values for the inverse functions are used.

Solution. Let $A = \tan^{-1}\left(-\frac{2}{9}\right)$, $B = \sin^{-1}\left(-1/\sqrt{17}\right)$, $C = \cos^{-1}(-0.6)$. From these and the conventions of §75, it appears that angles A, B, and C are correctly represented in Fig. 12. Inspection shows that the two members of equation (a) are

negative acute angles. Hence they are equal if a trigonometric function of one member is equal to the same trigonometric

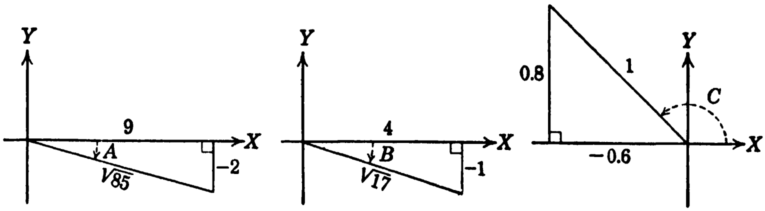

FIG. 12.

function of the other. Equation (a) may be written

$$A + B = \tfrac{1}{2}C - 90°. \qquad (b)$$

The cosine of the left-hand member of (b) is

$$\cos(A + B) = \cos A \cos B - \sin A \sin B, \qquad (c)$$

and the cosine of the right-hand member of (b) is

$$\cos(\tfrac{1}{2}C - 90°) = \sin \tfrac{1}{2}C = \sqrt{\tfrac{1}{2}(1 - \cos C)}. \qquad (d)$$

Replacing the functions in (c) and (d) by their values read from Fig. 12, we have

$$\cos(A + B) = \left(\frac{9}{\sqrt{85}}\right)\left(\frac{4}{\sqrt{17}}\right) - \left(\frac{-2}{\sqrt{85}}\right)\left(\frac{-1}{\sqrt{17}}\right)$$
$$= \frac{34}{17\sqrt{5}} = \frac{2}{\sqrt{5}},$$
$$\cos(\tfrac{1}{2}C - 90°) = \sqrt{\frac{1 + 0.6}{2}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}.$$

Since these values are equal, equation (a) is true.

EXERCISES

Using principal values for the inverse functions involved, evaluate the following expressions:

1. $\sin(\sin^{-1} \frac{2}{3})$.
2. $\cos(\cos^{-1} \frac{3}{5})$.
3. $\sin(\cos^{-1} \frac{5}{12})$.
4. $\cos(\sin^{-1} \frac{2}{3})$.
5. $\csc[\tan^{-1}(-\sqrt{7})]$.
6. $\sin[\sec^{-1}(-\frac{5}{3})]$.
7. $\cos[\csc^{-1}(-\frac{5}{4})]$.
8. $\cos[\cot^{-1}(-\frac{3}{4})]$.
9. $\cos[\tan^{-1}(-\frac{1}{2})]$.
10. $\sec(\cot^{-1} 2)$.
11. $\tan[\cot^{-1}(\pm 1)]$.
12. $\sec[\cot^{-1}(5.4)]$.
13. $\cos(2 \tan^{-1} 1)$.
14. $\tan(\cos^{-1} \frac{3}{5})$.
15. $\sin(\cot^{-1} \frac{1}{4})$.

16. Evaluate the following expressions, using principal values:

(*a*) $\tan\,[\tan^{-1}\frac{1}{2} + \tan^{-1}(-\frac{2}{3})]$.
(*b*) $\sec\,(\cos^{-1}\frac{1}{2} - \sin^{-1}\frac{1}{2})$.
(*c*) $\csc\,[\sin^{-1}(1/\sqrt{2}) + \tan^{-1} 1]$.
(*d*) $\sin\,[\sec^{-1}(-2) - \sin^{-1}(-\frac{3}{5})]$.

Using principal values for the inverse functions involved, verify the following equations:

17. $\sin^{-1} 1 - \tan^{-1} 1 = \dfrac{\pi}{4}$.

18. $\tan^{-1}\frac{1}{11} + \tan^{-1}\frac{5}{6} = \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2}$.
Hint. Take the tangent of both members.

19. $\tan^{-1}\frac{1}{2} + \sin^{-1}\frac{1}{10}\sqrt{10} = \frac{1}{4}\pi$.

20. $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{8}{17} + \csc^{-1}\frac{85}{13} = \csc^{-1} 1$.
Hint. Transpose $\csc^{-1}\frac{85}{13}$ to the right member and take the cosine of both members.

21. $\cos^{-1}\frac{12}{13} + \tan^{-1}\frac{1}{4} = \cot^{-1}\frac{43}{32}$.

22. $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \sec^{-1}\frac{1}{2}\sqrt{5}$.

23. $\cot^{-1} 7 + \tan^{-1}\frac{1}{8} + \cot^{-1} 18 = \cot^{-1} 3$.

24. $\tan^{-1}\frac{32}{43} - \cot^{-1} 4 = 2\tan^{-1}\frac{1}{3}$.

25. $\tan^{-1}\frac{2}{9} + \tan^{-1}\frac{1}{4} = \frac{1}{2}\sec^{-1}\frac{5}{3}$.

26. $\sin^{-1}\dfrac{3}{\sqrt{73}} + \sec^{-1}\dfrac{\sqrt{146}}{11} + \csc^{-1} 2 = \dfrac{5}{12}\pi$.

27. $\cos\,(2\sec^{-1}\frac{1}{7}\sqrt{50}) = \sin\,(4\sin^{-1}\frac{1}{10}\sqrt{10})$.

28. $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{1}{4}\pi$. (Clausen's formula for finding the value of π.)

29. $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = \frac{1}{4}\pi$. (Machin's formula for finding the value of π.)

30. $\tan^{-1}\frac{1}{239} = \tan^{-1}\frac{1}{70} - \tan^{-1}\frac{1}{99}$.

31. $\tan^{-1}\frac{5}{7} + \tan^{-1}\frac{1}{6} = \frac{1}{4}\pi$.

32. $\cot^{-1} 3 + \csc^{-1}\sqrt{5} = \frac{1}{4}\pi$.

33. $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\dfrac{x+y}{1-xy}$.

34. $3\sin^{-1} x = \sin^{-1}(3x - 4x^3),\ -\frac{1}{2} \leqq x \leqq \frac{1}{2}$.

35. $\sin\,(2\sin^{-1} x) = 2x\sqrt{1-x^2},\ -1 \leqq x \leqq 1$.

Find the value of the following expressions in terms of a and b; assume a and b positive, and use principal values for the inverse functions involved.

36. $\sin\left(2\cos^{-1} a + \frac{1}{2}\cos^{-1} b\right)$.

37. $\cos\left(\sec^{-1} a - \cos^{-1} \frac{1}{b}\right)$.

38. $\tan\left(\csc^{-1} \frac{1}{a} + \csc^{-1} \frac{1}{b}\right)$.

39. $\sin\left\{2\cos^{-1}\left[\tan\left(\frac{\pi}{2} - 2\tan^{-1} a\right)\right]\right\}$.

40. Solve Exercises 36 to 39, assuming that both a and b are negative.

78. Trigonometric equations. An equation which involves one or more trigonometric functions of a variable angle is a trigonometric equation. A trigonometric identity is a trigonometric equation which holds true for all values of the variable for which the members of the equation are defined. On the other hand, a trigonometric equation which is satisfied by only particular values of the variable is a trigonometric equation of condition. The problem connected with an identity concerns the proof that it is invariably true, whereas the problem associated with an equation of condition is to discover for what values it is true. By a solution of a trigonometric equation we mean general expressions defining all values of the variable which will satisfy the given equation. This will mean in many problems that a number n representing any integer must be used.

There are a number of methods for solving trigonometric equations. It is often possible to express all trigonometric functions involved in terms of a single function, solve the resulting equations for this function, and then write the angles associated with the values of the function. Another method consists in transferring all terms of the given equation to the left-hand member, factoring the resulting left-hand member, equating the factors to zero, and solving each equation thus obtained. The following examples will illustrate these methods of procedure.

Example 1. Solve $2\cos^2 x + \sin x - 1 = 0$.

Solution. Replacing $\cos^2 x$ by $1 - \sin^2 x$ and simplifying slightly, we obtain

$$2(\sin x)^2 - (\sin x)^1 - 1 = 0.$$

Evidently this is a quadratic equation with $\sin x$ appearing as the

unknown. Solving it by formula,* we obtain

$$\sin x = \frac{-(-1) \pm \sqrt{1+8}}{4} = 1 \text{ or } -\tfrac{1}{2}.$$

Hence $x = \sin^{-1} 1$ and $x = \sin^{-1}(-\frac{1}{2})$. Replacing these inverse functions by their general values, we get

$$x = \mathbf{90° + n360°}, \qquad x = \mathbf{210° + n360°}, \qquad x = \mathbf{330° + n360°}$$

or, in radians

$$x = \frac{\pi}{2} + 2n\pi, \qquad x = \tfrac{7}{6}\pi + 2n\pi, \qquad x = \frac{11\pi}{6} + 2n\pi.$$

Example 2. Solve $\sin 4\theta + \cos 2\theta = 0$.

Solution. Replacing $\sin 4\theta$ by $2 \sin 2\theta \cos 2\theta$ in the given equation and factoring, we obtain

$$\cos 2\theta\,(2 \sin 2\theta + 1) = 0.$$

Equating the factors to zero, we get

$$\cos 2\theta = 0, \qquad 2 \sin 2\theta + 1 = 0.$$

From $\cos 2\theta = 0$ we derive

$$2\theta = 90° + n360°, \quad \text{and} \quad 2\theta = 270° + n360°. \tag{a}$$

or

$$\theta = \mathbf{45° + n180°} \quad \text{and} \quad \theta = \mathbf{135° + n180°}.$$

From $2 \sin 2\theta + 1 = 0$, or $\sin 2\theta = -\frac{1}{2}$, we derive

$$2\theta = 210° + n360° \quad \text{and} \quad 2\theta = 330° + n360°,$$

or,

$$\theta = \mathbf{105° + n180°} \quad \text{and} \quad \theta = \mathbf{165° + n180°}.$$

EXERCISES

1. Find the values of x between 0° and 360° for which

(a) $\sin^2 x = \frac{1}{4}$.
(b) $\csc^2 x = 2$.
(c) $\tan^2 x - 3 = 0$.
(d) $\sec^2 x - 4 = 0$.
(e) $\tan 2x = 1$.
(f) $2 \sin 3x = 1$.

* The solution of $ay^2 + by + c = 0$ is $y = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

2. Find the values of the unknown between 0° and 360° for which

(a) $2\sin^2 x + 3\cos x = 0$.
(b) $\cos^2 \alpha - \sin^2 \alpha = \frac{1}{2}$.
(c) $2\sqrt{3}\cos^2 \alpha = \sin \alpha$.
(d) $\sin^2 y - 2\cos y + \frac{1}{4} = 0$.
(e) $4\sec^2 y - 7\tan^2 y = 3$.
(f) $\tan B + \cot B = 2$.
(g) $\sin x + \cos x = 0$.

3. Find, in radians, all angles between 0 and 2π that satisfy the following equations:

(a) $(\tan x + 1)(\sqrt{3}\cot x - 1) = 0$.
(b) $(2\cos x + 1)(\sin x - 1) = 0$.
(c) $(4\cos^2 \theta - 3)(\csc \theta + 2) = 0$.
(d) $2\cot \theta \sin \theta + \cot \theta = 0$.

4. For each of the following equations, find all values of the unknown that satisfy it:

(a) $2\sin^2 x + \cos x - 1 = 0$.
(b) $2\cos^2 \theta + 5\sin \theta - 4 = 0$.
(c) $\cos^2 x + 2\sin x + 2 = 0$.
(d) $2\cos^2 2\alpha + \sin 2\alpha - 1 = 0$.
(e) $2\sec^2 \theta - \tan \theta = 5$.
(f) $2\csc^2 \phi - 5\cot \phi + 1 = 0$.
(g) $4\sec^2 2A = 8 + 15\tan 2A$.
(h) $\cos^2 x(4\cos^2 x - 1) = 0$.
(i) $4\cos 2x + 3\cos x = 1$.
(j) $\cot^2 \theta - 3\csc \theta + 3 = 0$.
(k) $\tan^2 x + \cot^2 x - 2 = 0$.
(l) $\tan x + 3\cot x = 4$.
(m) $2\tan^2 x + 3\sec x = 0$.
(n) $\cos \theta + 6\sin \theta = 2$.
(o) $\sin x + \cos x = 1$.
(p) $\csc x \cot x = 2\sqrt{3}$.
(q) $\sin x \cos x + \frac{1}{4} = 0$.
(r) $\cos 2x + \cos x = -1$.
(s) $\tan 2\theta \tan \theta = 1$.

5. Solve for the unknown:

(a) $2\sin \theta = \tan \theta$.
(b) $\sin 2x - \cos x = 0$.
(c) $4\sin^4 \theta = 3\sin^2 \theta$.
(d) $\sin 2\alpha + \cos \alpha = 0$.
(e) $\sin 4x = \cos 2x$.
(f) $\sin 2\theta = \sqrt{3}\sin \theta$.
(g) $\sin^2 4\alpha = \sin^2 2\alpha$.
(h) $2\sin 4\theta + \sin 2\theta = 0$.
(i) $\cos 4\alpha = \cos 2\alpha$.

6. Find the abscissas of the points where each of the following curves crosses the x-axis:

(a) $y = 2\sin x - \sin 2x$.
(b) $y = \cos 2x - \cos x$.
(c) $y = \cos 2x - \cos^2 x$.
(d) $y = \tan(x + 45°) - 1 + \sin 2x$.

7. Plot each of the following pairs of curves on the same set of axes and find their points of intersection for values of x between 0° and 360°.

(a) $y = \sin 2x$, $y = \sin x$.
(b) $y = \cos 2x$, $y = \cos x$.

(c) $y = \sec x,$ $\quad y = 2 \cos x.$
(d) $y = \tan x,$ $\quad y = 3 \cot x.$
(e) $y = 2 \sin x,$ $\quad y = \tan x.$
(f) $y = \tan^2 x,$ $\quad y = 2 - \cot^2 x.$

79. Special types of trigonometric equation. The solution of certain types of trigonometric equation may often be obtained by transforming the equation or by some other device. The following examples will illustrate two methods.

Example 1. Solve $\cos 6x = \cos 4x$ for x.

Solution. Write the given equation in the form

$$\cos 6x - \cos 4x = 0,$$

and apply the conversion formula

$$\cos A - \cos B = -2 \sin \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B)$$

to the left-hand member and get

$$-2 \sin \tfrac{1}{2}(6x + 4x) \sin \tfrac{1}{2}(6x - 4x) = 0$$

or

$$-2 \sin 5x \sin x = 0.$$

Equate the factors $\sin 5x$ and $\sin x$ to zero and obtain

$$\sin 5x = 0, \qquad \sin x = 0. \tag{a}$$

From the first of equations (a) we get

$$5x = 0° + n360°, \qquad \text{and} \qquad 5x = 180° + n360°$$

or

$$x = \mathbf{n72°} \qquad \text{and} \qquad x = \mathbf{36° + n72°}.$$

From the second of equations (a) we get

$$x = \mathbf{0° + n360°} \qquad \text{and} \qquad x = \mathbf{180° + n360°}.$$

Example 2. Solve $\sin 9x = \cos 4x$ for x.

Solution. Write the given equation in the form

$$\sin 9x - \sin (90° - 4x) = 0,$$

and apply the conversion formula

$$\sin A - \sin B = 2 \cos \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B)$$

to the left-hand member and obtain

$$2 \cos \left(\tfrac{5}{2}x + 45°\right) \sin \left(\tfrac{13}{2}x - 45°\right) = 0.$$

Set the factors equal to zero and get

$$\cos \left(\tfrac{5}{2}x + 45°\right) = 0, \qquad \sin \left(\tfrac{13}{2}x - 45°\right) = 0. \qquad (a)$$

From the first of equations (a) we get

$$\tfrac{5}{2}x + 45° = 90° + n360°, \qquad \text{and} \qquad \tfrac{5}{2}x + 45° = 270° + n360°,$$

or

$$x = \mathbf{18° + \mathit{n}144°}, \qquad \text{and} \qquad x = \mathbf{90° + \mathit{n}144°}.$$

From the second of equations (a) we get

$$\tfrac{13}{2}x - 45° = 0° + n360° \qquad \text{and} \qquad \tfrac{13}{2}x - 45° = 180° + n360°$$

or

$$x = \frac{\mathbf{90° + \mathit{n}720°}}{\mathbf{13}} \qquad \text{and} \qquad x = \frac{\mathbf{450° + \mathit{n}720°}}{\mathbf{13}}.$$

In accordance with Exercise 4, §74, the complete answer could be written in the form

$$x = 18° + n72°, \qquad x = \frac{90° + 360°n}{13}.$$

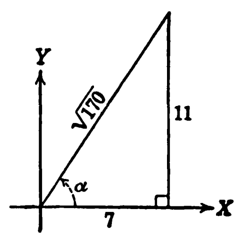

FIG. 13.

Example 3. Solve $7 \sin 3x - 11 \cos 3x = 12$ for x.

Solution. To solve this equation first transform the left-hand member into the sine of the difference of two angles. To do this let $\alpha = \tan^{-1} \frac{11}{7}$, and construct Fig. 13. Divide the given equation through by $\sqrt{170}$ to obtain

$$\frac{7}{\sqrt{170}} \sin 3x - \frac{11}{\sqrt{170}} \cos 3x = \frac{12}{\sqrt{170}}. \qquad (a)$$

In (a) replace $7/\sqrt{170}$ by $\cos \alpha$ and $11/\sqrt{170}$ by $\sin \alpha$, their values from Fig. 13, to get

$$\sin 3x \cos \alpha - \cos 3x \sin \alpha = \frac{12}{\sqrt{170}} \qquad (b)$$

or

$$\sin(3x - \alpha) = \frac{12}{\sqrt{170}}.$$

Use the slide rule or natural function table to obtain

$$\alpha = \tan^{-1}\tfrac{11}{7} = 57°32', \qquad \sin^{-1}\frac{12}{\sqrt{170}} = 66°59', \text{ and } 113°1'. \quad (c)$$

Use these angles to get

$$3x - 57°32' = 66°59' + n360°, \qquad (d)$$

$$3x - 57°32' = 113°1' + n360°. \qquad (e)$$

Solve (d) and (e) for x to obtain

$$x = \mathbf{41°30' + n120°}, \qquad x = \mathbf{56°51' + n120°}.$$

EXERCISES

1. Solve for the unknown:

(a) $\sin 3\theta - \sin 9\theta = 0$.
(b) $\cos 6\theta = \cos 2\theta$.
(c) $\sin 11x = \cos 7x$.
(d) $\sec 9x = \sec 5x$.
(e) $\tan 4x = \cot 6x$.
(f) $\sec 8x = \csc 10x$.
(g) $4\sin x + 3\cos x = 1$.
(h) $3\sin\theta - 4\cos\theta = 3$
(i) $12\cos\alpha + 5\sin\alpha = -6.5$.
(j) $5\cos\phi - 12\sin\phi = 3\frac{1}{4}$.
(k) $\cos 2x - 2\sin 2x = 2$.
(l) $12\sin 3\theta - 5\cos 3\theta = 5$.
(m) $\sin 4x - \sin 2x - \cos 3x = 0$.
(n) $\cos 5\theta + \cos 3\theta + \cos\theta = 0$.
(o) $\sin 4\theta = \sin 9\theta - \sin\theta$.
(p) $2\sin 3\theta\cos\theta - 2\sin\theta\cos 3\theta + 1 = 0$.
(q) $\tan 4\theta = \tan 10\theta$.
(r) $2\sin A\cos A - 2\cos A + \sin A - 1 = 0$.
(s) $3\sin\theta + \cos\theta = 2x, \quad \sin\theta + 2\cos\theta = x$.

2. Solve the equations

$$r\cos\phi\cos\theta = 2,$$
$$r\cos\phi\sin\theta = 3,$$
$$r\sin\phi = 5.$$

Hint. Divide the first equation by the second, member by member.

3. Solve the equation

$$\sin(\alpha + x) = m \sin x,$$

for $\tan(x + \frac{1}{2}\alpha)$.

4. Solve the equations

$$m \sin(\theta + x) = a,$$
$$m \sin(\phi + x) = b,$$

for m and x, the other four quantities, θ, ϕ, a, b, being known.

Hint. Expand $\sin(\varphi + x)$, $\sin(\theta + x)$ and solve for $\sin x$ and $\cos x$.

5. Solve $m \cos(\theta + x) = a$, and $m \sin(\phi + x) = b$, for $m \sin x$ and $m \cos x$.

6. Solve $m \cos(\theta + x) = a$, and $m \cos(\phi - x) = b$, for $m \sin x$ and $m \cos x$.

7. Solve the equations

$$x \cos\alpha + y \sin\alpha = m,$$
$$x \sin\alpha - y \cos\alpha = n,$$

for x and y.

80. Equations involving inverse functions. The following example will furnish an illustration of the method of solving an equation involving inverse trigonometric functions. In solving problems of this type, we shall understand that principal values only are to be considered.

Example. Solve the equation

$$\cos^{-1} x + \sin^{-1} 2x = -\tan^{-1} \frac{\sqrt{8x^4 - 5x^2 + 1}}{x\sqrt{5 - 8x^2}}. \qquad (a)$$

Solution. If we let

$$\cos^{-1} x = \alpha, \qquad \sin^{-1} 2x = \beta, \qquad \tan^{-1} \frac{\sqrt{8x^4 - 5x^2 + 1}}{x\sqrt{5 - 8x^2}} = \gamma$$

and if we substitute these values in (a), we have

$$\alpha + \beta = -\gamma.$$

Taking the cosine of both members of this equation, we obtain

$$\cos(\alpha + \beta) = \cos -\gamma,$$

or

$$\cos \alpha \cos \beta - \sin \alpha \sin \beta = \cos \gamma. \qquad (b)$$

The three triangles exhibiting α, β, and γ are shown in Fig. 14. Reading direct from the triangles the values of the functions

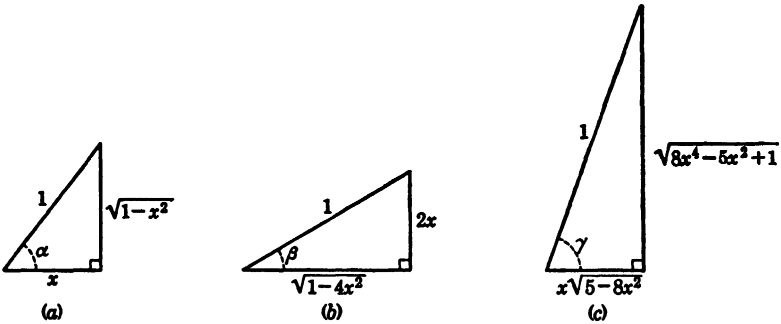

FIG. 14.

involved in (*b*), and substituting these values in (*b*), we obtain

$$x\sqrt{1-4x^2} - \sqrt{1-x^2}(2x) = x\sqrt{5-8x^2}.$$

Solving this equation, we get

$$x = 0, \qquad x = \pm\tfrac{1}{2}, \qquad \text{and} \qquad x = \pm 1.$$

Substituting these values of x in the original equation, we find that only $x = -\frac{1}{2}$ satisfies it for principal values. Hence the solution is

$$x = -\frac{1}{2}.$$

EXERCISES

1. Verify that $x = \frac{1}{2}$ does not satisfy (*a*) of the foregoing example if principal values only are considered.

2. Solve the following equations for the unknown, using principal values only:

(*a*) $\sin^{-1} y + \sin^{-1} 2y = \dfrac{\pi}{2}$.

(*b*) $\tan^{-1} 2x + \tan^{-1} 3x = \dfrac{3\pi}{4}$.

(*c*) $\tan(\sin^{-1}\sqrt{1-x^2}) - \sin(\tan^{-1} 2) = 0$.

(*d*) $\tan^{-1} y = \sin^{-1} a + \cos^{-1} b, 1 > b > 0$ and, numerically, $b > a$.

(*e*) $2 \tan^{-1} y = \frac{\pi}{2} - \cot^{-1} 3y.$

(*f*) $2 \tan^{-1} \frac{1}{2} + \cos^{-1} \frac{3}{5} = \sin^{-1} \frac{1}{x}.$

(*g*) $\tan^{-1} x + \tan^{-1} (1 - x) = 2 \tan^{-1} \sqrt{x(1 - x)}.$

(*h*) $\sin^{-1} \frac{5}{x} + \sin^{-1} \frac{12}{x} = \frac{\pi}{2}.$

(*i*) $\sin^{-1} \frac{m}{x} + \sin^{-1} \frac{n}{x} = \frac{\pi}{2}.$

(*j*) $\sin^{-1} x = 2 \cos^{-1} x.$

(*k*) $\sin^{-1} x = 2 \tan^{-1} x.$

(*l*) $\tan^{-1} x = 2 \sin^{-1} x.$

(*m*) $\cot^{-1} x - \cot^{-1} (x + 2) = 15°.$

(*n*) $\left\{\begin{matrix} a \sin^{-1} x + b \cos^{-1} y = \alpha \\ a \cos^{-1} x - b \sin^{-1} y = \beta \end{matrix}\right\}.$

81. MISCELLANEOUS EXERCISES

1. Find the values of the following:

(*a*) $\sin (\tan^{-1} \frac{5}{12}).$

(*b*) $\sin (\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}).$

(*c*) $\tan (2 \tan^{-1} a).$

(*d*) $\cot (2 \text{ arc } \sin \frac{3}{5}).$

(*e*) $\cos (2 \text{ arc } \cos a).$

(*f*) $\cos (2 \text{ arc } \tan a).$

(*g*) $\text{arc } \tan \frac{1}{\sqrt{3}}.$

(*h*) $\cot^{-1} (\pm 1).$

2. Prove the following using principal values:

(*a*) $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi.$

(*b*) $\text{arc } \cos \frac{4}{5} + \text{arc } \tan \frac{3}{5} = \text{arc } \tan \frac{27}{11}.$

(*c*) $2 \tan^{-1} \frac{2}{3} = \tan^{-1} \frac{12}{5}.$

(*d*) $\sin^{-1} \frac{3}{5} + \sin^{-1} \frac{8}{17} = \sin^{-1} \frac{77}{85}.$

(*e*) $\text{arc } \cos \frac{4}{5} + \text{arc } \cos \frac{12}{13} = \text{arc } \cos \frac{33}{65}.$

(*f*) $\text{arc } \tan \frac{1}{7} + \text{arc } \tan \frac{1}{13} = \text{arc } \tan \frac{2}{9}.$

Solve the following equations:

3. (*a*) $\sin x = 3 \cos x.$

(*b*) $2 \cos x = \cos 2x.$

(*c*) $\tan x = \tan 2x.$

4. (*a*) $3 \cos^2 x + 5 \sin x - 1 = 0.$
(*b*) $3 \sin x \tan x - 5 \sec x + 7 = 0.$
(*c*) $\tan x + \sec^2 x - 3 = 0.$
(*d*) $\sin x + \cos 2x = 4 \sin^2 x - 1.$
(*e*) $\sin (2x - 180°) = \cos x.$
(*f*) $\cos^2 x + 2 \sin x = 0.$
(*g*) $\sec^2 x - 4 \tan x = 0.$
(*h*) $\sin^2 2x - \sin 2x - 2 = 0.$
(*i*) $\tan^2 \frac{x}{2} - \tan \frac{x}{2} - 2 = 0.$
(*j*) $\sin x \sin \frac{x}{2} = 1 - \cos x.$
(*k*) $\csc y + \cot y = \sqrt{3}.$
(*l*) $6 \sec^2 \alpha + \cot^2 \alpha = 11.$

5. (*a*) $\cot 5x = \cot 7x.$
(*b*) $\sec 3x = \csc 5x.$
(*c*) $\sin 3x - \sin x = \sin 5x.$

6. $\cos 5x + \cos 6x = \sin 5x + \sin 6x.$

7. (*a*) $4 \sin x + 3 \cos x = 3.$
(*b*) $5 \sin x = 4 \cos x + 4.$

8. (*a*) $\sin (60° - x) - \sin (60° + x) = \frac{\sqrt{3}}{2}.$
(*b*) $\sin (30° + x) - \cos (60° + x) = -\frac{\sqrt{3}}{2}.$
(*c*) $\tan (45° - x) + \cot (45° - x) = 4.$
(*d*) $\sec (x + 120°) + \sec (x - 120°) = 2.$
(*e*) $\csc^2 x(1 + \sin x \cot x) = 2.$

9. (*a*) $\sin x + \sin 2x + \sin 3x = 0.$
(*b*) $\tan x + \tan 2x + \tan 3x = 0.$
(*c*) $\sin 4x - \cos 3x = \sin 2x.$

10. (*a*) If $x = a \cos \varphi$, $y = b \sin \varphi$, prove that $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

Hint. Solve for $\sin \varphi$ and $\cos \varphi$ and then use $\sin^2 \varphi + \cos^2 \varphi = 1.$

(*b*) If $x = a \sec \varphi$, $y = a \tan \varphi$, prove that $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

(*c*) From $x = a \cos^3 \varphi$, $y = a \sin^3 \varphi$, deduce $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$

(*d*) If $x = a + b \cos \varphi$, $y = c + d \sin \varphi$, find a relation between x and y.

(*e*) From $x = a \tan^3 \varphi$, $y = b \sec^3 \varphi$ deduce a relation between x and y.

(*f*) If $a \sin \theta + b \cos \theta = h$, $a \cos \theta - b \sin \theta = k$, prove that $a^2 + b^2 = h^2 + k^2.$

11. Solve the following equations:

(a) $\tan^{-1} x + \tan^{-1} (1 - x) = \tan^{-1} (\frac{4}{3})$.

(b) $\text{arc tan } x + 2 \text{ arc cot } x = \dfrac{2\pi}{3}$.

(c) $\tan^{-1} \dfrac{x - 1}{x + 2} + \tan^{-1} \dfrac{x + 1}{x + 2} = \dfrac{\pi}{4}$.

(d) $\cos^{-1} \dfrac{x^2 - 1}{x^2 + 1} + \tan^{-1} \dfrac{2x}{x^2 - 1} = \dfrac{2\pi}{3}$.

(e) $\text{arc tan } \dfrac{x + 1}{x - 1} + \text{arc tan } \dfrac{x - 1}{x} = \text{arc tan } (-7)$.

(f) $\tan^{-1} (x + 1) + \tan^{-1} (x - 1) = \tan^{-1} \frac{8}{31}$.

(g) $\sin^{-1} x + \sin^{-1} 2x = \dfrac{\pi}{3}$.

(h) $\text{arc sin } \dfrac{5}{x} + \text{arc sin } \dfrac{12}{x} = \dfrac{\pi}{2}$.

12. Plot each of the following pairs of curves on the same set of axes, and find their points of intersection between 0° and 360°.

(a) $y = \sin x$,	$y = \tan x$.
(b) $y = 2 \sin x$,	$y = \tan 2x$.
(c) $y = \tan x$,	$y = 4 - 3 \cot x$.
(d) $y = \cos 2x$,	$y = -(1 + \cos x)$.

CHAPTER X

COMPLEX NUMBERS

82. Pure imaginary numbers. In algebra it was found necessary to extend the number system to include imaginary numbers. A *pure* imaginary number is the indicated square root of a negative number. Thus $\sqrt{-5}$ is a pure imaginary number.

It is customary to reduce a pure imaginary number to the form $b\sqrt{-1}$ where b is a real number, to substitute the letter i for $\sqrt{-1}$, and then to treat i as a literal algebraic quantity that obeys all the laws of algebra in addition to the law $i^2 = -1$. It follows that a power of i is equal to one of the following: $i, -1, -i, 1$. Thus

$$i = i, \quad i^2 = -1, \quad i^3 = i^2 i = (-1)i = -i, \quad i^4 = i^2 i^2 = (-1)(-1) = 1, \quad i^5 = i^4 i = i, \quad i^6 = i^4 i^2 = -1,$$
$$i^7 = i^4 i^3 = -i, \quad i^{47} = (i^4)^{11} i^3 = -i, \quad i^{78} = (i^4)^{19} i^2 = -1.$$

EXERCISES

1. Express each radical in terms of i and simplify, noting that

$$\sqrt{-P} = \sqrt{P}\sqrt{-1} = i\sqrt{P},$$

if P is real and positive.

(*a*) $\sqrt{-36}$. (*d*) $\sqrt{-\frac{5}{12}}$. (*g*) $\sqrt{-125x^4y^2}$.

(*b*) $\sqrt{-27}$. (*e*) $\sqrt{-16x^2}$. (*h*) $\sqrt{b^2 - 4ac}$, $4ac > b^2$.

(*c*) $\sqrt{-49}$. (*f*) $\sqrt{-\dfrac{8}{2x^2}}$.

2. Write the two square roots of each of the following quantities:

(*a*) -16. (*b*) $-9x^2$. (*c*) -13. (*d*) $-7a^4x^2$

3. Simplify

(*a*) i^{21}. (*c*) i^{66}. (*e*) i^{131}. (*g*) i^{403}.

(*b*) i^{456}. (*d*) $i^3 i^{19}$. (*f*) $i^{191} i^{13}$. (*h*) $\dfrac{i^2 i^9}{i^3}$.

83. Complex numbers. *A complex number is one having the form $a + bi$ where a and b represent real numbers and $i = \sqrt{-1}$;* bi is termed the *imaginary part.* Any real number may be considered as a complex number in which the coefficient b of i is zero.

Two complex numbers are said to be equal if their real parts are equal and their imaginary parts are equal. Thus $a + bi = c + di$ if $a = c$ and $b = d$. Conversely, if $a + bi = c + di$, then $a = c$ and $b = d$. It therefore follows in particular that, if $a + bi = 0$, then $a = 0$ and $b = 0$.

In what follows we shall find it convenient to use the term *conjugate complex number. Two complex numbers that differ only in the signs of their pure imaginary parts are called conjugate complex numbers.* Thus $(2 + 3i)$ and $(2 - 3i)$ are conjugate.

84. Operations involving complex numbers. Since i obeys all the laws of algebra and since a and b are real numbers, we may operate with the complex number $a + bi$ in the usual way. In adding (and subtracting) complex numbers, it is necessary to *add* (or *subtract*) *the real parts and the imaginary parts separately.* Thus

$$(4 + 6i) + (5 - 7i) = [4 + 5 + (6 - 7)i] = 9 - i,$$
$$(7 - 2i) - (9 + 4i) = [7 - 9 - (2 + 4)i] = -2 - 6i.$$

In performing a multiplication one should replace i^2 by -1 whenever i^2 occurs. Thus

$$(6 - 5i)(9 + 2i) = 54 + 12i - 45i - 10i^2 = \mathbf{64 - 33i}.$$

The quotient of two complex numbers can be obtained in the form $a + bi$ by multiplying both numerator and denominator by the conjugate of the denominator. Thus

$$\frac{4 - 7i}{6 + i} = \frac{(4 - 7i)(6 - i)}{(6 + i)(6 - i)} = \frac{24 - 4i - 42i + 7i^2}{36 - i^2}$$
$$= \frac{(24 - 7) - 46i}{37} = \frac{\mathbf{17}}{\mathbf{37}} - \frac{\mathbf{46}}{\mathbf{37}}\mathbf{i}.$$

EXERCISES

1. Find real values for x and y if

(*a*) $x + yi = 2 - 3i$.
(*b*) $3x - 2yi = 5 + 7i$.
(*c*) $(3x - 2) - (4 - y)i = 0$.
(*d*) $2x - 4yi = 6 - 2xi$.
(*e*) $7x + 6y + 2xi - 3yi + 9 = x + yi - y + 3 - 2i$.

2. Write the conjugate of each of the following complex numbers:

(a) $7 + 2i$. (b) $x - yi$. (c) $3i$. (d) 14.

3. Perform the indicated operations.

(a) $(2 - 5i) + (3 + 4i)$.
(b) $(7 - 5i) - (11 - 13i)$.
(c) $(2 + 3i) + (4 - 6i)$.
(d) $(2 + 3i) + (1 + i)$.
(e) $(3 - 5i) + (3 + 5i)$.
(f) $(6 + 0i) - (3 - 7i)$.
(g) $(4 + 2i) + (-2 - 4i)$.
(h) $(3 + 4i) - (3 - 4i)$.

4. Show that the sum of two conjugate complex numbers is a real number and that the difference is a pure imaginary number.

5. Perform the indicated operations.

(a) $(3 + 5i)(6 - 2i)$.
(b) $(4i - 6)^2$.
(c) $(2 - 4i)(-3 + 2i)$.
(d) $(7 - 4i)(7 + 4i)$.
(e) $i(2 - 5i)$.
(f) $(7 - i)(1 + i)(1 - 4i)$.

6. Show that the product of two conjugate complex numbers is a real number.

7. Reduce the following quotients to the form $a + bi$.

(a) $\dfrac{4 - 7i}{9 + 2i}$.
(b) $\dfrac{3 + i}{2 + i}$.
(c) $\dfrac{2 + i}{(3 - 2i)(1 + i)}$.
(d) $\dfrac{1}{5 - 4i}$.
(e) $\dfrac{5 + 4i}{i}$.
(f) $\dfrac{i}{3 - 4i}$.
(g) $\dfrac{(3 - 4i)}{(2 + i)(2 - 3i)}$.
(h) $\dfrac{(3 + 7i)(8 + 6i)}{(5 - 7i)(4 + 6i)}$.
(i) $\dfrac{(4 - 5i)}{i(6 - 8i)}$.

85. Geometrical representation of complex numbers. In §19 it was pointed out that all real numbers may be represented by points on a straight line. Since complex numbers depend on two real numbers, it is necessary to use two dimensions in order to represent a complex number graphically. Accordingly, using the system of rectangular coordinates explained in Chap. III, we may represent the complex number $x + yi$ by a point P whose coordinates are x and y. The x-axis is called the *axis of reals*, and the y-axis the *axis of imaginaries*. Evidently a real

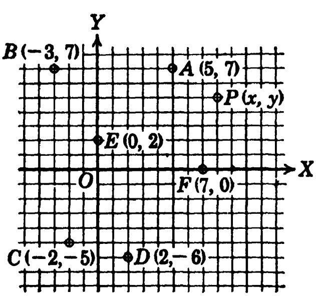

FIG. 1.

number is plotted on the *axis of reals* and a pure imaginary number is plotted on the *axis of imaginaries.*

For example in Fig. 1, point P (x, y) represents the complex number $x + yi$; point $A(5, 7)$ represents $5 + 7i$; $B(-3, 7)$ represents $-3 + 7i$; $C(-2, -5)$ represents $-2 - 5i$; $D(2, -6)$ represents $2 - 6i$; $E(0, 2)$ represents the pure imaginary number $2i$ and, $F(7, 0)$ represents the real number 7.

EXERCISES

1. Represent graphically the following complex numbers:

(*a*) $3 - 2i$. (*b*) $-4 + i$. (*c*) $6i$. (*d*) 0. (*e*) $1 - \sqrt{-2}$.

2. Plot the conjugates of the numbers in Exercise 1.

3. Find the sum of the numbers in Exercise 1 and plot the result.

86. Polar form of a complex number. Complex numbers can be represented in another form involving trigonometric functions. In Fig. 2 let $P(x, y)$ represent the complex number $x + yi$. Connect P with the origin of coordinates; denote by r the length of the connecting line OP and by θ the angle that OP makes with the *axis of reals*. Then $P(x, y)$ is determined by r and θ.

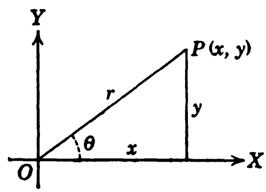

FIG. 2.

From the figure we have

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Replacing x and y in $x + yi$ by these values, we obtain

$$\boldsymbol{x + yi = r(\cos\theta + i\sin\theta).}$$

The form $r(\cos\theta + i\sin\theta)$ is called the *polar form* of a complex number. The angle θ is called the *amplitude* and the length r the *modulus*. Here r is positive and θ is any angle that is generated by the positive half of the x-axis when it is turned about the origin until its terminal position passes through $P(x, y)$. From this it appears that if α is one amplitude of a complex number, the other permissible amplitudes are $(\alpha + 2\pi n)$, where n is any integer.

In finding the values of r and θ it is well to solve* the right triangle of which the lengths x and y are the legs (see Fig. 2).

For convenience some writers use the notation **cis θ** as an abbreviation for $\cos\theta + i\sin\theta$. We shall use this notation occasionally.

Example. Write the complex number $-4 + 3i$ in the polar form.

Solution. We first plot $-4 + 3i$ and form the right triangle shown in Fig. 3. Solving this triangle in the usual way (§128, §127) we find that $r = 5$ and $\alpha = 36°52'$. The amplitude is found from the figure to be $\theta = 180° - \alpha = 143°8'$. Hence, using the notation cis θ for $\cos\theta + i\sin\theta$, we have

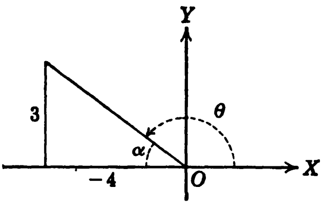

FIG. 3.

$$-4 + 3i = \mathbf{5\ cis\ 143°8'}.$$

If the slide rule is not used for solving the triangle, we may write

$$r = \sqrt{(-3)^2 + 4^2} = 5 \qquad \text{and} \qquad \theta = \tan^{-1}(-\tfrac{3}{4}) = 143°8'.$$

Evidently the amplitude may be taken as $(143°8' + n360°)$, where n is any integer.

EXERCISES

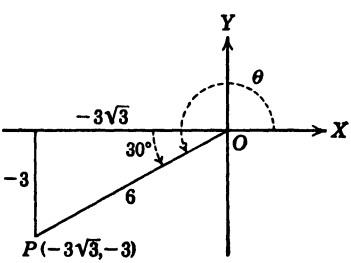

FIG. 4.

1. Write both forms of the complex number represented by point P of Fig. 4.

2. Write the polar form of the complex number represented by the point $P(1, 1)$.

* For the method of solving a right triangle by means of the slide rule, see §§127, 128.

3. Plot the following complex numbers and write them in the form $x + yi$:

(a) $2(\cos 30° + i \sin 30°)$.
(b) $3(\cos 60 + i \sin 60°)$.
(c) $2(\cos \frac{3}{4}\pi + i \sin \frac{3}{4}\pi)$.
(d) $4(\cos 180° + i \sin 180°)$.
(e) $11 \operatorname{cis} 210°$.
(f) $7 \operatorname{cis} 270°$.
(g) $6 \operatorname{cis} 300°$.
(h) $6 \operatorname{cis} 60°$.

4. Write the following complex numbers in the polar form:

(a) $1 - i$.
(b) $-2 - 3i$.
(c) $-2 + 3i$.
(d) $4 + 0i$.
(e) $0 + 4i$.
(f) 5.
(g) $7i$.
(h) $0.7 + 1.1i$.
(i) $3/(2i)$.
(j) $-i$.
(k) $7 - 5i$.
(l) $3.2 - 5.4i$.
(m) $-6.1 + 4.2i$.
(n) $-3.3 - 6.6i$.
(o) $7.1 - 4.4i$.

87. Multiplication of complex numbers in polar form. Multiplying the two complex numbers $r_1(\cos \alpha + i \sin \alpha)$ and $r_2(\cos \beta + i \sin \beta)$ in the usual way, we obtain

$$\begin{aligned} &r_1(\cos \alpha + i \sin \alpha) \cdot r_2(\cos \beta + i \sin \beta) \\ &= r_1 r_2(\cos \alpha \cos \beta + i \sin \alpha \cos \beta + i \cos \alpha \sin \beta - \sin \alpha \sin \beta) \\ &= r_1 r_2[(\cos \alpha \cos \beta - \sin \alpha \sin \beta) + i(\sin \alpha \cos \beta + \cos \alpha \sin \beta)]. \end{aligned}$$

This can be reduced, by using formulas (1) §52, to

$$r_1 r_2[\cos (\alpha + \beta) + i \sin (\alpha + \beta)].$$

Using the notation cis θ for $\cos \theta + i \sin \theta$, we may write

$$(r_1 \operatorname{cis} \alpha)(r_2 \operatorname{cis} \beta) = r_1 r_2 \operatorname{cis} (\alpha + \beta). \tag{1}$$

Or, stated in words,

The modulus of the product of two complex numbers is the product of their moduli, and the amplitude of the product is the sum of their amplitudes.

By using this italicized statement with the first two of three complex numbers we get

$$\begin{aligned} &[r_1(\cos \alpha_1 + i \sin \alpha_1) r_2(\cos \alpha_2 + i \sin \alpha_2)] r_3(\cos \alpha_3 + i \sin \alpha_3) \\ &\qquad = r_1 r_2[\cos (\alpha_1 + \alpha_2) + i \sin (\alpha_1 + \alpha_2)] r_3(\cos \alpha_3 + i \sin \alpha_3), \end{aligned}$$

and this last line is equal to

$$r_1 r_2 r_3[\cos (\alpha_1 + \alpha_2 + \alpha_3) + i \sin (\alpha_1 + \alpha_2 + \alpha_3)].$$

Continuing this process repeatedly for the product of n complex numbers, we should finally obtain

$$(r_1 r_2 \cdots r_n) \cos [(\alpha_1 + \alpha_2 + \cdots \alpha_n) + i \sin (\alpha_1 + \alpha_2 + \cdots \alpha_n)]$$

Using the notation cis θ for $\cos \theta + i \sin \theta$, we may write

$$(r_1 \operatorname{cis} \alpha)(r_2 \operatorname{cis} \alpha_2) \cdots (r_n \operatorname{cis} \alpha_n) = (r_1 r_2 \cdots r_n) \operatorname{cis} (\alpha_1 + \alpha_2 + \cdots + \alpha_n) \qquad (2)$$

or, stated in words:

The modulus of the product of n complex numbers is the product of their moduli, and the amplitude of the product is the sum of their amplitudes.

Example. Find the product of 3(cis 30°), 4(cis 150°), and 7(cis 72°).

Solution. The moduli of the given number are 4, 3, and 7. Hence in accordance with the theorem just stated the modulus of the product is

$$4 \times 3 \times 7 = 84.$$

The amplitudes of the given numbers are 30°, 150°, and 72°. Hence, in accordance with the theorem just stated, the amplitude of the product is

$$30° + 150° + 72° = 252°.$$

Therefore we have

$$(3 \operatorname{cis} 30°)(4 \operatorname{cis} 150°)(7 \operatorname{cis} 72°) = 84(\operatorname{cis} 252°).$$

88. The quotient of two complex numbers in polar form. To express the quotient $\dfrac{r_1(\cos \alpha + i \sin \alpha)}{r_2(\cos \beta + i \sin \beta)}$ in the polar form we first multiply both numerator and denominator by $\cos \beta - i \sin \beta$ and obtain

$$\frac{r_1(\cos \alpha + i \sin \alpha)(\cos \beta - i \sin \beta)}{r_2(\cos \beta + i \sin \beta)(\cos \beta - i \sin \beta)}$$

or

$$\frac{r_1}{r_2}\left[\frac{\cos \alpha \cos \beta + \sin \alpha \sin \beta + i(\sin \alpha \cos \beta - \cos \alpha \sin \beta)}{\cos^2 \beta + \sin^2 \beta}\right].$$

Using the subtraction formulas (10) and (11) of §53, we reduce this expression to

$$\frac{r_1}{r_2}[\cos(\alpha - \beta) + i \sin(\alpha - \beta)].$$

Using the notation cis θ for $\cos\theta + i \sin\theta$, we have

$$\frac{r_1 \text{ cis } \alpha}{r_2 \text{ cis } \beta} = \frac{r_1}{r_2} \text{ cis } (\alpha - \beta); \tag{3}$$

or, stated in words:

The modulus of the quotient of two complex numbers is the quotient of their moduli, and the amplitude of the quotient is the difference of their amplitudes.

Evidently multiplication and division are very simply performed when the numbers are in the polar form. If the numbers are in the rectangular form $a + bi$ and the amount of multiplication and division involved is extensive, the numbers should be changed to the polar form and then combined in accordance with the theorems just stated.

EXERCISES

In this set of exercises give your results in the $a + bi$ form.

1. Perform the indicated operations:

(*a*) $4(\cos 27° + i \sin 27°)5(\cos 34° + i \sin 34°)$.

(*b*) $7(\text{cis } 129°)4(\text{cis } 311°)$.

(*c*) $\dfrac{6 \text{ cis } 43°}{2 \text{ cis } 87°}$. (*d*) $\dfrac{7 \text{ cis } 143°}{5 \text{ cis } 17°}$.

2. Perform the indicated operations:

(*a*) $(\frac{1}{2}\sqrt{3} + \frac{1}{2}i)(\frac{1}{2} + \frac{1}{2}\sqrt{3}i)$. (*g*) $\dfrac{1 + i}{3 + 3\sqrt{3}i}$.

(*b*) $(\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i)(1 + i)$.

(*c*) $(1 - i)(\sqrt{2} + \sqrt{2}i)$. (*h*) $\dfrac{\sqrt{3} + i}{\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}i}$.

(*d*) $(1 + \sqrt{3}i)(-\frac{1}{2} + \frac{1}{2}\sqrt{3}i)$.

(*e*) $(-\frac{1}{2} - \frac{1}{2}\sqrt{3}i)(-\sqrt{2} - \sqrt{2}i)$. (*i*) $\dfrac{6(\cos 230° - i \sin 230°)}{2 + 2i}$.

(*f*) $(-2 + 2i)(3 - 3\sqrt{3}i)$. (*j*) $\dfrac{5(\cos 80° + i \sin 80°)}{2 - 2\sqrt{3}i}$.

3. Perform the indicated operations:

(a) $\dfrac{7(\text{cis } 30°)6(\text{cis } 45°)}{2 - 2i}$.

(b) $\dfrac{(\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}i)(1 - i)}{7 \text{ cis } 150°}$.

(c) $\dfrac{(\sqrt{2} - \sqrt{2}i)(3 - 3\sqrt{3}i)}{(\sqrt{2} + \sqrt{2}i)(\text{cis } 120°)}$.

(d) $(1 + i)(\sqrt{2} - \sqrt{2}i)^2(3 + 3\sqrt{3}i)3 \text{ cis } 225°$.

4. Perform the indicated operations.

(a) $\dfrac{(5 \text{ cis } 32°)^5(4 \text{ cis } 40°)^4}{(20 \text{ cis } 10°)^4}$.

(b) $\dfrac{(5.2 - 7.1i)(6.4 + 5.2i)}{8.3 + 4.6i}$.

(c) $7(\text{cis } 330°)6(\text{cis } 1764°)$.

89. Powers and roots of complex numbers. De Moivre's theorem. If, in (2), all the values of r be taken equal to unity and all the angles equal to θ, we obtain $(\text{cis } \theta)^n = \text{cis } n\theta$, or

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta. \tag{4}$$

This relation is known as De Moivre's theorem. Although we have proved it only when n is an integer, it is true for all real values of n.

Since the sine and the cosine of an angle are unchanged when the angle is changed by any multiple of 360°, formula (4) holds true when θ is replaced by

$$\theta + 2k\pi, \text{ or } \theta + k\,360°, \ k \text{ is an integer.} \tag{5}$$

When n is an integer the addition of k 360° to θ gives rise to nothing new; but when n is fractional a number of values of cis $(n\theta + kn360°)$ may be found by assigning different values to k. Thus, to find the nth root of $x + yi$ where n is an integer, write

$$\begin{aligned}(x + yi)^{\frac{1}{n}} &= \{r[\cos(\theta + k360°) + i \sin(\theta + k\,360°)]\}^{\frac{1}{n}} \\ &= r^{\frac{1}{n}}\left[\cos\left(\frac{\theta}{n} + \frac{k\,360°}{n}\right) + i \sin\left(\frac{\theta}{n} + \frac{k\,360°}{n}\right)\right.\end{aligned}$$

or, using the notation cis θ for $\cos\theta + i\sin\theta$

$$(x + yi)^{\frac{1}{n}} = r^{\frac{1}{n}} \text{ cis } \left(\frac{\theta}{n} + \frac{k\,360^\circ}{n}\right), \tag{6}$$

where k may be any integer. By letting k assume in succession the values $0, 1, 2, \cdots, n-1$, we obtain from (6), n distinct results, that is, n distinct complex numbers, each one of which is an nth root of $x + yi$. If k be assigned an additional value, the amplitude of the resulting number will differ from the amplitude of one of the roots just found by a multiple of 2π; that is, this new number will be equivalent to one of the roots already found. Also it can easily be proved that a complex number cannot have more than n different nth roots. Therefore, if n is an integer, every complex number different from zero has n and only n distinct nth roots.

Example 1. Find the three cube roots of -8.

Solution. Expressing the number $-8 + 0i$ in the polar form and using the general value of the amplitude, we obtain

$$-8 = 8 \text{ cis } (180^\circ + k360^\circ) \tag{a}$$

Extracting the cube root of (a) and using (6), we obtain

$$(-8)^{\frac{1}{3}} = 8^{\frac{1}{3}} \text{ cis } \left(\frac{180^\circ}{3} + k\,\frac{360^\circ}{3}\right).$$

Giving k the values 0, 1, 2 in succession, we obtain

$$\begin{aligned} 2 \text{ cis } \frac{\pi}{3} &= 2(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}i) = 1 + i\sqrt{3}, \\ 2 \text{ cis } \pi &= 2(-1 + 0i) = -2, \\ 2 \text{ cis } \frac{5\pi}{3} &= 2(\tfrac{1}{2} - \tfrac{1}{2}\sqrt{3}i) = 1 - i\sqrt{3}. \end{aligned}$$

Example 2. Find the four fourth roots of $-3 + 3\sqrt{3}i$.

Solution. Plotting the given number and solving the triangle exhibited in Fig. 5, we write direct from the figure the polar form of $(-3 + 3\sqrt{3}i)$, using the general value of the amplitude. This gives

$$-3 + 3\sqrt{3}i = 6 \text{ cis } (120^\circ + k360^\circ).$$

Extracting the fourth root and using (6), we obtain

$$(-3 + 3\sqrt{3}i)^{\frac{1}{4}} = 6^{\frac{1}{4}} \text{ cis } \left(\frac{120^\circ}{4} + k\frac{360^\circ}{4}\right)$$
$$= 1.565 \text{ cis } (30^\circ + k90^\circ). \qquad (a)$$

Assigning to k in (a) the values 0, 1, 2, 3, we obtain as the roots of $-3 + 3\sqrt{3}i$

1.565 cis 30°, 1.565 cis 120°, 1.565 cis 210°, 1.565 cis 300°

or in the $a + bi$ form

$\mathbf{1.355 + 0.782}\boldsymbol{i}$, $\mathbf{-0.782 + 1.355}\boldsymbol{i}$, $\mathbf{-1.355 - 0.782}\boldsymbol{i}$, $\mathbf{0.782 - 1.355}\boldsymbol{i}$

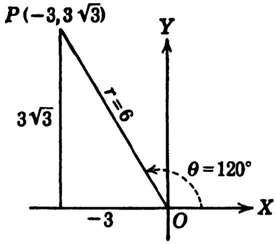

FIG. 5.

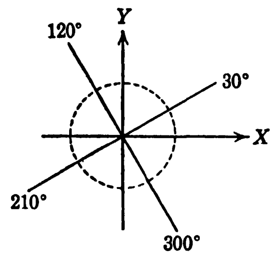

FIG. 6.

Since the moduli of the roots are equal, the points representing these roots will be on the circumference of a circle (see Fig. 6) having its radius equal to the common modulus of the roots and having its center at the origin. Since the amplitudes of any pair of successive roots differ by $360^\circ/n$, the points representing the roots are equally spaced along the circumference of the circle. Hence, after one root is located, it is easy to plot the remaining roots and to express them from the graph in the polar form.

EXERCISES

1. Find the values of each of the following numbers giving the results in polar form:

(a) $[2 \text{ cis } 120^\circ]^4$.
(b) $[4 \text{ cis } \frac{4}{5}\pi]^7$.
(c) $(\text{cis } 10^\circ)^3$.
(d) $(\frac{1}{2} + \frac{1}{2}\sqrt{3}i)^3$.
(e) $(3 - 3i)^5$.
(f) $(1 + i)^{-4}$.

2. Find the indicated roots, giving the results in polar form:

(a) $(10 - 6i)^{\frac{1}{2}}$.
(b) $(\frac{1}{2} - \frac{1}{2}\sqrt{3}i)^{\frac{1}{3}}$.
(c) $i^{\frac{1}{3}}$.
(d) $(-1)^{\frac{1}{3}}$.
(e) $(5.6 - 7.2i)^{\frac{1}{4}}$.
(f) $[14(\text{cis } 45^\circ + k360^\circ)]^{\frac{1}{5}}$.
(g) $[\text{cis } (\pi + 2k\pi)]^{\frac{1}{6}}$.

3. Solve the following equations:

(a) $x^3 + 1 = 0$.
(b) $x^5 = -32$.
(c) $x^3 = -i$.
(d) $x^6 - 2x^3 - 35 = 0$.
(e) $x^7 - x^4 + x^3 - 1 = 0$.

4. Derive the formula for $\cos 2\theta$ and $\sin 2\theta$ by expanding the left-hand member of $(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$, and then equating the real parts and the imaginary parts of the two members.

5. Using the formula $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ and giving n appropriate values, derive formulas for $\cos 3\theta$,* $\sin 3\theta$, $\cos 5\theta$, and $\sin 5\theta$.

Hint. Letting $n = 3$, we have

$$[\cos\theta + i\sin\theta]^3 = \cos 3\theta + i\sin 3\theta$$

or, expanding the left-hand member,

$$\cos^3\theta + i\,3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta + i\sin^3\theta = \cos 3\theta + i\sin 3\theta$$

or

$$(\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta + \sin^3\theta) = \cos 3\theta + i\sin 3\theta.$$

Now equate the real part of the left-hand member of the above equation to the real part of the right-hand member to obtain the formula for $\cos 3\theta$.

90. Exponential forms of a complex number. In higher mathematics we find justification for the equation

$$r(\cos\theta + i\sin\theta) = re^{i\theta}, \tag{7}$$

where θ is expressed in radians and $e (= 2.71828$, approximately) is the base of the system of natural logarithms. Thus we have another form in which to write a complex number.

From (7) we write

$$\cos\theta + i\sin\theta = e^{i\theta},$$
$$\cos\theta - i\sin\theta = e^{-i\theta}.$$

Solving these simultaneously for $\cos\theta$ and $\sin\theta$, we obtain

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{8}$$

* This formula may be used to obtain an elegant solution of the cubic equation.

These relations were stated by Euler in 1743. Taking them as fundamental definitions and further defining $\tan\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$ by the equations

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta},$$

we may develop, independent of any geometric meaning attached to the functions or their arguments, all the formulas of trigonometry. It is also interesting to observe that the theorems relating to multiplication, division, involution, and evolution of complex numbers are easily proved by using this exponential form.

EXERCISES

1. Use (7) to evaluate $e^{i\pi}$, $e^{i\frac{\pi}{2}}$, e^{i2}, $e^{-i\frac{3\pi}{4}}$.

2. Use (8) to find $\cos 2i$ and $\sin 2i$.

3. Prove that $\cos(i\log_e k) = \dfrac{k^2+1}{2k}$.

4. Assume that (7) holds true, and use it to prove De Moivre's theorem.

5. Use (8) to prove

(a) $\cos^2\theta + \sin^2\theta = 1$.
(b) $\cos(A+B) = \cos A\cos B - \sin A\sin B$.

6. Use (7) to evaluate $e^{(2k+1)\pi i}$, where k is an integer; then show that

$$\log_e(-1) = (2k+1)\pi i.$$

91. The hyperbolic functions. A class of functions very useful in many fields is analogous to the trigonometric functions. The function $\cos i\theta$ is called the hyperbolic cosine of θ and is written $\cosh\theta$. Similarly $-i\sin i\theta$ is called the hyperbolic sine of θ and is written $\sinh\theta$. Using (7) with θ replaced by iA, $\cos iA$ by $\cosh A$, and $-i\sin iA$ by $\sinh A$, we have

$$\cosh A = \frac{e^A + e^{-A}}{2}, \qquad \sinh A = \frac{e^A - e^{-A}}{2}. \tag{9}$$

Corresponding to other trigonometric functions, there are four other hyperbolic functions defined as

$$\left.\begin{aligned} \tanh A &= \frac{\sinh A}{\cosh A}, \qquad \coth A = \frac{\cosh A}{\sinh A} = \frac{1}{\tanh A} \\ \operatorname{sech} A &= \frac{1}{\cosh A}, \qquad \operatorname{csch} A = \frac{1}{\sinh A}, \end{aligned}\right\} \tag{10}$$

and named by prefixing the word hyperbolic to the names of their trigonometric counterparts.

Example. Using the definitions (9), verify

$$\cosh^2 A - \sinh^2 A = 1. \tag{a}$$

Solution. From (9)

$$\cosh^2 A = \left(\frac{e^A + e^{-A}}{2}\right)^2 = \tfrac{1}{4}e^{2A} + \tfrac{1}{2} + \tfrac{1}{4}e^{-2A}. \tag{b}$$

$$\sinh^2 A = \left(\frac{e^A - e^{-A}}{2}\right)^2 = \tfrac{1}{4}e^{2A} - \tfrac{1}{2} + \tfrac{1}{4}e^{-2A}. \tag{c}$$

Subtracting (c) from (b), member by member, we obtain

$$\cosh^2 A - \sinh^2 A = 1.$$

EXERCISES

1. Find cosh 0, sinh 0, cosh 1, sinh 1.

2. Prove that $\cosh x$ is always positive and greater than 1 if x is a real number.

3. Prove that the value of $\tanh x$ is numerically less than 1 for all real values of x. What other hyperbolic function is always less than 1?

4. Using definitions (9) and (10), show that
$\sinh(-x) = -\sinh x$, $\cosh(-x) = \cosh x$, $\tanh(-x) = -\tanh x$.

5. Show that $\cosh x + \sinh x = e^x$, $\cosh x - \sinh x = e^{-x}$.

6. Using definitions (9) and (10), verify the following identities:

(a) $\tanh^2 x + \operatorname{sech}^2 x = 1$.
(b) $\coth^2 x - \operatorname{csch}^2 x = 1$.
(c) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$.
(d) $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$.
(e) $\sinh x + \sinh y = 2 \sinh\left(\frac{x+y}{2}\right) \cosh\left(\frac{x-y}{2}\right)$.
(f) $\sinh x - \sinh y = 2 \cosh\left(\frac{x+y}{2}\right) \sinh\left(\frac{x-y}{2}\right)$.
(g) $\cosh x + \cosh y = 2 \cosh\left(\frac{x+y}{2}\right) \cosh\left(\frac{x-y}{2}\right)$.
(h) $\cosh x - \cosh y = 2 \sinh \frac{x+y}{2} \sinh \frac{x-y}{2}$.

7. In the equation

$$x = \sinh y, \tag{a}$$

y is a number whose hyperbolic sine is x. We express this by writing

$$y = \sinh^{-1} x,$$

and define the symbol *$\sinh^{-1} x$* to be the *number whose hyperbolic sine is x.*

In equation (a) replace $\sinh y$ by $\frac{e^y - e^{-y}}{2}$, solve the result for y, and show that

$$\sinh^{-1} x = \log (x + \sqrt{x^2 + 1}).$$

8. The symbol *$\cosh^{-1} x$* means the number whose hyperbolic cosine is x and is read the *number whose hyperbolic cosine is x.* The $\tanh^{-1} x$, $\coth^{-1} x$, $\operatorname{sech}^{-1} x$, $\operatorname{csch}^{-1} x$ are defined and read in an analogous manner.

Proceed in a manner similar to that of problem (7) and show that

$$\cosh^{-1} x = \pm\log (x + \sqrt{x^2 + 1}),$$

$$\tanh^{-1} x = \frac{1}{2} \log \frac{1 + x}{1 - x}.$$

9. Show that

$$\sinh^{-1} x = \operatorname{csch}^{-1} \frac{1}{x},$$

$$\cosh^{-1} x = \operatorname{sech}^{-1} \frac{1}{x},$$

$$\tanh^{-1} x = \coth^{-1} \frac{1}{x}.$$

92. MISCELLANEOUS EXERCISES

1. Plot the following complex numbers and write them in the form $x + yi$:

(a) 3 cis 45°.
(b) 4 cis 150°.
(c) 5 cis 300°.
(d) 7 cis 90°.
(e) 5 cis 58°.
(f) 8 cis 124°.
(g) 6 cis 324°.
(h) 2 cis 220°20′.

2. Write the following complex numbers in the polar form:

(a) $2 + 2i$.
(b) $3 - 3i$.
(c) $-3 + i$.
(d) $2 - 3i$.
(e) $-3 - 4i$.
(f) $5 - 6i$.
(g) $6 - 2i$.
(h) $-3.2 - 2.4i$.
(i) $-4.2 + 1.4i$.

3. Perform the indicated operations:

(a) $\dfrac{(7 \text{ cis } 45°)(8 \text{ cis } 300°)}{4 \text{ cis } 135°}$.

(b) $\dfrac{(8 \text{ cis } 47°)(3 \text{ cis } 200°)}{-2 + 7i}$.

(c) $\dfrac{(2 - 6i)(-3 + i)}{(7 - 6i)(4 - i)}$.

(d) $\dfrac{(8.2 - 3.4i)(7.1 + 3.8i)}{-6.3 - 3.1i}$.

4. Find the values of each of the following numbers, giving the results in polar form:

(a) $[2 \text{ cis } 45°]^5$.
(b) $[2.6 \text{ cis } 73°]^3$.
(c) $(1 - \sqrt{3}i)^4$.
(d) $(-3 + 4i)^5$.

5. Find the indicated roots, giving the results in polar form:

(a) $\sqrt{\sqrt{3} - i}$.
(b) $\sqrt[4]{4 - 3i}$.
(c) $\sqrt[3]{-3.4 - 5.1i}$.
(d) $\sqrt[5]{5.8 + 3i}$.
(e) $\sqrt[3]{-i}$.
(f) $\sqrt[9]{-3.6 + 5.6i}$.

6. Solve the following equations:

(a) $x^3 - 8 = 0$.
(b) $x^3 = i$.
(c) $x^6 = 3 - 4i$.
(d) $x^7 = -3.8 - 7i$.

7. Show that

$$\tan x = \frac{1}{i}\left(\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}}\right).$$

8. Prove that

$$\sec x = \frac{2e^{ix}}{e^{2ix} + 1}.$$

9. Using definitions (9) and (10), verify the following identities.

(a) $\tanh(x + y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \cdot \tanh y}$.

(b) $\tanh(x - y) = \dfrac{\tanh x - \tanh y}{1 - \tanh x \cdot \tanh y}$.

(c) $\sinh 2x = 2 \sinh x \cosh x$.

(d) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$.

CHAPTER XI

LOGARITHMS

93. Introduction. The labor involved in many numerical computations is considerably lessened by the use of logarithms. In the following articles we shall discover that in a sense the use of logarithms reduces multiplication to addition, division to subtraction, raising to a power to multiplication, and extracting a root to division. For this reason logarithms constitute a remarkable labor-saving device in computation.

We shall learn presently that logarithms are exponents and that the laws that govern the use of exponents are the ones that govern the use of logarithms. Hence, before discussing logarithms, we shall recall from algebra the laws of exponents.

94. Laws of exponents. It is proved in algebra that, when the exponents m and n are any numbers, the following laws hold:

(I) $a^m a^n = a^{m+n}$. (IV) $(ab)^m = a^m b^m$.

(II) $\dfrac{a^m}{a^n} = a^{m-n}$. (V) $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$.

(III) $(a^m)^n = a^{mn}$.

EXERCISES

1. Evaluate the following:

(a) $3^2 3^{-3}$. (d) $3^{-\frac{3}{2}} 3^{\frac{7}{2}}$. (g) $(25 \times 49)^{-\frac{1}{2}}$.

(b) $7^{-\frac{3}{7}} \sqrt[7]{7^{10}}$. (e) $\dfrac{5^{-\frac{3}{2}}}{\sqrt{5}}$. (h) $(\frac{3}{2})^{-3}$.

(c) $3^{-\frac{1}{2}} 3^0$. (f) $(3^{-1})^{\frac{27}{8}}$. (i) $(\frac{8}{27})^{-\frac{2}{3}}$.

2. Find, in each case, the value of x which satisfies the equation:

(a) $10^x = 1000$. (f) $x^{-2} = 100$. (k) $7^x = 1$.
(b) $3^{-3} = x$. (g) $10^0 = x$. (l) $x^{-1} = 0.01$.
(c) $x^4 = 10{,}000$. (h) $x^{-2} = 10°$. (m) $7^x = 343$.
(d) $x^{-\frac{1}{2}} = 3$. (i) $(36)^x = \frac{1}{6}$. (n) $\left(\dfrac{1}{x}\right)^{-2} = 16$.
(e) $4^x = \frac{1}{2}$. (j) $x^{-\frac{1}{3}} = \sqrt{7}$. (o) $2^{\frac{1}{x}} = 4^3$.

3. Find x if

(*a*) $10^x = \frac{1}{10}$.	(*d*) $10^x = 1000$.
(*b*) $10^x = 0.001$.	(*e*) $10^x = 1$.
(*c*) $10^x = 0.0001$.	(*f*) $10^x = 100{,}000$.

4. Solve each of the following equations for x:

(*a*) $(3)(2)^x + 4 = 100$.	(*g*) $(3^{\frac{1}{2}})(9)^{2x} = 3^{-\frac{3}{2}}$.
(*b*) $5^{x+3} - 5^{2x} = 0$.	(*h*) $(\frac{16}{25})^{-\frac{1}{2}} = 5\sqrt{x}$.
(*c*) $(8)(2)^x - 2^{4x} = 0$.	(*i*) $(\frac{8}{27})^{-\frac{1}{3}} = 2x^{-1}$.
(*d*) $(8)(3^x) = (27)(2^x)$.	(*j*) $(7^{x^2-1})(49^{1-x}) = \sqrt{7}$.
(*e*) $(x - 2)^0 = x^2 + 1$.	(*k*) $\left(\dfrac{9x}{4}\right)^{-\frac{1}{2}} - 3^{-2} = 3^{-3}$.
(*f*) $27^x = 81$.	(*l*) $\frac{1}{2}\sqrt{x}\sqrt[3]{x} = 64$.

95. Definition of a logarithm. If b, L, and N are numbers such that b raised to the power L is equal to N, then L is called the logarithm of N to the base b. In symbols, if

$$b^L = N, \qquad \text{then} \qquad L = \log_b N. \tag{1}$$

Stated differently, *the logarithm of a number to a given base is the power to which the base must be raised to produce the number.*

The two equations in (1) express the same relation between the base b, the number N, and the logarithm L. The second one is read: L is the logarithm of N to the base b. Also N is called the antilogarithm of L (or the number whose logarithm is L) to the base b. Since $5^2 = 25$, 2 is the logarithm of 25 to the base 5, and 25 is the antilogarithm of 2 to the base 5. Similarly, we have

$$\begin{aligned}
10^3 &= 1000, &\quad \therefore \quad 3 &= \log_{10} 1000;\\
10^{-2} &= 0.01, &\quad \therefore \quad -2 &= \log_{10} 0.01;\\
3^{\frac{1}{2}} &= \sqrt{3}, &\quad \therefore \quad \tfrac{1}{2} &= \log_3 \sqrt{3}.
\end{aligned}$$

Since $1^x = 1$ for all values of x, 1 cannot be used as a base for logarithms. Also a negative number is not used as base; for many real numbers would have imaginary logarithms to a negative base. For example, if $(-3)^x = 27$, x is imaginary. Although any positive number different from 1 might be used as a base, 10 is often chosen for reasons that will appear as our study continues.

EXERCISES

Write each of the following exponential equations as a logarithmic equation:

1. $2^4 = 16$.

2. $10^2 = 100$.

3. $\sqrt{100} = 10$.

4. $(\frac{1}{2})^{-2} = 4$.

5. $8^{\frac{2}{3}} = 4$.

6. $10^{-2} = 0.01$.

7. $25^{-\frac{1}{2}} = \frac{1}{5}$.

8. $10^0 = 1$.

9. $10^{-3} = 0.001$.

Write each of the following equations as an exponential equation:

10. $\log_2 8 = 3$.

11. $\log_5 1 = 0$.

12. $\log_7 49 = 2$.

13. $\log_{10} 0.1 = -1$.

14. $\log_9 \frac{1}{3} = -\frac{1}{2}$.

15. $\log_9 1 = 0$.

In each of the following exercises, find the value of x:

16. $\log_6 x = 2$.

17. $\log_x \frac{1}{4} = 2$.

18. $\log_5 25 = x$.

19. $\log_x 15 = 1$.

20. $\log_2 x = 3$.

21. $\log_2 x = -2$.

22. $\log_4 x = -\frac{1}{2}$.

23. $\log_{10} 100 = x$.

24. $\log_2 32 = x$.

25. $\log_5 (\frac{1}{625}) = x$.

26. $\log_{10} x = 2$.

27. $\log_{10} x = -2$.

28. $\log_x 3 = -\frac{1}{2}$.

29. $\log_x 49 = -2$.

30. $\log_x 49 = 2$.

31. $\log_{27} 3 = x$.

32. $\log_2 \left(\frac{1}{\sqrt[3]{16}}\right) = x$.

33. $\log_5 x = 1$.

34. $\log_b x = 1$.

35. $\log_x (\frac{1}{9}) = 2$.

36. $\log_b x = 0$.

Show that

37. $(\log_b a)(\log_a b) = 1$.

38. $(\log_b a)(\log_c b)(\log_a c) = 1$.

39. $\log_b \left(\frac{1}{b}\right) = -1$.

40. Why cannot unity be used as a base for a system of logarithms?

41. Why cannot a negative number be used as a base for a system of logarithms?

96. Laws of logarithms. There are three fundamental laws of logarithms with which the student must be thoroughly familiar. These laws are easily derived from the laws of exponents.

I. *The logarithm of the product of two numbers is equal to the sum of the logarithms of the factors.*

Proof. Let M and N be any two positive numbers, and let

$$x = \log_b N, \qquad \text{and} \qquad y = \log_b M. \tag{2}$$

Then we may write

$$b^x = N, \qquad \text{and} \qquad b^y = M. \tag{3}$$

Multiplying member by member the first of equations (3) by the second, we get

$$b^x b^y = b^{x+y} = MN, \qquad \text{or} \qquad \log_b MN = x + y. \tag{4}$$

Substituting the values of x and y from (2) in (4), we get

$$\mathbf{\log_b MN = \log_b M + \log_b N.}$$

By repeated application of the first law it is readily proved that *the logarithm of the product of any finite number of factors is equal to the sum of the logarithms of the factors.*

II. *The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.*

Proof. Dividing member by member the first of equations (3) by the second, we get

$$\frac{N}{M} = \frac{b^x}{b^y} = b^{x-y}, \qquad \text{or} \qquad \log_b \frac{N}{M} = x - y. \tag{5}$$

Substituting the values of x and y from (2) in (5), we get

$$\mathbf{\log_b \frac{N}{M} = \log_b N - \log_b M.}$$

III. *The logarithm of a number affected by an exponent is the product of the exponent and the logarithm of the number.*

Proof. Let

$$x = \log_b N, \qquad \text{or} \qquad N = b^x. \tag{6}$$

Raising both members of $N = b^x$ to the pth power, we obtain

$$N^p = b^{px},$$

Therefore, in accordance with (1)

$$\log_b N^p = px. \tag{7}$$

Substitution of the value of x from (6) in (7) gives

$$\mathbf{\log_b N^p = p \log_b N.}$$

Example 1. Find the value of $\log_{10} \sqrt{0.001}$.

Solution. $\log_{10} \sqrt{0.001} = \log_{10} (0.001)^{\frac{1}{2}} = \frac{1}{2} \log_{10} 0.001$
$= \frac{1}{2} \log_{10} \frac{1}{1000} = \frac{1}{2}(-3) = -\frac{3}{2}.$

Example 2. Write $\log_b \sqrt[3]{\dfrac{a^2(c+d)^{\frac{1}{2}}}{c^5}}$ in expanded form.

Solution. $\log_b \sqrt[3]{\dfrac{a^2(c+d)^{\frac{1}{2}}}{c^5}} = \frac{1}{3} \log_b \dfrac{a^2(c+d)^{\frac{1}{2}}}{c^5}$

$$= \tfrac{1}{3}[\log_b a^2 + \log_b (c+d)^{\frac{1}{2}} - \log_b c^5]$$

$$= \tfrac{1}{3}[\mathbf{2 \log_b a + \tfrac{1}{2} \log_b (c+d) - 5 \log_b c}].$$

Example 3. Write $\frac{3}{2} \log_b (x+1) + \frac{1}{3} \log_b x - 2 \log_b (x^2+1)$ in contracted form.

Solution. $\frac{3}{2} \log_b (x+1) + \frac{1}{3} \log_b x - 2 \log_b (x^2+1)$

$$= \log_b (x+1)^{\frac{3}{2}} + \log_b x^{\frac{1}{3}} - \log_b (x^2+1)^2$$

$$= \mathbf{\log_b \frac{(x+1)^{\frac{3}{2}}x^{\frac{1}{3}}}{(x^2+1)^2}}.$$

Another form of the answer is found as follows:

$$\log_b \frac{(x+1)^{\frac{3}{2}}x^{\frac{1}{3}}}{(x^2+1)^2} = \log_b \left[\frac{(x+1)^9x^2}{(x^2+1)^{12}}\right]^{\frac{1}{6}} = \mathbf{\tfrac{1}{6} \log_b \frac{(x+1)^9x^2}{(x^2+1)^{12}}}.$$

EXERCISES

1. Verify the following:

(*a*) $\log_{10} \sqrt{1000} + \log_{10} \sqrt{0.1} = 1.$
(*b*) $\log_2 \sqrt{8} + \log_2 \sqrt{2} = 2.$
(*c*) $\log_8 (2)^5 + \log_7 (\frac{1}{49})^{\frac{1}{3}} = 1.$.
(*d*) $\log_2 \sqrt{8} + \log_3 (\frac{1}{3})^2 = -\frac{1}{2}.$
(*e*) $\log_5 \sqrt{125} + \log_{13} \sqrt[3]{169} = \frac{13}{6}.$
(*f*) $\log_{11} \frac{1}{11} + 2 \log_{11} \sqrt{11} = 0.$
(*g*) $\log_2 (0.5)^3 - \log_4 \sqrt[6]{64} = -\frac{7}{2}.$
(*h*) $\log_5 1 - \log_7 6^0 = 0.$
(*i*) $\log_{10} 10^5 - \log_{10} 10^2 + \log_{10} 10^{-2} + \log_{10} 1 = 1.$

2. Write the following logarithmic expressions in expanded form:

(*a*) $\log_b \dfrac{a^2b^{\frac{1}{2}}}{c^3}.$ (*e*) $\log_b \dfrac{a^3cd^5}{7\sqrt[4]{e}}.$ (*i*) $\log_b \left[\dfrac{(p^0-5)^{\frac{1}{2}}}{(p-7)^2}\right]^5.$

(*b*) $\log_b \left(\dfrac{a^3b^6}{c^2}\right)^{\frac{1}{2}}.$ (*f*) $\log_b \sqrt[3]{\dfrac{x(x-y)}{z(x+y)}}.$ (*j*) $\log_b \dfrac{(x+g)x^2}{\sqrt{x-y}(z+y)}.$

(*c*) $\log_b \sqrt[5]{\dfrac{a^{\frac{1}{3}}c^{\frac{5}{2}}}{d^7}}.$ (*g*) $\log_b \dfrac{\sqrt[3]{p^2(1-q)}}{p^{\frac{1}{2}}(1+q)}.$ (*k*) $\log_b \dfrac{a(c-d)^2}{6(a+f)}.$

(*d*) $\log_b P(1+r)^n.$ (*h*) $\log_b \dfrac{[\sqrt{p}-1]^3}{q^2}.$ (*l*) $\log_b \sqrt[5]{\left[\dfrac{a^2(c-d)^3}{c\sqrt{a-d}}\right]^2}.$

3. Write the following expressions in contracted form.

(a) $\log_b a + 2 \log_b c - \frac{1}{2} \log_b d$.
(b) $\frac{1}{2} \log_b a - 3 \log_b c - 4 \log_b (a + c)$.
(c) $\frac{1}{2} \log_b (a + c) + \frac{1}{2} \log_b (a - c)$.
(d) $\log_b 3c - \frac{4}{3} \log_b d + \log_b e$.
(e) $\frac{1}{3}[\log_b a + 2 \log_b (c - d) - 4 \log_b c - \frac{1}{3} \log_b (2 - a)]$.
(f) $5[\frac{1}{2} \log_b (a - c) + \log_b (a + d) - 6 \log_b d - 2 \log_b a]$.

4. Take from a five-place table the following logarithms:

$$\log_{10} 2 = 0.30103, \log_{10} 3 = 0.47712, \log_{10} 7 = 0.84510.$$

From these numbers find $\log_{10} 4$, $\log_{10} 9$, $\log_{10} 28$, $\log_{10} 32$, $\log_{10} \frac{4}{3}$, $\log_{10} \frac{3}{4}$.

5. Using the logarithms in Exercise 4, find $\log_{10} \frac{2}{3}$, $\log_{10} \frac{3}{2}$, $\log_{10} 343$, $\log_{10} \sqrt{2}$, $\log_{10} \sqrt[3]{7}$, $\log_{10} 5$.

6. Using the logarithms in Exercise 4, find the value of the logarithm of each of the following expressions:

(a) $\dfrac{(2)(5)}{3}$.

(b) $\dfrac{(10)(6)}{7}$.

(c) $\dfrac{(3)(9)(5)}{14}$.

(d) $\sqrt{\dfrac{(30)(21)}{8}}$.

(e) $\sqrt[5]{\dfrac{(6)(4)(7)^{\frac{1}{2}}}{28}}$.

(f) $\dfrac{(9)^{\frac{1}{2}}(12)(4)^{\frac{1}{3}}}{35}$.

97. Common logarithms. Characteristic. In computation, it is convenient and customary to employ logarithms to the base 10. Logarithms to this base are called *common logarithms*. Throughout this text we shall use common logarithms only, and we shall write $\log N$ as an abbreviation of $\log_{10} N$. Thus when the base is omitted it will be understood that the base is 10.

In this system of logarithms, the logarithm of any integral power of 10 is an integer, while the logarithm of any positive number not an integral power of 10 may be written as an integer plus a decimal. In general, the logarithm of a number consists of two parts, an integer called the *characteristic*, and a decimal called the *mantissa*. The characteristic is found by inspection; the mantissa is found from a table. We shall now deduce rules for finding the characteristic.

Consider the following table:

$10^5 = 100{,}000$	or	$\log 100{,}000 =$	$5,$
$10^4 = 10{,}000$	or	$\log 10{,}000 =$	$4,$
$10^3 = 1000$	or	$\log 1000 =$	$3,$
$10^2 = 100$	or	$\log 100 =$	$2,$
$10^1 = 10$	or	$\log 10 =$	$1,$
$10^0 = 1$	or	$\log 1 =$	$0,$
$10^{-1} = 0.1$	or	$\log 0.1 =$	$-1,$
$10^{-2} = 0.01$	or	$\log 0.01 =$	$-2,$
$10^{-3} = 0.001$	or	$\log 0.001 =$	$-3,$
$10^{-4} = 0.0001$	or	$\log 0.0001 =$	$-4,$
$10^{-5} = 0.00001$	or	$\log 0.00001 =$	$-5.$

From the foregoing table, we get by inspection the following information:

Number	Number of digits to left of decimal point	Logarithm	Characteristic
$1 < N < 10$	1	$0 +$ a decimal	0
$10 < N < 100$	2	$1 +$ a decimal	1
$100 < N < 1000$	3	$2 +$ a decimal	2
$1000 < N < 10{,}000$	4	$3 +$ a decimal	3
$10^n < N < 10^{n+1}$	$n + 1$	$n +$ a decimal	n

From the data just tabulated, we infer the following rule:

Rule 1. *The characteristic of the common logarithm of a number greater than* 1 *is positive and is one less than the number of digits to the left of the decimal point.*

Similarly, we get

Number	Number of zeros to right of decimal point	Logarithm	Characteristic
$0.1 < N < 1$	0	$-1 +$ a decimal	-1 or $9 - 10$
$0.01 < N < 0.1$	1	$-2 +$ a decimal	-2 or $8 - 10$
$0.001 < N < 0.01$	2	$-3 +$ a decimal	-3 or $7 - 10$
$10^{-n} < N < 10^{-(n-1)}$	$n - 1$	$-n +$ a decimal	$-n$ or $(10 - n) - 10$

From the tabulated data, we infer the following rule:

Rule 2. *The characteristic of the common logarithm of a positive number less than* 1 *is negative and is numerically one greater than the number of zeros immediately following the decimal point.*

When the characteristic is negative, it is convenient to add 10 to the characteristic and subtract 10 at the right of the mantissa. Thus log 0.02545 = $-2 +$ a decimal $= 8 +$ a decimal -10. In general, if the characteristic $-n$ of log N is negative, change it to the equivalent value $(10 - n) - 10$, or $(20 - n) - 20$, etc. *To obtain directly the characteristic of the logarithm of a number less than* 1, *subtract from* 9 *the number of zeros immediately following the decimal point; write the result before the mantissa and* -10 *after it.*

Illustrations:

Number	Characteristic	Rule
4261	3	1
3.6121	0	1
0.1210	-1 or $9 - 10$	2
0.0025	-3 or $7 - 10$	2
0.00000345	-6 or $4 - 10$	2

EXERCISES

Write the characteristic of the logarithm of each number:

1. 7.613. **5.** 761.3. **9.** 89,261. **13.** 3101.

2. 467,916. **6.** 31.12. **10.** 412.16. **14.** 14,481.10.

3. 20.02. **7.** 0.0371. **11.** 0.0000309. **15.** 0.30001.

4. 3.00008. **8.** 0.81219. **12.** 0.003872. **16.** 0.000810.

98. Effect of changing the decimal point in a number. Any number may be written in the form $N \times 10^k$, where N is a number between 1 and 10 and k is an integer. Thus we may write $1{,}782{,}500 = 1.7825 \times 10^6$, $17825 = 1.7825 \times 10^4$. Evidently a shift of the decimal point appears in this notation as a change in k. Now $\log [N \times 10^k] = \log N + k \times 1$. Since a shift of the decimal point changes k, but not $\log N$, it appears that the mantissa of $\log N$ is not affected by the position of the decimal point. In other words, a change in the position of the decimal

point in a given sequence of figures has no effect on the mantissa; its sole effect is to change the characteristic. Because of this fact, 10 affords a particularly convenient base for a system of logarithms to be used for purposes of computation.

99. The mantissa. Mantissas can be computed by use of advanced mathematics and, except in special cases, are unending decimal fractions. Computed mantissas are tabulated in tables of logarithms, also called tables of mantissas. These tables are called "three-place," "four-place," "five-place," etc., according as the mantissas tabulated contain 3, 4, 5, etc., significant figures. The choice of a table of logarithms should depend upon the degree of accuracy required and the accuracy of the data. In this text we shall discuss and use a five-place table, thus obtaining results accurate to five significant figures.

100. To find the logarithm of a number. In general, a five-place table of logarithms gives the mantissas of all integral numbers lying between 999 and 10,000. The first three digits of the numbers are found in the left-hand column headed N, and the fourth digit is in the row at the top of the page. Therefore the mantissa of a number with four significant figures is in the row with the first three significant figures of the number and in the column headed by the fourth.

Example 1. Find log 42.43.

Solution. By the rule in §97, the characteristic is found to be 1. To find the mantissa, first find 424 in the left-hand column headed N, then follow the row containing 424 until the column headed by 3 is reached. Here we find 62767. Therefore the mantissa is 0.62767. Hence

$$\log 42.43 = \mathbf{1.62767.}$$

Example 2. Find log 0.0416.

Solution. By the rule in §97, the characteristic is found to be 8. -10. Using 4160, we find the mantissa to be 0.61909. Therefore

$$\log 0.0416 = \mathbf{8.61909 - 10.}$$

EXERCISES

Verify the following:

1. $\log 2934 = 3.46746$.
2. $\log 3.478 = 0.54133$.
3. $\log 28.7 = 1.45788$.
4. $\log 1.817 = 0.25935$.
5. $\log 981.7 = 2.99198$.
6. $\log 0.3132 = 9.49582 - 10$.
7. $\log 0.0003146 = 6.49776 - 10$.
8. $\log 0.03426 = 8.53479 - 10$.
9. $\log 0.272 = 9.43457 - 10$.
10. $\log 0.005075 = 7.70544 - 10$.

101. Interpolation. From the five-place table of logarithms we cannot obtain directly the logarithm of a number with five significant figures. However, by a process known as interpolation, we can find the mantissa of a number having a fifth significant figure. In this process we use the principle of proportional parts, which states that, for small changes in N, the corresponding changes in $\log N$ are proportional to the changes in N. Although this principle is not strictly true, it is sufficiently accurate to lead to results correct to the number of figures given in the table.

The process of interpolation is illustrated by means of the following example:

Example. Find log 235.47.

Solution. From the table of logarithms we find the logarithms in the following form and then compute the differences exhibited.

$$\left.\begin{array}{l} \left.\begin{array}{l}\log 235.40 \\ \log 235.47\end{array}\right\} 7 \\ \log 235.50 \end{array}\right\} 10 \quad \left.\begin{array}{l} \left.\begin{array}{l}= 2.37181 \\ = \;?\end{array}\right\} d \\ = 2.37199 \end{array}\right\} 18 \text{ (tabular difference)}$$

By the principle of proportional parts, we have

$$\frac{7}{10} = \frac{d}{18}, \qquad \text{or} \qquad d = \left(\frac{7}{10}\right)(18) = 13 \text{ (nearly)}.$$

We add $d = 13$ to the last two figures of 2.37181 to obtain

$$\log 235.47 = \mathbf{2.37194}.$$

Notice that the value used for d was 13 instead of 12.6 because the table of logarithms is accurate only to five decimal places.

In order to save work in interpolating when finding the mantissas of five-place numbers, each tabular difference occurring in the table has been multiplied by 0.1, 0.2, . . . 0.9, and the results placed on the right-hand sides of the pages where these tabular differences occur. These tabulated results, called tables of proportional parts (P.P.), are headed by the tabular difference for which they have been formed, and the decimal points have been omitted. To interpolate in the example just solved, we locate the proportional parts table headed 18, and opposite 7 in the left-hand column we find $d = 13$.

EXERCISES

Find the logarithm of each of the following:

1. 40.488.

2. 3.0473.

3. 10,201.

4. 108.17.

5. 0.21544.

6. 0.0038345.

7. 0.086452.

8. 0.000076123.

9. 0.027038.

10. 0.18253.

102. To find the number corresponding to a given logarithm. Generally in every problem involving logarithms, it is necessary not only to find the logarithms of numbers but also to perform the inverse process, that of finding a number corresponding to a given logarithm.

If $\log N = L$, then N is the number corresponding to the logarithm L. The number N is called the *antilogarithm* of L. To find the antilogarithm N of the logarithm L, first use the given mantissa to find the sequence of figures in N, and then use the given characteristic to place the decimal point so as to agree with the rule of §97.

Example. Given $\log N = 1.60334$, find N.

Solution. The mantissa .60334 is not found exactly in the table, but we find the two successive mantissas .60325 and .60336, between which the given mantissa lies. From the table we find the numbers in the following form and then compute the differences exhibited.

$$\left.\begin{array}{l}\left.\begin{array}{l}1.60325\\1.60334\end{array}\right\}9\\1.60336\end{array}\right\}11\quad\begin{array}{l}=\log 40.110\\=\log N\\=\log 40.120\end{array}\quad\left.\begin{array}{l}\left.\begin{array}{l}\\ \\\end{array}\right\}x\\ \\\end{array}\right\}10$$

By the principle of proportional parts, we have

$$\frac{x}{10} = \frac{9}{11}, \qquad \text{or} \qquad x = \frac{(9)(10)}{11} = 8 \text{ (nearly)}.$$

We add $x = 8$ to the last figure of 40.110 to obtain

$$N = \mathbf{40.118}.$$

This interpolation should be performed by means of the table of proportional parts. In the P.P. column under the block corresponding to the tabular difference 11, we find the difference 9; immediately to the left of this we find 8, the fifth significant figure in the number N.

EXERCISES

Find x in each of the following:

1. $\log x = 8.66200 - 10.$

2. $\log x = 3.89779.$

3. $\log x = 5.31664.$

4. $\log x = 9.70000 - 10.$

5. $\log x = 7.97295 - 10.$

6. $\log x = 2.99876.$

7. $\log x = 0.87484.$

8. $\log x = 0.42239.$

9. $\log x = 1.11240.$

10. $\log x = 6.54782 - 10.$

11. Find x in each of the following:

(*a*) $\log x = -0.34345.$

(*b*) $\log x = -2.41325.$

(*c*) $\log x = -3.12864.$

(*d*) $\log x = -0.16132.$

103. The use of logarithms in computations. The following examples will illustrate how logarithms are used.

Example 1. Evaluate (461)(4.321).

Solution. Denoting the product by x, we may write

$$x = (461)(4.321).$$

Equating the logarithms of the two members of this equation, we get

$$\log x = \log 461 + \log 4.321.$$

Looking up the logarithms of the numbers, we obtain

$$\begin{aligned} \log 461 &= 2.66370 \\ \log 4.321 &= \underline{0.63558} \end{aligned}$$

Adding, we have $\qquad \log x = 3.29928.$

The antilogarithm of 3.29928, is

$$x = \mathbf{1992.0.}$$

Example 2. Evaluate $\dfrac{(217)(3.18)}{62.142}$.

Solution. Let $x = \dfrac{(217)(3.18)}{62.142}$.

Then $\log x = \log 217 + \log 3.18 - \log 62.142$.

$$\begin{aligned} \log 217 &= 2.33646 \\ \log 3.18 &= \underline{0.50243} \\ \text{Sum} &= 2.83889 \\ \log 62.142 &= \underline{1.79338} \end{aligned}$$

Subtracting, we obtain $\log x = 1.04551$

The antilogarithm of 1.04551 is

$$x = \mathbf{11.105.}$$

Example 3. Evaluate $(2.713)^3$.

Solution. Let $x = (2.713)^3$. Then

$$\log x = 3 \log 2.713 = 3(0.43345) = 1.30035.$$
$$\therefore x = \mathbf{19.969.}$$

Example 4. Evaluate $\sqrt[3]{0.7214}$.

Solution. Let $x = \sqrt[3]{0.7214} = (0.7214)^{\frac{1}{3}}$. Then

$$\log x = \tfrac{1}{3} \log 0.7214 = \tfrac{1}{3}(9.85818 - 10).$$

If we should divide this logarithm by 3, the characteristic of the resulting logarithm would not be in the standard form. Hence we first add 20 and then subtract 20, writing the logarithm in the form $29.85818 - 30$. Then we write

$$3\underline{)29.85818 - 30}$$

Dividing, we get $\log x = 9.95273 - 10$

or $x = 0.89688.$

EXERCISES

Evaluate the following:

1. $52{,}564 \times 0.0082546$.
2. $\dfrac{0.0031593 \times 684.82}{0.0096548}$.
3. $(1.045)^{26}$.
4. $7^{\frac{1}{7}}$.
5. $(0.03628)^{\frac{1}{5}}$.
6. $\sqrt[11]{(442.84)^3}$.
7. $(33.982)^{-\frac{2}{5}}$.
8. $\dfrac{75{,}859 \times 0.0028242}{37{,}568 \times 0.09185}$.

104. Cologarithms. Subtracting a first number from a second is equivalent to adding the negative of the first to the second. Hence, to avoid subtraction in dealing with logarithms, we introduce cologarithms.

The cologarithm of a number is the negative of its logarithm. Therefore adding the cologarithm of a number is equivalent to subtracting its logarithm.

To avoid negative mantissas, the cologarithm of a number n, written colog n, is found by using the form colog $n = 10 - \log n - 10$. Thus colog $2 = 10 - \log 2 - 10 = 10 - 0.30103 - 10 = 9.69897 - 10$, and colog $0.3 = 10 - (9.47712 - 10) - 10 = 0.52288$. The student will find it convenient in getting colog n to *begin at the left of log n, subtract each of its digits from* 9 *except the last significant one, and subtract that from* 10.

The following example will illustrate the use of cologarithms.

Example. Find x if $x = \dfrac{342.10}{(6710)(0.31820)}$.

Solution.
$$\begin{aligned}\log x &= \log 342.10 - \log 6710 - \log 0.31820\\ &= \log 342.10 + \text{colog } 6710 + \text{colog } 0.31820\end{aligned}$$

$$\begin{aligned}
& & \log 342.10 &= 2.53415\\
\log 6710 &= 3.82672, & \text{colog } 6710 &= 6.17328 - 10\\
\log 0.31820 &= 9.50270 - 10, & \text{colog } 0.31820 &= 0.49730\\
& & \log x &= 9.20473 - 10
\end{aligned}$$

and $x =$ **0.16023.**

EXERCISES

1. Verify the following:

(*a*) colog $179.82 = 7.74516 - 10$.
(*b*) colog $0.63273 = 0.19878$.
(*c*) colog $7.5328 = 9.12304 - 10$.
(*d*) colog $23.975 = 8.62024 - 10$.

2. Using cologarithms, find the value of

(*a*) $\dfrac{36.21}{7.215}$. (*b*) $\dfrac{42.21}{0.2861}$. (*c*) $\dfrac{41.262}{(61.84)(1612)}$. (*d*) $\dfrac{142.3}{0.02813}$.

105. Computation by logarithms. In solving complicated problems, the computer is helped materially by a good form. The one discussed below has the advantages of simplicity, completeness of record, and brevity. It is practically self-explanatory since the main feature consists in reference of every function on a line to the first number in the line; a complete record of logarithms and operations is tabulated, and little writing is required. Since the outline of the form can always be made in advance, the student should first make this outline and then perform the computation without interruption. Speed and accuracy are gained by this method.

The form will be used in the following solution.

Example 1. Find x if $x = \dfrac{a^{\frac{1}{3}}\sqrt[5]{bc^2}}{de^4}$ and $a = 8.1632$, $b = 729.77$, $c = 0.46330$, $d = 5.2133$, $e = 0.32411$.

Solution. First write the formula

$$\log x = \tfrac{1}{3}\log a + \tfrac{1}{5}\log b + 2\log c + \operatorname{colog} d + 4\operatorname{colog} e.$$

The following form contains the solution:

$a = 8.1632$	$\log a = 0.91186$	$\frac{1}{3}\log a = 0.30395$
$b = 729.77$	$\log b = 2.86318$	$\frac{1}{5}\log b = 0.57264$
$c = 0.04633$	$\log c = 8.66586 - 10$	$2\log c = 7.33172 - 10$
$d = 5.2133$	$\log d = 0.71711$	$\operatorname{colog} d = 9.28289 - 10$
$e = 0.32411$	$\log e = 9.51069 - 10$	$4\operatorname{colog} e = 1.95724$
$x = \mathbf{0.28083}$		$\log x = 9.44844 - 10$

Note that each number in any line relates to the first number in the line, and the relation is indicated that the record of operations is complete, that little writing is required, and that an examiner could easily perceive and follow the steps taken.

In the following solution a form is indicated, but the computation is left as in exercise to the student.

Example 2. Find x if $x = \left[\dfrac{\sqrt{c} \times a^2}{a + \sqrt{e}}\right]^{\frac{1}{3}}$ where $a = 61.214$, $c = 12.112$, and $e = 139.02$.

Solution. First we write the formula

$$\log x = \tfrac{1}{3}[\tfrac{1}{2} \log c + 2 \log a + \operatorname{colog} (a + \sqrt{e})]$$

and then make the following form:

$e = 139.02$	$\log e =$	$\frac{1}{2} \log e =$
$\sqrt{e} =$		$\log \sqrt{e} =$
$a = 61.214$	$\log a =$	$2 \log a =$
$a + \sqrt{e} =$	$\log (a + \sqrt{e}) =$	$\operatorname{colog} (a + \sqrt{e}) =$
$c = 12.112$	$\log c =$	$\frac{1}{2} \log c$
		3)
$x =$		$\log x =$

The student should perform the computation to obtain $x =$ **5.6319.**

EXERCISES

Make a form or outline for computing each of the following:

1. $\dfrac{(32.861)^2(3.1416)^{\frac{1}{3}}}{(62.181)^3}$.
2. $\sqrt[3]{\dfrac{(31.64)^2(62.12)}{(9.31)^5}}$.
3. $\left[\dfrac{a^2b^3c^{\frac{1}{2}}}{d^5e}\right]^2$.
4. $\sqrt[5]{\dfrac{a^2\sqrt{b}\sqrt[3]{c}}{d^3\sqrt{e}}}$.

106. Remarks on computation by logarithms.

(*a*) *When interpolating, do not carry logarithms beyond the number of decimal places given in the table used.*

(*b*) *When evaluating an expression containing negative numbers, use logarithms to compute desired positive components, and then combine the results with appropriate signs. In this text a symbol* (−) *before a logarithm will indicate that a negative number is under consideration; thus if* $\log x = (-)9.87123 - 10$, $x = -0.74342$.*

(*c*) *Make a form like that of Example* 1, §105, *before beginning computation.*

(*d*) *Strive for accuracy in computation. Speed comes with practice.*

* This does not mean that a negative number has a real logarithm. The minus symbols serve merely to keep a record of the signs involved in the given expression.

Example. Find the value of x if $x = \sqrt[5]{\dfrac{(-47.123)^2(-36.184)^{\frac{1}{3}}}{\sqrt{31.118}}}$.

Solution.

$$\log(-x) = \tfrac{1}{5}[2 \log 47.123 + \tfrac{1}{3} \log 36.184 + \tfrac{1}{2} \operatorname{colog} 31.118].$$

$a = -47.123$	$\log a = (-)1.67324$	$2 \log a = 3.34648$
$b = -36.184$	$\log b = (-)1.55852$	$\frac{1}{3}\log b = (-)0.51951$
$c = 31.118$	$\log c = 1.49301$	$\frac{1}{2} \operatorname{colog} c = 9.25350 - 10$
		$5)\ (-)3.11949$
$x = \mathbf{-4.2063}$		$\log x = (-)0.62390$

EXERCISES

Find by use of logarithms the results of the following exercises. In each case make a complete outline or form before using the tables.

1. 3.1416×2.7183.

2. 29.572×0.00036841.

3. $335{,}000{,}000 \times 0.000099854$.

4. 2727.5×0.37375.

5. $1487 \times 3.139 \times 42.96$.

6. $\dfrac{2.9275 \times 34.278}{505.92}$.

7. $\dfrac{48.962 \times 39.595}{78.545}$.

8. $\dfrac{2964.5 \times 38.423}{75.65 \times 84.384}$.

9. $\dfrac{2954.5 \times 64.532}{911.36 \times 318.5}$.

10. $\dfrac{26.893 \times 0.0000545}{319.62 \times 0.00068432}$.

11. $(1.5)^{15}$.

12. $\sqrt[3]{31}$.

13. $\sqrt{347.3}$.

14. $\sqrt[3]{0.17638 \times 2.1279}$.

15. $\left[\dfrac{19.876}{38.345}\right]^2$.

16. $(0.00062584)^{\frac{1}{8}}$.

17. $(665.35)^{-\frac{1}{7}}$.

18. $\sqrt{\dfrac{(57.45)(423.34)}{(178)(89)}}$.

19. $\dfrac{(-80{,}941)\sqrt[5]{-0.031}}{(54{,}082)\sqrt[6]{0.0712}}$.

20. $\dfrac{4 \times 28.7 \times \sqrt{345}}{29 \times 137}$.

21. $\sqrt{(67.811)^2 + (83.314)^2}$.

22. $\sqrt{(7631.25)^2 - (6712.15)^2}$.*

23. $\sqrt[3]{\dfrac{(23.975)(5.793)^2}{179.82}}$.

24. $[(-8.90172)(732.95)^{\frac{1}{2}}(0.0954)^{\frac{3}{5}}]^2$.

25. $\sqrt{(27.5)^2 - (3.483)^2}$.*

26. $\dfrac{5086(-0.0008769)^3}{(9802)(0.001984)^4}$.

* *Hint.* First factor the radicand.

27. $\dfrac{1954.7 \times \sqrt[5]{0.0030121}}{\sqrt[4]{17{,}959 \times (0.84132)^8(560.63)}}$.

28. $\dfrac{(0.04)^{\frac{2}{3}}(0.057897)^{\frac{4}{5}}}{(87.67)^{0.9}}$.

29. $\sqrt[4]{\dfrac{(348.7)^2(-2.685)^3(3.08212)}{(2.678)^{\frac{1}{2}}(0.08216)^4(-800{,}013)}}$.

30. $\sqrt[3]{\dfrac{(0.002452)^{\frac{1}{4}}(86.47)^3(-128.721)}{(-5280)(-0.07115)^2(-62.472)}}$.

31. $\sqrt[3]{\dfrac{a^{\frac{1}{3}}b}{a^2 - b}}$, $a = 7.5328$, $b = 6384$.

32. $\sqrt[5]{\dfrac{b}{a^3} - \sqrt{a^2c}}$; $a = 735.9$, $b = 0.198$, $c = 27$.

33. $\dfrac{a^2c^{\frac{1}{2}}}{bD}$; $D = a + c^2$, $a = 23.722$, $b = 571.17$, $c = 0.03218$.

34. Given $a = 3.7124$, $b = 32.617$, find $\log (a + b)$, $\log (a - b)$, $\log \dfrac{a}{b}$, $\log ab$.

35. Find K, given $s = \frac{1}{2}(a + b + c + d)$,

$$K = \sqrt{(s - a)(s - b)(s - c)(s - d)},$$

$a = 6.3246$, $b = 7.7459$, $c = 8.5441$, $d = 5.1961$.

36. $\dfrac{a^3b^2c}{d^{\frac{1}{3}}}$, given $a = 0.00275$, $b = 100.5$, $c = 5075.5$, $d = 0.001875$.

37. $\left[\dfrac{a^5b^3c^2d^{\frac{1}{5}}}{e^2f^3g^4}\right]^{\frac{1}{8}}$, given $a = 301.03$, $b = 0.00036954$, $c = 0.0028182$, $d = 35{,}890{,}000$, $e = 0.000002814$, $f = 561.29$, $g = 2718.3$.

38. Find the weight of a steel sphere 1.0127 ft. in diameter if steel weighs 490 lb. per cu. ft.

39. Find the weight of a cube of metal weighing 530 lb. per cu. ft. if the edge of the cube is 1.6271 ft.

40. A conical piece of wood weighs 92 lb. If the area of the base of the solid is 1.3341 sq. ft., find the altitude. (The wood weighs 33 lb. per cu. ft.)

41. During a rain 0.521 in. of water fell. Find how many gallons of water fell on a level 10.7-acre park. (Take 1 cu. ft. = 7.48 gal., 1 acre = 43,560 sq. ft.)

42. The time t of oscillation of a simple pendulum of length l ft. is given in seconds by the formula

$$t = \pi\sqrt{\frac{l}{32.16}}.$$

Find the time of oscillation of a pendulum 3.326 ft. long. (Take $\pi = 3.142$.)

43. What is the weight in tons of a solid cast-iron sphere whose radius is 5.343 ft. if the weight of 1 cu. ft. of water is 62.355 lb. and the specific gravity of cast iron is 7.154?

44. Find the volume and surface of a sphere of radius 14.71.

45. The stretch of a brass wire when a weight is hung at its free end is given by the relation

$$S = \frac{mgl}{\pi r^2 k},$$

where m is the weight applied, $g = 980$, l is the length of the wire, r is its radius, and k is a constant. Find k for the following values: $m =$ 944.2 g., $l = 219.2$ cm., $r = 0.32$ cm., and $S = 0.060$ cm.

46. Find the length l of a wire that stretches 5.9 cm. for a weight of 1826.5 g. hanging at its free end, when the diameter of the wire is 0.064 cm. and $k = 1.1 \times 10^{12}$.

47. The weight P in pounds that will crush a solid cylindrical cast-iron column is given by the formula

$$P = 98{,}920\,\frac{d^{3.55}}{l^{1.7}},$$

where d is the diameter in inches and l the length in feet. What weight will crush a cast-iron column 6 ft. long and 4.3 in. in diameter?

48. For wrought-iron columns the crushing weight is given by

$$P = 299{,}600\,\frac{d^{3.55}}{l^2}.$$

What weight will crush a wrought-iron column of the same dimensions as that in Problem 47?

49. The weight W of 1 cu. ft. of saturated steam depends upon the pressure in the boiler according to the formula

$$W = \frac{P^{0.941}}{330.36},$$

where P is the pressure in pounds per square inch. What is W if the pressure is 280 lb. per sq. in.?

107. Change of base in logarithms. Occasionally it is necessary to find the logarithm of a number N to a base b other than 10. To do this we let

$$\log_b N = x, \qquad \text{or} \qquad b^x = N.$$

Equating the logarithms to the base 10 of the two members of this equation, we get

$$x \log_{10} b = \log_{10} N, \qquad \text{or} \qquad x = \frac{\log_{10} N}{\log_{10} b}.$$

Since the divisor and dividend of this fraction are logarithms, they will generally be numbers of several digits. Therefore it is advisable to perform the indicated division by means of logarithms.

Example. Find the value of $\log_3 0.092118$.

Solution. Let $x = \log_3 0.092118$. Then $3^x = 0.092118$. Equating the logarithms to the base 10 of the two members of this equation, we obtain

$$x \log_{10} 3 = \log_{10} 0.092118$$

or

$$x = \frac{\log_{10} 0.092118}{\log_{10} 3} = \frac{8.96434 - 10}{0.47712} = \frac{-1.03566}{0.47712}.$$

This quotient is evaluated as follows:

$a = -1.0357$		$\log a = (-)0.01523$
$b = 0.47712$	$\log b = 9.67863 - 10$	$\operatorname{colog} b = 0.32137$
$x = \mathbf{-2.1707}$		$\log x = (-)0.33660$

108. Solution of equations of the form $x = a^b$, $a = x^b$. We shall now illustrate the method of solving equations of the form $x = a^b$, and $a = x^b$, in which a and b are given numbers.

Example 1. Find x if $x = (3.21)^{8.27}$.

Solution. $\log x = 8.27 \log 3.21 = (8.27)(0.50651)$.

The solution is displayed below.

$a = 8.27$	$\log a = 0.91751$
$b = 0.50651$	$\log b = 9.70459 - 10$
$\log x = 4.1889$	$\log (\log x) = 0.62210$

Therefore $\log x = 4.1889$, from which we get $x =$ **15,449.**

Example 2. Find x if $x^{7.2143} = 0.080133$.

Solution. Equate the logarithms of the two members of the given equation and solve for $\log x$ to obtain

$$7.2143 \log x = \log 0.080133$$

or

$$\log x = \frac{\log 0.080133}{7.2143} = \frac{8.90381 - 10}{7.2143} = \frac{-1.09619}{7.2143}$$

The evaluation of the quotient for $\log x$ follows:

$a = -1.0962$		$\log a = (-)0.03989$
$b = 7.2143$	$\log b = 0.85820$	$\text{colog } b = 9.14180 - 10$
$\log x = -0.15195$		$\log (\log x) = (-)9.18169 - 10$

To make the mantissa of $\log x$ positive add it to $10 - 10$ to obtain

$$\log x = 10 - 0.15195 - 10 = 9.84805 - 10.$$

Therefore

$$x = \mathbf{0.70477.}$$

EXERCISES

1. $x = \log_7 100$.
2. $x = \log_{0.88} 99{,}324$.
3. $x = \log_{27} 0.00328$.
4. $x = \log_{0.0954} 87.543$.
5. $x = \log_{20} 100$.
6. $x = \log_8 27{,}569$.
7. $x = \log_{3.7} 0.8173$.
8. $x = \log_{21} 0.09827$.
9. $5^{\frac{1}{x}} = 1.307$.
10. $5^{2x} = 317.46$.
11. $\log_x 8 = 0.35678$.
12. $\log_x 2 = 0.69315$.
13. $\log_x 0.07936 = 2.983$.
14. $x^{2.892} = 0.07936$.
15. $(1.5)^{\frac{1}{x}} = 32$.
16. $4.02 = (2.37)^{\frac{1}{x+1}}$.
17. Given $3^{x+y} = 2(5^x)$, $x - y = 1$, find x and y.
18. How long will it take a sum of money to double itself if put at 4 per cent compound interest? This is represented by $(1.04)^x = 2$ where x is the number of years. Solve for x.
19. Solve the equation $e^x + e^{-x} = y$, for x (*a*) when $y = 2$, (*b*) when $y = 4$. $e = 2.7183$.

20. If fluid friction is used to retard the motion of a flywheel making V_0 revolutions per min., the formula $V = V_0e^{-kt}$ gives the number of revolutions per minute after the friction has been applied t seconds. If the constant $k = 0.35$, how long must the friction be applied to reduce the number of revolutions from 200 to 50 per min.? $e = 2.7183$.

21. The pressure, P, of the atmosphere in pounds per square inch, at a height of z ft. is given approximately by the relation

$$P = P_0e^{-kz},$$

where P_0 is the pressure at sea level and k is a constant. Observations at sea level give $P_0 = 14.72$, and at a height of 1122 ft., $P = 14.11$. What is the value of k?

22. Assuming the law in Exercise 21 to hold, at what height will the pressure be half as great as at sea level?

23. If a body of temperature $T_1°$ is surrounded by cooler air of temperature $T_0°$, the body will gradually become cooler, and its temperature, $T°$, after a certain time, say t min., is given by Newton's law of cooling, that is,

$$T = T_0 + (T_1 - T_0)e^{-kt},$$

where k is a constant. In an experiment a body of temperature 55°C. was left to itself in air whose temperature was 15°C. After 11 min. the temperature was found to be 25°. What is the value of k?

24. Assuming the value of k found in Exercise 23, what time will elapse before the temperature of the body drops from 25° to 20°?

25. Solve the equation $\log_e (3x + 1) = 2$ for x.

26. Solve the equation $\log_{10} (x^2 - 21x) = 2$ for x.

109. Graph of $y = \log_{10} x$. If we assign values to x in the equation $y = \log_{10} x$ and find the corresponding values of y, we shall obtain the coordinates of points on the curve $y = \log_{10} x$. A few of these values are tabulated in the accompanying table. Plotting these points and drawing a smooth curve through

x	0.5	1	3	5	8	10	15	20	25	30	35	40
y	−0.3	0	0.48	0.70	0.9	1	1.17	1.3	1.4	1.48	1.54	1.6

them, we obtain the graph shown in Fig. 1. For convenience, the unit on the y-axis has been taken ten times as large as the unit on the x-axis.

If the student retains a mental picture of this graph, he will find it easy to recall the following facts:

(*a*) A negative number has no real number for its logarithm.

(*b*) The logarithm of a positive number is negative or positive according as the number is less than or greater than 1.

(*c*) If the number x approaches zero, $\log x$ decreases without limit.

(*d*) If the number x increases indefinitely, $\log x$ increases without limit.

In the process of interpolation in logarithms, values are inserted as if the change in the logarithm between the nearest

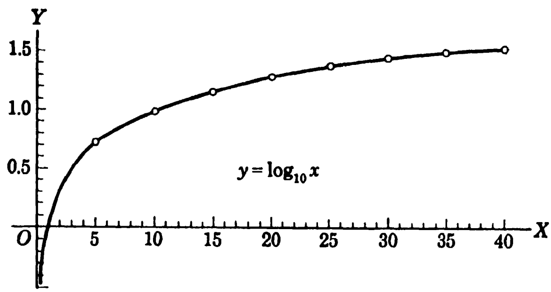

FIG. 1.

tabulated values were directly proportional to the change in the number. This assumes that the graph of $y = \log x$ for the interval concerned is a straight line. From the graph it is apparent this would be approximately true. In other words, when a number is changed by an amount that is very small in comparison with the number itself, the change in the value of the logarithm of the number is very nearly proportional to the change in the number.

EXERCISES

1. Plot the graph of $y = \log_5 x$.

Hint. $\log_5 x = \dfrac{\log_{10} x}{\log_{10} 5}$.

2. Plot the graph of $x = \log_5 y$.

3. Plot the graph of $x = \log_2 y$.

110. MISCELLANEOUS EXERCISES

Find by use of logarithms the results of the following exercises. In each case make a complete outline or form before using the tables.

1. 3.87×57.6.

2. 7.0928×0.0052683.

3. $22.9 \times 4.95 \times 0.643$.

4. $0.0063982 \times 23.473 \times 0.062547$.

5. $\dfrac{76.9}{3.14}$.

6. $\dfrac{1}{0.8236}$.

7. $\dfrac{8.211}{0.6634}$.

8. $\dfrac{49.36 \times 0.7657}{8.439}$.

9. $\dfrac{6.47 \times 12.93 \times 0.2462}{896 \times 0.0074939}$.

10. $(0.09245)^3$.

11. $\sqrt[6]{0.002855}$.

12. $\sqrt[4]{0.0070008}$.

13. $(0.935)^{\frac{3}{5}}$.

14. $(4.267)^{0.4}$.

15. $(19.26)^{1.2}$.

16. $\dfrac{(41.911)^{\frac{5}{4}}}{\sqrt[5]{(3.215)^3} \times 0.78356}$.

17. $\dfrac{(89.1)^{\frac{2}{3}} \times (0.764)^{0.2}}{\sqrt[4]{0.0387}}$.

18. $\dfrac{(7.9036)^{1.1} \times \sqrt[5]{(0.50267^3}}{(0.0014123)^{0.9}}$.

19. $(-0.091111)^{-\frac{3}{5}}$.

20. $\dfrac{45.86 \times (0.7288)^{\frac{3}{4}}}{(-9.423)^{\frac{5}{3}}}$.

21. $\dfrac{(-0.49173)^{\frac{2}{3}}}{\sqrt[5]{-207.99}}$.

22. $\dfrac{1}{\sqrt[4]{(170.5)^3 - 15}}$.

23. $\dfrac{\sqrt{0.7285} + (2.706)^{\frac{3}{2}}}{318.2 \times (0.06004)^2}$.

24. $\dfrac{(0.8195)^{-0.3} + (0.9713)^{0.4}}{(5.004)^{-\frac{1}{3}}}$.

25. $\dfrac{\log 9.5}{\log 4.27}$.

26. $\dfrac{\log 0.87189}{\log 0.022223}$.

27. The radius r of the inscribed circle of a triangle in terms of its sides a, b, and c is given by

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

where $s = \frac{1}{2}(a + b + c)$. Compute r when (a) $a = 0.525$, $b = 0.261$, $c = 0.438$; (b) $a = 698.2$, $b = 476.3$, $c = 744.9$; (c) $a = 3.0023$, $b = 2.1128$, $c = 1.5007$.

28. The number n of revolutions per minute of a certain water turbine is given by

$$n = \frac{400}{61.3} h^{1.3} P^{-0.4},$$

where h is the height of fall in feet, and P is the horsepower developed. Compute n when $h = 15$ ft. and $P = 98$ hp.

29. The formula $y = 0.0263x^{1.1}$ gives the relation between y and x when x stands for the stress in kilograms per square centimeter of cross section of a hollow cast-iron tube subject to tensile stress and y for the elongation of the tube in terms of $\frac{1}{800}$ cm. as a unit. Compute y when $x = 101.8$.

30. The formula $y = ks^x g^{c^x}$, where $\log k = 5.03370116$, $\log s = -0.003296862$, $\log g = -0.00013205$, $\log c = 0.04579609$, gives the number living at age x in Hunter's Makehamized American Experience Table of Mortality. Find, to such a degree of accuracy as you can secure with a five-place table of logarithms, the number living (*a*) at age ten, (*b*) at age thirty.

31. Given that 1 km. = 0.6214 mile. Find the number of miles in 2489 km.

32. Given that 1 km. = 0.6214 mile and that the area of Illinois is 56,625 square miles. Express the area of Illinois in square kilometers (to four significant figures).

CHAPTER XII

THE SLIDE RULE

111. Introduction. This chapter, while giving a brief review of the method of using a slide rule, stresses the settings relating to trigonometry. The settings given apply to most slide rules, but the explanation is based on the manuals written by the authors of this text for the slide rules manufactured by the Keuffel and Esser Company. For a logarithmic explanation of this slide rule and more detail concerning the settings, the student is referred to the manuals just cited.

Efficient operation of a slide rule is a comparatively simple matter. Since nearly every setting is based on one principle called the *proportion principle*, it is easy to recall forgotten settings and devise new ones especially suited to the work at hand. The first step is to learn to read the scales on the rule.

112. Reading the scales.* Figure 1 represents, in skeleton form, the fundamental scale of the slide rule, namely the D scale.

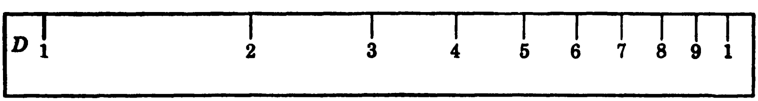

FIG. 1.

An examination of this actual scale on the slide rule will show that it is divided into 9 parts by primary marks that are numbered 1, 2, 3, . . . , 9, 1. The space between any two primary marks is divided into ten parts by nine secondary marks. These are not numbered on the actual scale except between the primary marks numbered 1 and 2. Figure 2 shows the secondary marks lying between the primary marks of the D scale. On this scale each italicized number gives the reading to be associated with

* The description here given has reference to the 10-in. slide rule. However, anyone having a rule of different length will be able to understand his rule in the light of the explanation given.

its corresponding secondary mark. Thus, the first secondary mark after 2 is numbered 21, the second 22, the third 23, etc.; the first secondary mark after 3 is numbered 31, the second 32, etc. Between the primary marks numbered 1 and 2 the secondary marks are numbered 1, 2, . . . , 9. Evidently the readings associated with these marks are 11, 12, 13, . . . , 19. Finally between the secondary marks, see Fig. 3, appear smaller or tertiary marks that aid in obtaining the third digit of a reading. Thus between the secondary marks numbered 22 and 23 there are four tertiary marks. If we think of the end marks as repre-

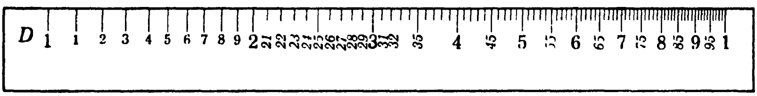

FIG. 2.

senting 220 and 230, the four tertiary marks divide the interval into five parts, each representing two units. Hence with these marks we associate the numbers 222, 224, 226, and 228; similarly the tertiary marks between the secondary marks numbered 32 and 33 are read 322, 324, 326, and 328, and the tertiary marks between the primary marks numbered 3 and the first succeeding secondary mark are read 302, 304, 306, and 308. Between any pair of secondary marks to the right of the primary mark numbered 4, there is only one tertiary mark. Hence, each smallest space represents five units. Thus the primary mark between the secondary marks representing 41 and 42 is read 415, that between

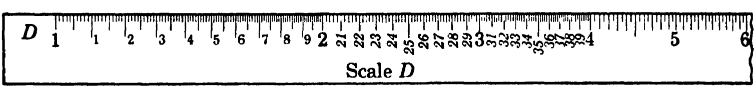

FIG. 3.

the secondary marks representing 55 and 56 is read 555, and the first tertiary mark to the right of the primary mark numbered 4 is read 405. The reading of any position between a pair of successive tertiary marks must be based on an estimate. Thus a position halfway between the tertiary marks associated with 222 and 224 is read 223, and a position two-fifths of the way from the tertiary mark numbered 415 to the next mark is read 417. The principle illustrated by these readings applies in all cases.

It is important to note that the decimal point has no bearing upon the position associated with a number on the *C* and *D* scales.

Consequently, the number G in Fig. 4 may be read 207, 2.07, 0.000207, 20,700, or any other number whose principal digits are 2, 0, and 7. The placing of the decimal point will be explained later in this chapter.

For a position between the primary marks numbered 1 and 2, four digits should be read; the first three will be exact and the last one estimated. No attempt should be made to read more than three digits for positions to the right of the primary mark numbered 4.

While making a reading, the learner should have definitely in mind the number associated with the smallest space under consideration. Thus between 1 and 2, the smallest division is associated with 10 in the fourth place; between 2 and 3, the smallest division has a

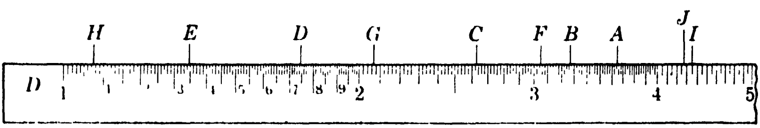

FIG. 4.

value 2 in the third place; while to the right of 4, the smallest division has a value 5 in the third place.

The learner should read from Fig. 4 the numbers associated with the marks lettered A, B, C, . . . and compare his readings with the following numbers: A 365, B 327, C 263, D 1745, E 1347, F 305, G 207, H 1078, I 435, J 427.

113. Accuracy of the slide rule. From the discussion of §112, it appears that we read four figures of a result on one part of the scale and three figures on the remaining part. This means an attainable accuracy of roughly one part in 1000 or one-tenth of 1 per cent. The accuracy is nearly proportional to the length of the scale. Hence we associate with the 20-in scale an accuracy of about one part in 2000, and with the Thacher Cylindrical slide rule, an accuracy of about one part in 10,000. The accuracy obtainable with the 10-in. slide rule is sufficient for most practical purposes; in any case the slide rule result serves as a check.

114. Definitions. The central sliding part of the rule is called the *slide*, the other part, the *body*. The glass runner is called the

indicator, and the line on the indicator is referred to as the *hairline.*

The mark associated with the primary number 1 on any scale is called the *index* of the scale. An examination of the *D* scale shows that it has two indices, one at the left end and the other at the right end.

Two positions on different scales are said to be *opposite* if, without moving the slide, the hairline may be brought to cover both positions at the same time.

115. Multiplication. The process of multiplication may be performed by using scales *C* and *D*. The *C* scale is on the slide, but in other respects it is like the *D* scale and is read in the same manner.

To multiply 2 by 4,

to 2 on *D* set index of *C*,
push hairline to 4 on *C*,
at the hairline read **8** on *D*.

Figure 5 shows the setting in skeleton form.

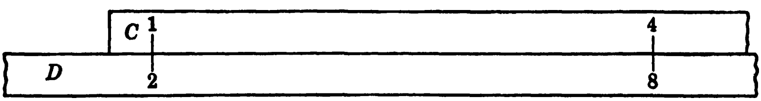

FIG. 5.

To multiply 3×3,

to 3 on *D* set index of *C*,
push hairline to 3 on *C*,
at the hairline read **9** on *D*.

See Fig. 6 for the setting in skeleton form.

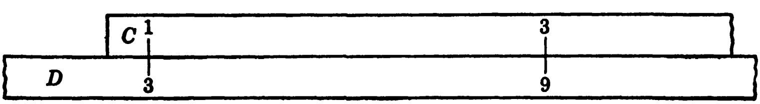

FIG. 6.

To multiply 1.5×3.5, disregard the decimal point and

to 15 on *D* set index of *C*,
push hairline to 35 on *C*,
at the hairline read **525** on *D*.

By inspection we know that the answer is near 5 and is therefore **5.25.**

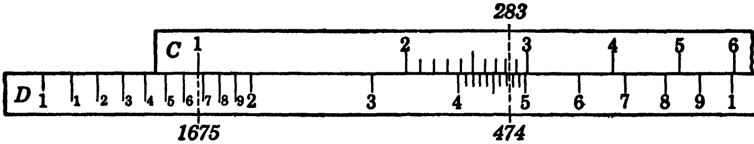

FIG. 7.

To find the value of 16.75×2.83 (see Fig. 7) disregard the decimal point and

> to 1675 on *D* set index of *C*,
> push hairline to 283 on *C*,
> at the hairline read **474** on *D*.

To place the decimal point we approximate the answer by noting that it is near to $3 \times 16 = 48$. Hence the answer is **47.4.**

These examples illustrate the use of the following rule.

Rule. *To find the product of two numbers: To either number on scale D set index of scale C, push hairline to second number on scale C, at the hairline read product on scale D.* Disregard the decimal point while making the settings and readings; finally place the decimal point in accordance with the result of a rough approximation.

EXERCISES

1. 3×2.

2. 3.5×2.

3. 5×2.

4. 2×4.55.

5. 4.5×1.5.

6. 1.75×5.5.

7. 4.33×11.5.

8. 2.03×167.3.

9. 1.536×30.6.

10. 0.0756×1.093.

11. 1.047×3080.

12. 0.00205×408.

13. $(3.142)^2$.

14. $(1.756)^2$.

116. Either index may be used. It may happen that a product cannot be read when the left index of the *C* scale is used in the rule of §115. This will be due to the fact that the second number of the product is on the part of the slide projecting beyond the body. In this case reset the slide using the right index of the *C* scale in place of the left, or use the following rule:

When a number is to be read on the D scale opposite a number on the slide scale and cannot be read, push the hairline to the index of the C scale inside the body and draw the other index of the C scale under the hairline. The desired reading can then be made. This very important rule applies generally.

If, to find the product of 2 and 6, we set the left index of the *C* scale opposite 2 on the *D* scale, we cannot read the answer on the *D* scale opposite 6 on the *C* scale. Hence, we set the right index of *C* opposite 2 on *D*; opposite 6 on *C* read the answer, **12,** on *D*.

Again, to find 0.0314 × 564,

to 314 on *D* set the right index of *C*,
push hairline to 564 on *C*,
at the hairline read **1771** on *D*.

A rough approximation is obtained by finding 0.03 × 600 = *18*. Hence the product is **17.71.**

EXERCISES

Perform the indicated multiplications.

1. 3 × 5.
2. 3.05 × 5.17.
3. 5.56 × 634.
4. 743 × 0.0567.
5. 0.0495 × 0.0267.
6. 1.876 × 926.
7. 1.876 × 5.32.
8. 42.3 × 31.7.

117. Division. The process of division is performed by using the *C* and *D* scales.

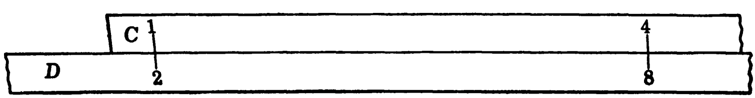

FIG. 8.

To divide 8 by 4 (see Fig. 8)

push hairline to 8 on *D*,
draw 4 of *C* under the hairline,
opposite index of *C* read **2** on *D*.

To divide 876 by 20.4,

push hairline to 876 on *D*,
draw 204 of *C* under the hairline,
opposite index of *C* read **429** on *D*.

The rough calculation $800 \div 20 = 40$ shows that the decimal point must be placed after the 2. Hence the answer is **42.9.**

EXERCISES

Perform the indicated operations.

1. $87.5 \div 37.7$.

2. $3.75 \div 0.0227$.

3. $0.685 \div 8.93$.

4. $1029 \div 9.70$.

5. $0.00377 \div 5.29$.

6. $2875 \div 37.1$.

7. $871 \div 0.468$.

8. $0.0385 \div 0.001462$.

9. $3.14 \div 2.72$.

10. $3.42 \div 81.7$.

118. Use of scales DF and CF (folded scales). If your slide rule contains folded scales, they may often be used to save using the italicized rule of §116 to move the slide its own length leftward or rightward. These folded scales are used precisely like the other scales. The following rule will indicate how one may transfer operations from the *C* and *D* scales to the *CF* and *DF* scales.

Rule. *Shifting an operation from the C and D scales to the CF and DF scales or vice versa may be made whenever the process is pushing the hairline to a number, never when a number on the slide is to be drawn under the hairline.*

For example, to find 2×6,

> to 2 on *D* set left index of *C*,
> push hairline to 6 on *CF*,
> at the hairline read **12** on *DF*.

To find 6.17×7.34,

> to 617 on *DF* set index of *CF*,
> push hairline to 734 on *C*,
> at the hairline read **45.3** on *D*.

By using the *CF* and *DF* scales we saved the trouble of moving the slide as well as the attendant source of error. This saving, entering as it does in many ways, is a main reason for using the folded scales.

The folded scales may be used to perform multiplications and divisions just as the *C* and *D* scales are used. Thus, to find 6.17×7.34,

to 617 on *DF* set index of *CF*,
push hairline to 734 on *CF*,
at the hairline read **45.3** on *DF*;

or

to 617 on *DF* set index of *CF*,
push hairline to 734 on *C*,
at the hairline read **45.3** on *D*.

Again to find the quotient 7.68/8.43,

push hairline to 768 on *DF*,
draw 843 of *CF* under the hairline,
opposite the index of *CF* read **0.912** on *DF*;

or

push hairline to 768 on *DF*,
draw 843 of *CF* under the hairline,
opposite the index of *C* read **0.912** on *D*.

It now appears that we may perform a multiplication or a division in several ways by using two or more of the scales *C*, *D*, *CF*, and *DF*. The sentence written in italics near the beginning of the article sets forth the guiding principle. A convenient method of multiplying or dividing a number by π ($= 3.14$ approx.) is based on the statement: any number on *DF* is π times its opposite on *D*, and any number on *D* is $1/\pi$ times its opposite on *DF*.

EXERCISES

Perform each of the operations indicated in exercises 1 to 11 in four ways; *first* by using the *C* and *D* scales only; *second* by using the *CF* and *DF* scales only; *third* by using the *C* and *D* scales for the initial setting and the *CF* and *DF* scales for completing the solution; *fourth* by using the *CF* and *DF* scales for the initial setting and the *C* and *D* scales for completing the solution.

1. 5.78×6.35.
2. 7.84×1.065.
3. $0.00465 \div 73.6$.
4. $0.0634 \times 53{,}600$.
5. $1.769 \div 496$.
6. $946 \div 0.0677$.
7. 813×1.951.
8. $0.00755 \div 0.338$.
9. $0.0948 \div 7.23$.
10. $149.0 \div 63.3$.
11. $2.718 \div 65.7$.
12. 783π.
13. $783 \div \pi$.
14. 0.0876π.
15. $0.504 \div \pi$.
16. $1.072 \div 10.97$.

119. The proportion principle. The proportion principle is very important because settings can be devised and recalled by using it. *When the slide is set in any position, the ratio of any number on the D scale to its opposite on the C scale is the same as the ratio of any other number on D to its opposite on C.* This is true because each ratio, in accordance with the setting for division is equal to the number on D opposite the index of C. For example, draw 1 of C opposite 2 on D and find the opposites indicated in the following table:

C (or CF)	1	1.5	2 5	3	4	5
D (or DF)	2	3	5	6	8	10

Now consider the proportion

$$\frac{x}{56} = \frac{9}{7}. \tag{1}$$

If 9 on C be set opposite 7 on D, then x will appear on C opposite 56 on D. Hence, to find x in (1),

push hairline to 7 on D,
draw 9 of C under the hairline,
push hairline to 56 on D,
at the hairline read **72** on C,

or

push hairline to 9 on D,
draw 7 of C under the hairline,
push hairline to 56 on C,
at the hairline read **72** on D.

Again consider the continued proportion

$$\frac{C}{D}: \quad \frac{3.15}{5.29} = \frac{x}{4.35} = \frac{57.6}{y} = \frac{z}{183.4}.$$

Observe that 3.15/5.29 is the known ratio, and

push hairline to 529 on D,
draw 315 of C under the hairline;
opposite 435 on D, read x = **2.59** on C,
opposite 576 on C, read y = **96.7** on D,
opposite 1834 on D, read z = **109.2** on C.

The positions of the decimal points were determined by noticing that each denominator had to be approximately twice its numerator since 5.29 is approximately twice 3.15. The position of the decimal point is always determined by a rough approximation.

Whenever an answer cannot be read because the slide projects beyond the body, use the italicized rules of §§116 and 118.

EXERCISES

Find, in each of the following equations, the values of the unknowns.

1. $\dfrac{2}{3} = \dfrac{x}{7.83}$.

2. $\dfrac{x}{1.804} = \dfrac{y}{25} = \dfrac{1}{0.785}$.

3. $\dfrac{x}{709} = \dfrac{246}{y} = \dfrac{28}{384}$.

4. $\dfrac{x}{0.204} = \dfrac{y}{0.506} = \dfrac{5.28}{z} = \dfrac{2.01}{0.1034}$.

5. $\dfrac{x}{2.07} = \dfrac{3}{61.3} = \dfrac{z}{1.571}$.

6. $\dfrac{8.51}{1.5} = \dfrac{9}{x} = \dfrac{235}{y}$.

7. $\dfrac{17}{x} = \dfrac{1.365}{8.53} = \dfrac{4.86}{y}$.

8. $\dfrac{x}{y} = \dfrac{y}{7.34} = \dfrac{3.75}{29.7}$.

9. $\dfrac{x}{49.6} = \dfrac{z}{y} = \dfrac{y}{3.58} = \dfrac{1.076}{0.287}$.

120. Use of the *CI* scale. The scale marked *CI* is designed so that *when the hairline is set to a number on the CI scale, its reciprocal* (1 divided by the number) *is set on the C scale.* Accordingly this scale may be used to deal with reciprocals. Thus, to find x when

$$x = 415 \times 1.87 \times 2.54,$$

divide through by 415 and replace 2.54 by 1 ÷ (1/2.54) to get

$$\frac{D}{C}: \qquad \frac{x}{415} = \frac{1.87}{1/2.54}.$$

Hence, in accordance with the proportion principle,

push hairline to 1.87 on D,
draw 2.54 of CI under the hairline,
push hairline to 415 on C,
at the hairline read $x =$ **1970** on D.

Observe that 1/2.54 of C was drawn under the hairline indirectly by drawing 2.54 on CI under the hairline. If one keeps in mind the italicized statement he will find that he can multiply by the reciprocal of a number, divide by it, or use it in a proportion by using the CI scale for the number instead of the C scale. The same principle governs the use of the CIF scale.

EXERCISES

In each of the following equations find the value of the unknown:

1. $\dfrac{y}{28} = \dfrac{3.2}{\frac{1}{118}}$.

2. $\dfrac{y}{42} = \dfrac{39.2}{\frac{1}{56}}$.

3. $y = 25(\frac{1}{742})$.

4. $y = 74.5\left(\dfrac{1}{42.3}\right)$.

5. $y = (321)(46.2)(4.93)$.

6. $y = (62)(49)(82)$.

7. $(36.2)(47.2)y = 3.8$.

8. $y = \dfrac{3.41}{(1.72)(6.31)}$.

9. $y = \dfrac{(6.72)}{(5.81)(6.43)}$.

10. $y = (\frac{1}{8})(14)(\frac{1}{15})$.

121. Combined multiplication and division. The importance of this article is secondary only to §119, which relates to the proportion principle.

Example 1. Find the value of $\dfrac{7.36 \times 8.44}{92}$.

Solution. Reason as follows: first divide 7.36 by 92, and then multiply the result by 8.44. This would suggest that we

push hairline to 736 on D,
draw 92 of C under the hairline;
opposite 8.44 on C, read **0.675** on D.

Example 2. Find the value of $\dfrac{18 \times 45 \times 37}{23 \times 29}$.

Solution. Reason as follows: (a) divide 18 by 23, (b) multiply the result by 45, (c) divide this second result by 29, (d) multiply this third result by 37. This argument suggests that we

push hairline to 18 on D,
draw 23 of C under the hairline,

push hairline to 45 on C,
draw 29 of C under the hairline,
push hairline to 37 on C,
at the hairline read **449** on D.

To determine the position of the decimal point write $\frac{20 \times 40 \times 40}{20 \times 30}$ = about 50. Hence the answer is **44.9.**

A little reflection on the procedure of Example 2 will enable the operator to evaluate by the shortest method expressions similar to the one just considered. He should observe that: the D scale was used only twice, once at the beginning of the process and once at its end; *the process for each number of the denominator consisted in drawing that number, located on the C scale, under the hairline; the process for each number of the numerator consisted in pushing the hairline to that number located on the C scale.*

If at any time the indicator cannot be placed because of the projection of the slide, apply the rule of §116, *or carry on the operations using the folded scales.*

Example 3. Find the value of $1.843 \times 92 \times 2.45 \times 0.584 \times 365$.

Solution. Write the given expression in the form

$$\frac{1.843 \times 2.45 \times 365}{(1/92)\,(1/0.584)}$$

and reason as follows: (*a*) divide 1.843 by (1/92), (*b*) multiply the result by 2.45, (*c*) divide this second result by (1/0.584), (*d*) multiply the third result by 365. This argument suggests that we

push hairline to 1843 on D,
draw 92 of CI under the hairline,
push hairline to 245 on C,
draw 584 of CI under the hairline,
push hairline to 365 on C,
at the hairline read **886** on D.

To approximate the answer we write 2(90) (5/2) (6/10) 300 = 81,000. Hence the answer is **88,600.**

EXERCISES

1. $\dfrac{1375 \times 0.0642}{76{,}400}$.

2. $\dfrac{45.2 \times 11.24}{336}$.

3. $\dfrac{218}{4.23 \times 50.8}$.

4. $\dfrac{235}{3.86 \times 3.54}$.

5. $2.84 \times 6.52 \times 5.19$.

6. $9.21 \times 0.1795 \times 0.0672$.

7. $37.7 \times 4.82 \times 830$.

8. $\dfrac{65.7 \times 0.835}{3.58}$.

9. $\dfrac{362}{3.86 \times 9.61}$.

10. $\dfrac{24.1}{261 \times 32.1}$.

11. $\dfrac{75.5 \times 63.4 \times 95}{3.14}$.

12. $\dfrac{3.97}{51.2 \times 0.925 \times 3.14}$.

13. $\dfrac{47.3 \times 3.14}{32.5 \times 16.4}$.

14. $\dfrac{3.82 \times 6.95 \times 7.85 \times 436}{79.8 \times 0.0317 \times 870}$.

15. $187 \times 0.00236 \times 0.0768 \times 1047 \times 3.14$.

16. $\dfrac{0.917 \times 8.65 \times 1076 \times 3152}{7840}$.

122. Square roots. The square root of a given number is a second number whose square is the given number. Thus the square root of 4 is 2, and the square root of 9 is 3, or, using the symbol for square root, $\sqrt{4} = 2$, and $\sqrt{9} = 3$.

The A scale consists of two parts that differ only in slight details. We shall refer to the left-hand part as A *left* and to the right-hand part as A *right.* Similar reference will be made to the B scale.

Rule. *To find the square root of a number between* 1 *and* 10, *set the hairline to the number on scale A left and read its square root at the hairline on the D scale. To find the square root of a number between* 10 *and* 100, *set the hairline to the number on scale A right and read its square root at the hairline on the D scale.* In either case place the decimal point after the first digit. A similar statement relating to the B scale and the C scale holds true. For example, set the hairline to 9 on scale A left, read 3 ($= \sqrt{9}$) at the hairline on D, set the hairline to 25 on scale B right, read 5 ($= \sqrt{25}$) at the hairline on C.

To obtain the square root of any number, *move the decimal point an even number of places to obtain a number between* 1 *and* 100; *then apply the rule written above in italics; finally move the decimal point one half as many places as it was moved in the original number but in the opposite direction.** The learner may also place the decimal point in accordance with information derived from a rough approximation.

For example, to find the square root of 23,400, move the decimal point four places to the left, thus getting 2.34 (a number between 1 and 10); set the hairline to 2.34 on scale *A* left; read 1.530 at the hairline on the *D* scale; finally, move the decimal point $\frac{1}{2}$ of 4 or two places to the right to obtain the answer 153.0. The decimal point could have been placed after observing that $\sqrt{10,\bar{0}0\bar{0}} = 100$ or that $\sqrt{40,00\bar{0}} = 200$. Also, the left *B* scale and the *C* scale could have been used instead of the left *A* scale and the *D* scale.

To find $\sqrt{3850}$, move the decimal point two places to the left to obtain $\sqrt{38.50}$; set the hairline to 38.50 on scale *A* right; read 6.20 at the hairline on the *D* scale; move the decimal point one place to the right to obtain the answer **62.0.** The decimal point could have been placed by observing that $\sqrt{3600} = 60$.

To find $\sqrt{0.\bar{0}0\bar{0}585}$, move the decimal point four places to the right to obtain $\sqrt{5.\bar{8}5}$; find $\sqrt{5.85} = 2.42$; move the decimal point two places to the left to obtain the answer **0.0242.**

EXERCISES

1. Find the square root of each of the following numbers: 8, 12, 17, 89, 8.90, 890, 0.89, 7280, 0.0635, 0.0000635, 63,500, 100,000.

2. Find the length of the side of a square whose area is (*a*) 53,500 ft.^{2}; (*b*) 0.0776 ft.^{2}; (*c*) 3.27×10^7 ft.^{2}

3. Find the diameter of a circle having area (*a*) 256 ft.^{2}; (*b*) 0.773 ft.^{2}; (*c*) 1950 ft.^{2}

123. Combined operations involving square roots. When the hairline is set to a number on the *B* scale it is automatically set on the *C* scale to the square root of the number. Therefore the

* The following rule may also be used: If the square root of a number greater than unity is desired, use *A* left when it contains an odd number of digits to the left of the decimal point; otherwise use *A* right. For a number less than unity use *A* left if the number of zeros immediately following the decimal point is odd; otherwise, use *A* right.

B scale can be used in combined operations like the *CI* scale. Naturally, the rule for square-root settings should be used to determine whether *B* left or *B* right is to be used in any particular case. The following example will illustrate the method of procedure.

Example. Evaluate $\dfrac{\sqrt{832} \times \sqrt{365} \times 1863}{(\frac{1}{736}) \times 89{,}400}$.

Solution. In accordance with italicized statement of §121,

push hairline to 832 on *A* left,
draw 736 of *CI* under the hairline,
push hairline to 365 on *B* left,
draw 894 of *C* under the hairline,
push hairline to 1863 on *CF*,
at the hairline read **8450** on *DF*.

The method of finding cube roots is much like that of finding square roots. The following rule may be used:

Rule. *To obtain the cube root of a number, move the decimal point over three places (or digits) at a time until a number between* 1 *and* 1000 *is obtained. Then push the hairline to the new number on K left, K middle, or K right according as it lies between* 1 *and* 10, 10 *and* 100, *or* 100 *and* 1000. *Read the cube root on scale D at the hairline and place the decimal point after the first digit. Then move the decimal point one-third as many places as it was moved in the original number but in the opposite direction.*

EXERCISES

1. $\dfrac{7.87 \times \sqrt{377}}{2.38}$.

2. $\dfrac{86 \times \sqrt{734} \times \pi}{775 \times \sqrt{0.685}}$.

3. $\dfrac{4.25 \times \sqrt{63.5} \times \sqrt{7.75}}{0.275 \times \pi}$.

4. $\dfrac{(2.60)^2}{2.17 \times 7.28}$.

5. $\dfrac{20.6 \times 7.89^2 \times 6.79^2}{4.67^2 \times 281}$.

6. $\dfrac{189.7 \times \sqrt{0.00296} \times \sqrt{347} \times 0.274}{\sqrt{2.85} \times 165 \times \pi}$.

7. $\sqrt{285} \times 667 \times \sqrt{6.65} \times 78.4 \times \sqrt{0.00449}$.

8. $\dfrac{239 \times \sqrt{0.677} \times 374 \times 9.45 \times \pi}{84.3 \times \sqrt{9350} \times \sqrt{28400}}$.

124. The S (sine) and ST (sine tangent) scales. The numbers on the sine scales S and ST* represent angles. In order to set the indicator to an angle on the sine scales it is necessary to determine the value of the angles represented by the subdivisions. Thus, since there are six primary intervals between 4° and 5°, each represents 10′; since each of the primary intervals is subdivided into five secondary intervals, each of the latter represents 2′. Again, since there are five primary intervals between 20° and 25°, each represents 1°; since each primary interval here is subdivided into two secondary intervals, each of the latter represents 30′; since each secondary interval is subdivided into three parts, these smallest intervals represent 10′. These illustrations indicate the manner in which the learner should analyze the part of the scale involved to find the value of the smallest interval to be con-

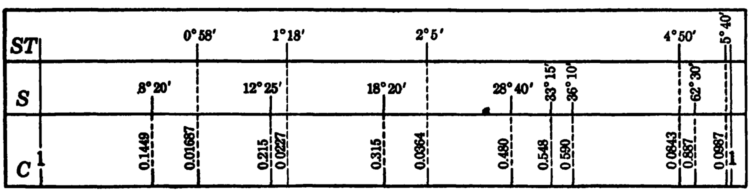

FIG. 9.

sidered. In general, when setting the hairline to an angle, the student should always have in mind the value of the smallest interval on the part of the slide rule under consideration.

When the indicator is set to a black number (angle) on scale S or ST, the sine of the angle is on scale C at the hairline and hence on scale D when the indices on scales C and D coincide.

When scale C is used to read sines of angles on ST, the left index of C is taken as 0.01, the right index as 0.1. In reading sines of angles on S, the left index of C is taken as 0.1, the right index as 1. Thus, to find sin 36°26′, opposite 36°26′ on scale S, read 0.594 on scale C; to find sin 3°24′, opposite 3°24′ on scale ST, read 0.0593 on scale C. Figure 9 shows scales ST, S, and C on which certain angles and their sines are indicated. As an exercise, read from your slide rule the sines of the angles shown in the figure and compare your results with those given.

* The ST scale is a sine scale, but since it is also used as a tangent scale it is designated ST.

EXERCISES

1. By examination of the slide rule verify that on the S scale from the left index to 16° the smallest subdivision represents 5′; from 16° to 30° it represents 10′; from 30° to 60° it represents 30′; from 60° to 80° it represents 1°; and from 80° to 90° it represents 5°.

2. Find the sine of each of the following angles:

(*a*) 30°.	(*c*) 3°20′.	(*e*) 87°45′.	(*g*) 14°38′.	(*i*) 11°48′.
(*b*) 38°.	(*d*) 90°.	(*f*) 1°35′.	(*h*) 22°25′.	(*j*) 51°30′.

3. Find the cosine of each of the angles in Exercise 2 by using the relation $\cos \varphi = \sin (90° - \varphi)$.

4. For each of the following values of x,

(*a*) 0.5,	(*c*) 0.375,	(*e*) 0.015,	(*g*) 0.062,	(*i*) 0.92,
(*b*) 0.875,	(*d*) 0.1,	(*f*) 0.62,	(*h*) 0.031,	(*j*) 0.885,

find the value of φ less than 90°, (A) if $\varphi = \sin^{-1} x$, where $\sin^{-1} x$ means "the angle whose sine is x"; (B) if $\varphi = \cos^{-1} x$.

5. Find the cosecant of each of the angles in Exercise 2 by using the relation $\csc \varphi = \dfrac{1}{\sin \varphi}$.

Hint. Set the angle on S, read the cosecant on CI (or on DI when the rule is closed).

6. Find the secant of each of the angles in Exercise 2 by using the relation $\sec \varphi = \dfrac{1}{\cos \varphi}$.

7. For each of the following values of x,

(*a*) 2.	(*b*) 2.4.	(*c*) 1.7.	(*d*) 6.12.	(*e*) 80.2.	(*f*) 4.72.

find the value of φ less than 90°, (A) if $\varphi = \csc^{-1} x$; (B) if $\varphi = \sec^{-1} x$.

125. The T (tangent) scale. *When the indicator is set to a black angle on scale T, the tangent of the angle is on scale C at the hairline* and hence on scale D when the indices of scales T and D coincide. Also *when the indicator is set to a black angle on scale T, the cotangent of the angle is on scale CI at the hairline.* Thus, to find tan 36°, push the hairline to 36° on T; at the hairline read **0.727** on C. To find cot 27°10′, push the hairline to 27°10′ on T; at the hairline read **1.949** on CI.

When scale C is used to read tangents, the left index is taken as 0.1 and the right index as 1.0. Only those angles that range

from 5°43′ to 45° appear on scale T. It is shown in trigonometry that for angles less than 5°43′, the sine and tangent are approximately equal. Hence, so far as the slide rule is concerned, the tangent of an angle less than 5°43′ may be replaced by the sine of the angle. Thus to find tan 2°15′, push the hairline to 2°15′ on ST, at the hairline read **0.0393** on C. To find the tangent of an angle greater than 45°, use the relation

$$\cot \theta = \tan (90° - \theta).$$

To find tan 56°, push the hairline to 34° (= 90° − 56°) on T, at the hairline read **1.483** on CI. The student should observe that he could have set the hairline to 56° in red on the T scale and thus have avoided subtracting 34° from 90°.

EXERCISES

1. Fill out the following table:

φ	8°6′	27°15′	62°19′	1°7′	74°15′	87°	47°28′
tan φ							
cot φ							

2. The following numbers are tangents of angles. Find the angles.

(*a*) 0.24.	(*d*) 0.54.	(*g*) 0.432.	(*j*) 0.374.	(*m*) 17.01.
(*b*) 0.785.	(*e*) 0.059.	(*h*) 0.043.	(*k*) 3.72.	(*n*) 1.03.
(*c*) 0.92.	(*f*) 0.082.	(*i*) 0.0149.	(*l*) 4.67.	(*o*) 1.232.

3. The numbers in Exercise 2 are cotangents of angles. Find the angles.

126. Combined operations. The method for evaluating expressions involving combined operations as stated in §§121 and 123 applies without change when some of the numbers are trigonometric functions. This is illustrated in the following example.

Example. Evaluate $\dfrac{6.1\sqrt{17}\,\sin 72°\,\tan 20°}{2.2}$.

Solution. Write

$$\frac{\sqrt{17}\ \sin 72^\circ\ \tan 20^\circ}{2.2\left(\frac{1}{6.1}\right)}.$$

Push hairline to 17 on A right,
draw 2.2 of C under the hairline,
push hairline to 20° on T,
draw 6.1 of CI under the hairline,
push hairline to 72° on S,
at the hairline read **3.96** on D.

EXERCISES

Evaluate the following:

1. $\dfrac{18.6 \sin 36^\circ}{\sin 21^\circ}$.

2. $\dfrac{32 \sin 18^\circ}{27.5}$.

3. $\dfrac{4.2 \tan 38^\circ}{\sin 45^\circ 30'}$.

4. $\dfrac{34.3 \sin 17^\circ}{\tan 22^\circ 30'}$.

5. $\dfrac{13.1 \cos 40^\circ}{\tan 35^\circ 10'}$.

6. $\dfrac{17.2 \cos 35^\circ}{\cot 50^\circ}$.

7. $\dfrac{7.8 \csc 35^\circ 30'}{\cot 21^\circ 25'}$.

8. $\dfrac{63.1 \sec 80^\circ}{\tan 55^\circ}$.

9. $\dfrac{\sin 18^\circ \tan 20^\circ}{3.7 \tan 41^\circ \sin 31^\circ}$.

10. $\dfrac{\sin 26^\circ 25'}{8.1 \tan 22^\circ 18'}$.

11. $3.14 \sin 13^\circ 10' \csc 32^\circ$.

12. $7.1\pi \sin 47^\circ 35'$.

13. $\dfrac{0.61 \csc 12^\circ 15'}{\cot 35^\circ 16'}$.

14. $\dfrac{1 \sin 22^\circ 40'}{\tan 28^\circ 10'}$.

15. $\dfrac{3.1 \sin 61^\circ 35' \csc 15^\circ 18'}{\cos 27^\circ 40' \cot 20^\circ}$.

16. $\dfrac{13.1 \sin 3^\circ 7'}{\tan 30^\circ 10'}$.

17. $\dfrac{0.0037 \sin 49^\circ 50'}{\tan 2^\circ 6'}$.

18. $\dfrac{\sqrt{16.5}\ \sin 45^\circ 30'}{\sqrt{4.6}\ 41.2 \cot 71^\circ 10'}$.

19. $\dfrac{\sqrt[3]{6.1}\ 4.91}{\tan 13^\circ 14'}$.

20. $\dfrac{\sin 51^\circ 30'}{(39.1)(6.28)}$.

21. $\dfrac{\csc 49^\circ 30'}{(19.1)(7.61)\sqrt{69.4}}$.

22. $(48.1)(1.68) \sin 39^\circ$.

23. $0.0121 \sin 81^\circ \cot 41^\circ$.

24. $\dfrac{1.01 \cos 71^\circ 10' \sin 15^\circ}{\sqrt{4.81}\ \cos 27^\circ 10'}$.

127. Solving a triangle by means of the law of sines. If the sides and angles of a triangle are lettered as indicated in Fig. 10,

the law of sines is written

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}. \tag{2}$$

This law is the basis of most slide-rule solutions of triangles.

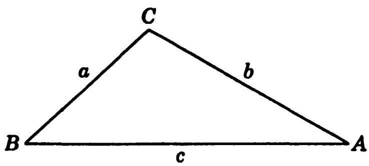

FIG. 10.

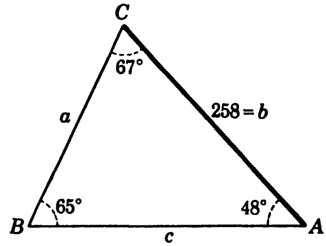

FIG. 11.

To solve the triangle shown in Fig. 11 for a and c, write

$$\frac{\sin 65^\circ}{258} = \frac{\sin 48^\circ}{a} = \frac{\sin 67^\circ}{c},$$

and, using the setting based on the proportion principle,

push hairline to 258 on D,
draw 65° of S under the hairline,
push hairline to 48° on S,
at the hairline read a = **212** on D,
push hairline to 67° on S,
at the hairline read c = **262** on D.

The decimal point was placed by inspection.

In general, to solve **any** triangle in which a side and the angle opposite are known,

push hairline to known side on D,
draw opposite angle of S under the hairline,
push hairline to any known side on D,
at the hairline read opposite angle on S,
push hairline to any known angle on S,
at the hairline read opposite side on D.

When an angle A of a triangle is greater than 90°, replace it by $180^\circ - A$. This is permissible since $\sin(180^\circ - A) = \sin A$. When the decimal point in a result cannot be placed by inspection, compute the part involved approximately by using (2) with the trigonometric functions replaced by their natural values.

When the given parts of a triangle are two sides and the angle opposite one of them, there may be two solutions. For example, if the given parts are $a = 175$, $b = 215$, $A = 35°30'$, the two

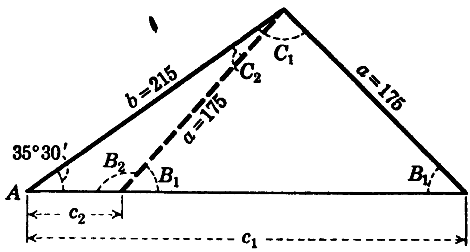

FIG. 12.

possible triangles are shown in Fig. 12. Using the setting (2) of §127,

push hairline to 175 on D,
draw 35°30′ of S under the hairline,
push hairline to 215 on D,
at the hairline read $B_1 =$ **45°30′** on S.
Compute $C_1 = 180° - 35°30' - 45°30' = 99°$
push hairline to 81° (= 180° − 99°) on S,
at the hairline read $c_1 =$ **298** on D,
Compute $C_2 = B_1 - 35°30' =$ **10°**,
push hairline to 10° on S,
at the hairline read $c_2 =$ **52.3** on D.

EXERCISES

Solve the following oblique triangles.

1. $a = 50$, $A = 65°$, $B = 40°$.

2. $c = 60$, $A = 50°$, $B = 75°$.

3. $a = 550$, $A = 10°12'$, $B = 46°36'$.

4. $a = 222$, $b = 4570$, $C = 90°$.

5. $a = 120$, $b = 80$, $A = 60°$.

6. $b = 0.234$, $c = 0.198$, $B = 109°$.

7. $a = 795$, $A = 79°59'$, $B = 44°41'$.

8. $a = 21$, $A = 4°10'$, $B = 75°$.

9. $b = 91.1$, $c = 77$, $B = 51°7'$.

10. $a = 50$, $c = 66$, $A = 123°11'$.

11. $a = 8.66$, $c = 10$, $A = 59°57'$.

12. $b = 8$, $a = 120$, $A = 60°$.

13. A ship at point S can be seen from each of two points, A and B, on the shore. If $AB = 800$ ft., angle $SAB = 67°43'$, and angle $SBA = 74°21'$, find the distance of the ship from A.

14. To determine the distance of an inaccessible tower A from a point B, a line BC and the angles ABC and BCA were measured and found to be 1000 yd., 44°, and 70°, respectively. Find the distance AB.

Solve the following oblique triangles.

15. $a = 18$, $b = 20$, $A = 55°24'$.

16. $b = 19$, $c = 18$, $C = 15°49'$.

17. $a = 32.2$, $c = 27.1$, $C = 52°24'$.

18. $b = 5.16$, $c = 6.84$, $B = 44°3'$.

19. $a = 177$, $b = 216$, $A = 35°36'$.

20. $a = 17{,}060$, $b = 14{,}050$, $B = 40°$.

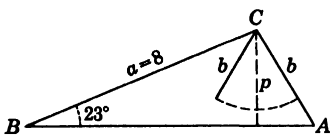

FIG. 13.

21. Find the length of the perpendicular p for the triangle of Fig. 13. How many solutions will there be for triangle ABC if (*a*) $b = 3$? (*b*) $b = 4$? (*c*) $b = p$?

128. To solve a right triangle when two legs are given. When the two legs of a right triangle are the given parts, first find the smaller acute angle from its tangent, and then apply the law of sines to complete the solution.

Example. Solve the right triangle of Fig. 14 in which $a = 3.18$, $b = 4.24$.

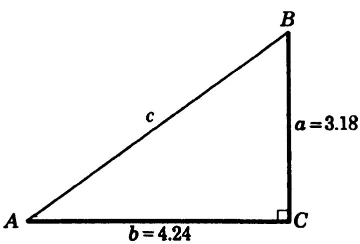

FIG. 14.

Solution. From the triangle read $\tan A = \left(\frac{3.18}{4.24}\right)$, and write this equation in the form

$$\frac{\tan A}{3.18} = \frac{1}{4.24}.$$

Using the setting based on the principle of proportion,

set index of C to 4.24 on D,
push hairline to 3.18 on D,
at the hairline read A = **36°52′** on T.

Since angle $A = 36°52'$ and $a = 3.18$, we know a pair of opposite parts and may proceed to use the law of sines. Since the hairline

is on 3.18 of D from the setting just made,

> draw 36°52′ of S under the hairline,
> at index of C read c = **5.31** on D.
> Evidently $B = 90° - A =$ **53°8′**.

The following rule states the method of solution.

Rule. *To solve a right triangle for which two legs are given,*

> *set index of C to larger leg on D,*
> *push hairline to smaller leg on D,*
> *at the hairline read the smaller acute angle on T,*
> *draw this angle on S under the hairline,*
> *at index of slide read hypotenuse on D.*

EXERCISES

Solve the following right triangles:

1. $a = 12.3$, $b = 20.2$.

2. $a = 101$, $b = 116$.

3. $a = 50$, $b = 23.3$.

4. $a = 273$, $b = 418$.

5. $a = 28$, $b = 34$.

6. $a = 12$, $b = 5$.

7. $a = 13.2$, $b = 13.2$.

8. $a = 42$, $b = 71$.

9. $a = 0.31$, $b = 4.8$.

129. To solve a triangle in which two sides and the included angle are given. The method here explained will consist in dividing the given triangle into two right triangles by means of an altitude to one of the known sides and then solving the two right triangles separately. The method is illustrated in the following example.

Example. Solve the triangle of Fig. 15 in which $a = 6.18$, $b = 9.27$, $C = 32°$.

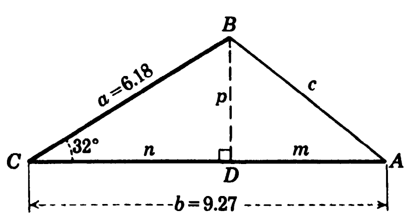

FIG. 15.

Solution. Draw the altitude BD to side AC, and observe that angle $BCD = 90°$ and $a = 6.18$ are known. Hence use the italicized rule of §127 and

set index of C to 6.18 on D,
push hairline to 32° on S,
at the hairline read $p = 3.27$ on D,
opposite 58° (= 90° − 32°) on S read $n = 5.24$ on D,
compute $m = 9.27 - 5.24 = 4.03$.

To solve triangle ABD, use the italicized rule of §128. Hence

set index of C to 4.03 on D,
push hairline to 3.27 on D,
at the hairline read A = **39°3′** on T,
draw 39°3′ on S under the hairline,
at index of C read c = **5.19** on D.
Evidently $B = 180° - 32° - 39°3'$ = **108°57′**.

If the given angle is obtuse the altitude lies outside the triangle, but the method is essentially the same as that used in the solution above.

EXERCISES

Solve the following triangles

1. $a = 94,$ $b = 56,$ $C = 29°.$	**4.** $b = 2.30,$ $c = 3.57,$ $A = 62°.$	**7.** $a = 0.085,$ $c = 0.0042,$ $B = 56°30'.$
2. $a = 100,$ $c = 130,$ $B = 51°49'.$	**5.** $a = 27,$ $c = 15,$ $B = 46°.$	**8.** $a = 17,$ $b = 12,$ $C = 59°18'.$
3. $a = 235,$ $b = 185,$ $C = 84°36'.$	**6.** $a = 6.75,$ $c = 1.04,$ $B = 127°9'.$	**9.** $b = 2580,$ $c = 5290,$ $A = 138°21'.$

10. The two diagonals of a parallelogram are 10 and 12 and they form an angle of 49°18′. Find the length of each side.

11. Two ships start from the same point at the same instant. One sails due north at the rate of 10.44 miles per hour, and the other due northeast at the rate of 7.71 miles per hour. How far apart are they at the end of 40 min.?

130. To solve a triangle in which three sides are given. When three sides of a triangle are given, one angle may be found

by using the law of cosines,

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

and the other parts may then be found by means of the law of sines.

Example. Solve the triangle of Fig. 16 in which $a = 15$, $b = 18$, $c = 20$.

Solution. From the law of cosines we write

$$\frac{\cos A}{1} = \frac{b^2 + c^2 - a^2}{2bc} = \frac{18^2 + 20^2 - 15^2}{2 \times 18 \times 20} = \frac{499}{720}.$$

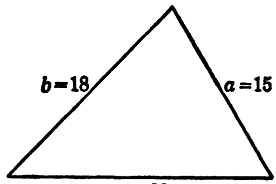

FIG. 16.

Hence, using a setting based on the proportion principle,

to 720 on D set 499 of C,
at index of D read $A =$ **46°6′** on S (red).

Now complete the solution by means of the law of sines to obtain $B =$ **59°54′**, $C =$ **74°**. When all three angles are read from the slide rule, the relation $A + B + C = 180°$ may be used as a check. Thus, for the solution just completed,

$$A + B + C = 46°6' + 59°54' + 74° = 180°.$$

EXERCISES

Solve the following triangles:

1. $a = 3.41$, $b = 2.60$, $c = 1.58$.	**3.** $a = 35$, $b = 38$, $c = 41$.	**5.** $a = 97.9$, $b = 106$, $c = 139$.
2. $a = 111$, $b = 145$, $c = 40$.	**4.** $a = 61.0$, $b = 49.2$, $c = 80.5$.	**6.** $a = 57.9$, $b = 50.1$, $c = 35.0$.

131. To change radians to degrees or degrees to radians. Since π (= 3.1416 approx.) radians equal 180°, we may write

$$\frac{\pi}{180} = \frac{r \text{ (number of radians)}}{d \text{ (number of degrees)}}.$$

Hence

> *draw* π *on C opposite* 180 *on D,*
> *push hairline to d (number of degrees given) on D,*
> *at the hairline read number of radians on C,*
> *push hairline to r (number of radians given) on C,*
> *at the hairline read number of degrees on D.*

EXERCISES

1. Express the following angles in radians:

(*a*) 45°. (*d*) 180°. (*g*) 22°30′.
(*b*) 60°. (*e*) 120°. (*h*) 200°.
(*c*) 90°. (*f*) 135°. (*i*) 3000°.

2. Express the following angles in degrees:

(*a*) $\pi/3$ radians. (*c*) $\pi/72$ radian. (*e*) $20\pi/3$ radians.
(*b*) $3\pi/4$ radians. (*d*) $7\pi/6$ radians. (*f*) 0.98π radians.

3. Express in radians the following angles:

(*a*) 1°. (*c*) 1″. (*e*) 180°34′20″.
(*b*) 1′. (*d*) 10°11′. (*f*) 300°25′43″.

4. Find the following angles in degrees and minutes:

(*a*) $\frac{1}{10}$ radian; (*b*) $2\frac{1}{2}$ radians; (*c*) 1.6 radians; (*d*) 6 radians.

SPHERICAL TRIGONOMETRY

CHAPTER XIII

THE RIGHT SPHERICAL TRIANGLE

132. Introduction. Just as plane trigonometry has for its object the study of the relations existing among the sides and angles of a plane triangle, so spherical trigonometry has for its

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object the study of the relations connecting the sides and angles of a spherical triangle. Since the earth is approximately a sphere, this theory will apply when distances and directions on the earth are in question. Hence the subject of spherical trigonometry is basic in navigation, geodesy, and astronomy.

133. The spherical triangle. The circle in which a plane through the center of a sphere intersects the sphere is called a

great circle. As in plane geometry, an arc on a great circle is measured by the angle that it subtends at the center and will be expressed in degrees, minutes, and seconds.

A spherical triangle consists of three arcs of great circles that form the boundaries of a portion of a spherical surface. As in plane geometry, the vertices of the spherical triangle will be denoted by capital letters A, B, and C and the sides opposite by a, b, and c, respectively. The magnitude of an angle of a spherical triangle is that of the plane angle formed by tangents to the sides of the angle at its vertex. *In general, we shall consider only spherical triangles, each of whose sides and each of whose angles is less than* 180°.

The planes of the great circles belonging to a spherical triangle form a trihedral angle at the center of the sphere (see Fig. 1).

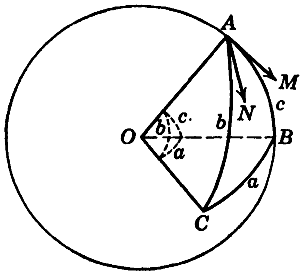

FIG. 1.

The face angles of this trihedral angle, being measured by their intercepted arcs, are designated by the same letters as the corresponding sides of the spherical triangle. The tangents to the arcs AB and AC at point A, being perpendicular to the radius OA, are the sides of the plane angle of dihedral angle M-AO-N. These tangents measure angle A of the spherical triangle ABC. Hence *an angle of the spherical triangle is measured by the dihedral angles made by the planes of its sides.*

Important propositions from solid geometry:

1. *The sum of the angles of a spherical triangle is greater than* 180° *and less than* 540°; *that is,* $180° < A + B + C < 540°$.

2. *If two angles of a spherical triangle are equal, the sides opposite are equal; and conversely.*

3. *If two angles of a spherical triangle are unequal, the sides opposite are unequal, and the greater side lies opposite the greater angle; and conversely.*

4. *The sum of two sides of a spherical triangle is greater than the third side.*

EXERCISES

1. If each angle of a spherical triangle is a right angle, what is the value of each side?

2. Show that if a spherical triangle has two right angles, the sides opposite these angles are quadrants and the third angle has the same measure as the opposite side.

3. The face angles of the trihedral angle associated with a spherical triangle are each 90° and the radius of the sphere is 10 in. Find the angles of the triangle in degrees, and find the sides both in degrees and in inches.

4. Find the magnitude of the face angles and of the dihedral angles of the trihedral angle associated with a spherical triangle whose sides are 90°, 90°, and 60°.

5. The face angles of a trihedral angle at the center of the earth are 50°, 60°38′, 45°50′20″. Find in nautical miles* the lengths of the sides of the associated spherical triangle on the surface of the earth.

6. Two great circles on a sphere intersect at an angle of 23°30′. Find the least great-circle distance from the pole of one to a point on the other.

7. What can be said regarding the size and shape of a spherical equiangular triangle if the sum of its angles is (*a*) nearly equal to 180°; (*b*) nearly equal to 540°?

8. Find all sides and angles of a spherical triangle having as angles $A = 90°$, $B = 90°$, and

(*a*) $C = 30°$.	(*c*) $C = 120°$.	(*e*) $C = 110°$.
(*b*) $C = 45°$.	(*d*) $C = 70°$.	(*f*) $C = 160°$.

9. Show that the sum of the angles of a right spherical triangle is greater than 180° and less than 360°.

134. Formulas relating to the right spherical triangle. Since spherical triangles having more than one right angle can be solved by inspection, we shall be concerned with right spherical triangles having only one right angle.

In this article, ten formulas relating to the right spherical triangle are derived, and in the next article simple rules for writing these formulas are given.

The solution of all cases of spherical triangles generally considered in spherical trigonometry can be solved by means of these formulas.

In Fig. 2 is represented a spherical pyramid that is part of a sphere having unit radius and center O. In the right spherical triangle ABC bounding the base of the pyramid, C is a right angle,

* A nautical mile is the length of an arc of a great circle on a sphere the size of the earth subtended by an angle of 1′ at its center.

and therefore the dihedral angle having edge OC is a right dihedral angle. From A, a plane is passed perpendicular to edge OB cutting the spherical pyramid in the triangle AED. Since OE is perpendicular to plane AED, it is perpendicular to lines EA and ED. Hence angle AED is the plane angle of the dihedral angle having OB as edge. Therefore angle AED is equal to angle B. Also, plane AED is perpendicular to plane COB, since it is perpendicular to a line in the plane. Therefore line AD is

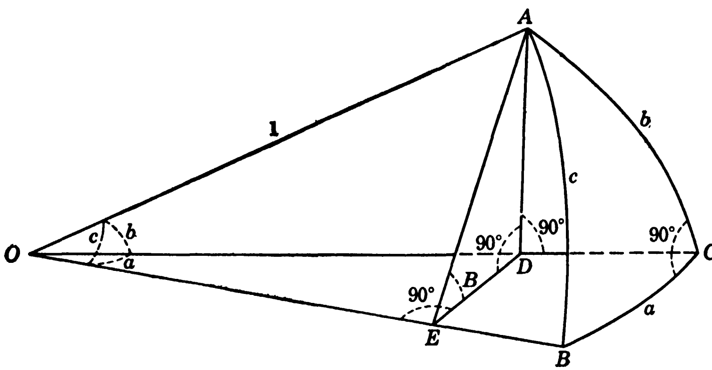

FIG. 2.

perpendicular to plane OBC because it is the intersection of the two planes OAD and ADE, both of which are perpendicular to OBC. Hence the angles ADO and ADE are right angles. Each face angle of the trihedral angle O-ABC is measured by the side of the spherical triangle intercepted by it and is therefore designated by the same letter as that side.

From Fig. 2 we read

$$\frac{DA}{1} = \sin b, \quad \frac{EA}{1} = \sin c, \quad \frac{OE}{1} = \cos c, \quad \frac{OD}{1} = \cos b. \qquad \text{(I)}$$

Also from triangle OED, $ED/OE = \tan a$. Replacing OE in this by $\cos c$ from (I) and simplifying slightly, we have

$$ED = OE \tan a = \cos c \tan a. \qquad \text{(II)}$$

Similarly, from triangle OED,

$$ED = OD \sin a = \cos b \sin a. \qquad \text{(III)}$$

Figure 3 is obtained from Fig. 2 by enlarging it and writing on it the values of the line segments just derived.

Both values for ED, one from (II) and the other from (III) are written on ED. From the triangle AED in Fig. 3, we read

$$\sin B = \frac{\sin b}{\sin c}, \tag{IV}$$

$$\cos B = \frac{\tan a \cos c}{\sin c},$$

$$\tan B = \frac{\sin b}{\sin a \cos b},$$

$$\tan a \cos c = \sin a \cos b.$$

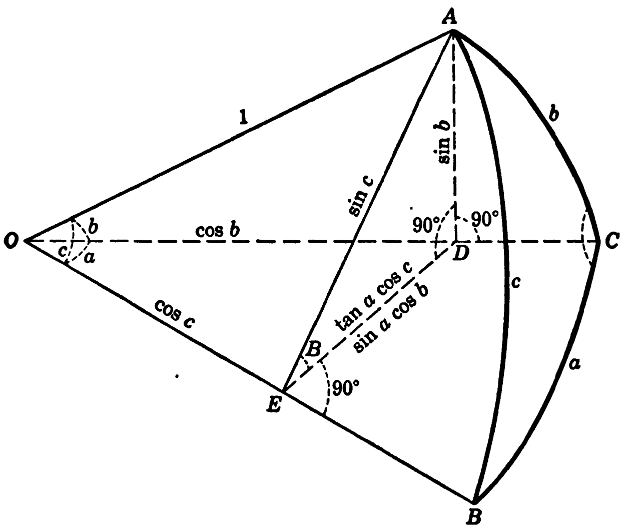

FIG. 3.

These last four equations may be written in the following form:

$$\sin b = \sin c \sin B, \tag{1}$$

$$\cos B = \tan a \cot c, \tag{2}$$

$$\sin a = \tan b \cot B, \tag{3}$$

$$\cos c = \cos a \cos b. \tag{4}$$

Similarly, by passing a plane through B of Fig. 2 perpendicular to OA and proceeding as above, we could prove the formulas

$$\sin a = \sin c \sin A, \tag{5}$$

$$\cos A = \tan b \cot c, \tag{6}$$

$$\sin b = \tan a \cot A. \tag{7}$$

Formulas (5), (6), and (7) are the result of interchanging a and b

and A and B in (1), (2), and (3), respectively. From (7) $\cot A = \sin b/\tan a$ and from (3) $\cot B = \sin a/\tan b$; multiplying these two equations member by member, we obtain

$$\cot A \cot B = \frac{\sin b}{\tan a} \times \frac{\sin a}{\tan b} = \cos b \cos a,$$

or, interchanging members and replacing $\cos b \cos a$ by $\cos c$ from (4),

$$\cos c = \cot A \cot B. \tag{8}$$

Similarly from (2), (5), and (4), we obtain

$$\cos B = \cos b \sin A \tag{9}$$

and from (6), (1), and (4),

$$\cos A = \cos a \sin B. \tag{10}$$

135. Napier's rules. The ten formulas derived in §134 need not be memorized, for it is easy to write them by using two rules devised by John Napier, the inventor of logarithms.

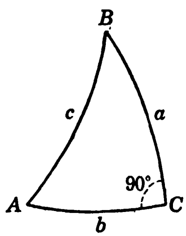

FIG. 4.

Figure 4 represents a right spherical triangle. Figure 5 contains the same letters as Fig. 4 except $C(= 90°)$, arranged in the same order. The bars on the letters c, B, and A mean *the complement of;* thus $\bar{B}$ means $90° - B$. *Note that the barred parts are the hypotenuse and the two angles each of which has a side along the hypotenuse.* The angular quantities a, b, $\bar{c}$, $\bar{A}$, $\bar{B}$ are called *the circular parts.* There are two circular parts contiguous with any given part and two parts that are not contiguous to it. Speaking of this given part as the *middle part,* we call the two contiguous parts the *adjacent* parts, and the two non-contiguous parts the *opposite parts.* Napier's rules may now be stated as follows:

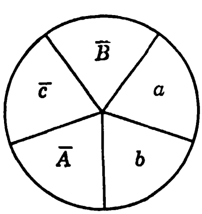

FIG. 5.

Napier's Rule I. The sine of any middle part is equal to the product of the cosines of the opposite parts.

Napier's Rule II. The sine of any middle part is equal to the product of the tangents of the adjacent parts.

We may use the expression ***sin middle = cos opposite = tan adjacent*** as an aid in recalling these rules.

Thinking of any part as the middle part, we can write two formulas, one from each of the two rules. Considering each of the five parts in turn as middle part, we may write ten formulas, and these are found to be the ten formulas numbered (1) to (10) in §134.*

Example. Use Napier's rules to write two formulas by using (*a*) b as middle part; (*b*) A as middle part.

Solution. Noting that $\sin \bar{A} = \sin (90° - A) = \cos A$, $\cos \bar{A} = \cos (90° - A) = \sin A$, etc., and applying the first rule to the parts b, $\bar{c}$, $\bar{B}$ (see Fig. 6), we obtain

$$\sin b = \cos \bar{c} \cos \bar{B},$$

or

$$\sin b = \sin c \sin B. \qquad (a)$$

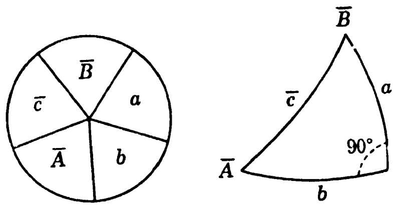

FIG. 6

Applying the second rule, using parts $\bar{A}$, b, a, we obtain

$$\sin b = \tan \bar{A} \tan a = \cot A \tan a. \qquad (b)$$

Similarly, using the parts $\bar{A}$, $\bar{B}$, a and the first rule, and afterwards the parts $\bar{c}$, $\bar{A}$, b and the second rule, we obtain

$$\sin \bar{A} = \cos \bar{B} \cos a, \quad \text{or} \quad \cos A = \sin B \cos a, \qquad (c)$$
$$\sin \bar{A} = \tan \bar{c} \tan b, \quad \text{or} \quad \cos A = \cot c \tan b. \qquad (d)$$

The formulas (*a*), (*b*), (*c*), and (*d*) are, respectively, the formulas (1), (7), (10), and (6) of §134.

EXERCISES

1. Solve each of the following right spherical triangles for the unknown part indicated.

(*a*) $a = 30°$, $b = 60°$, $c = ?$

(*b*) $c = 60°$, $a = 45°$, $B = ?$

(*c*) $a = 45°$, $B = 60°$, $c = ?$

(*d*) $a = 60°$, $B = 30°$, $A = ?$

(*e*) $c = 60°$, $A = 45°$, $b = ?$

(*f*) $A = 30°$, $B = 60°$, $a = ?$

* After the student has become familiar with the use of Napier's rules, he may save time by writing the desired formulas directly from the triangle on which the letters have been properly barred.

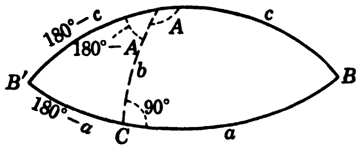

FIG. 7.

2. Using Fig. 7, show that formulas (1) to (10) hold true for the case a is greater than 90°, c is greater than 90°, b is less than 90°.

3. Solve each of the following right spherical triangles for the unknown part indicated:

(*a*) $a = 60°$, $b = 120°$,	$A = ?$	(*d*) $A = 135°$, $B = 60°$,	$c = ?$
(*b*) $c = 135°$, $b = 120°$,	$a = ?$	(*e*) $a = 30°$, $B = 120°$,	$A = ?$
(*c*) $B = 150°$, $c = 120°$,	$a = ?$	(*f*) $c = 120°$, $a = 135°$,	$B = ?$

4. Corresponding to each of the following formulas pertaining to a plane right triangle, write, using Napier's rules, an analogous formula pertaining to a right spherical triangle.

(*a*) $\sin A = a/c$. (*d*) $\cos A = b/c$. (*f*) $\tan A = a/b$.
(*b*) $\sin B = b/c$. (*e*) $\cos B = a/c$. (*g*) $\tan B = b/a$.
(*c*) $1 = \cot A \cot B$.

5. On Fig. 8 interchange A and B, also a and b. Then express the values of the line segments OD, OE, BE, BD, DE in terms of a, b, c,

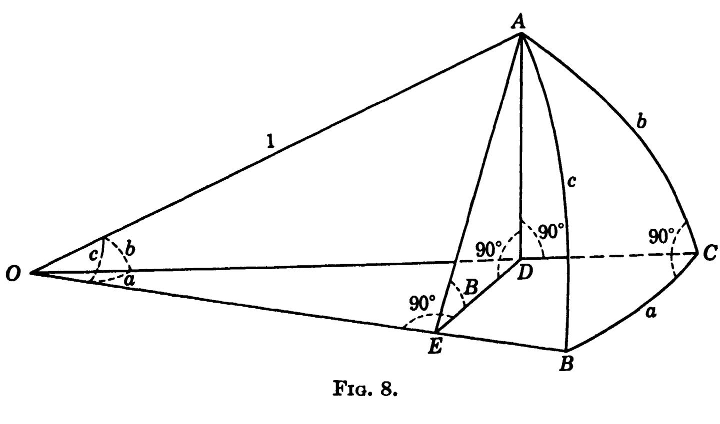

FIG. 8.

and write each of these line values on the figure. Equate two values of DE to obtain formula (4), and apply the definitions of the trigonometric functions to triangle BDE to obtain formulas (5), (6), and (7).

6. Using formula (4), show that the hypotenuse of a right spherical triangle is less than or greater than 90°, according as the two legs lie in the same quadrant or in different quadrants.

7. Using formula (10), show that in a right spherical triangle each leg and the opposite angle are of the same quadrant.

8. Use Napier's rules to write a formula involving the following, taking c as unknown part,

(a) c, B, A. (b) c, B, a. (c) c, B, b.

9. Use Napier's rules to write three formulas, each involving a and b.

10. Prove that $\tan A = \dfrac{\sin a}{\tan b \cos c}$.

11. Prove that $\cos A = \dfrac{\sin b \cos a}{\sin c}$.

136. Two important rules. In what follows it will be convenient to speak of an angle of the first quadrant or of the second quadrant. An angle is said to be of the first, second, third, or fourth quadrant according as its terminal side falls in the first, second, third, or fourth quadrant when laid off in the usual manner relative to rectangular coordinate axes.

From formula (10) of §134, namely,

$$\cos A = \cos a \sin B,$$

it follows that $\cos A$ and $\cos a$ must have the same sign since $\sin B$ is positive in all cases. Hence both A and a must be less than 90°, or both must be greater than 90°. Formula (9) may be used to show that B and b must be of the same quadrant. The following rule expresses the relation.

Rule (A). In a right spherical triangle an oblique angle and the side opposite are of the same quadrant.

From formula (4), namely,

$$\cos c = \cos a \cos b,$$

it appears that the product $\cos a \cos b$ must be positive when c is less than 90°; therefore $\cos a$ and $\cos b$ must have the same sign, and for that reason a and b are both of the first quadrant or both of the second quadrant. From the same formula it appears that $\cos a \cos b$ must be negative when c is greater than

90°; therefore cos a and cos b must have opposite signs, and a and b are of different quadrants. The following rule expresses the relation.

Rule (B). When the hypotenuse of a right spherical triangle is less than 90°, the two legs are of the same quadrant; when the hypotenuse is greater than 90°, one leg is of the first quadrant and the other of the second.

Rules (A) and (B) enable the computer to tell the quadrant of an angle found from its sine or its cosecant.

EXERCISES

State the quadrant of each of the unknown parts in each of the right spherical triangles indicated in the following diagram:

	a	b	c	A	B
1	30°	40°			
2	30°		120°		
3	120°				50°
4		140°	75°		
5				120°	130°
6		35°		100°	
7			100°	100°	
8			60°		60°

137. Solution of right spherical triangles. When two parts of a right spherical triangle in addition to the right angle are given, the remaining parts can be computed from formulas obtained by using Napier's rules. In solving the triangle it will be found advantageous to proceed as follows:

a. Draw a right spherical triangle lettered in the conventional way and encircle the given parts.

b. Write a formula for each unknown part by applying Napier's rules. *Each formula should contain the unknown part and both*

of the given parts. Then write a check formula connecting the three required parts.

c. Make a form.

d. Fill in the blank spaces of the form.

Example. Solve the right spherical triangle in which $a = 66°59'31''$, $b = 156°34'19''$.

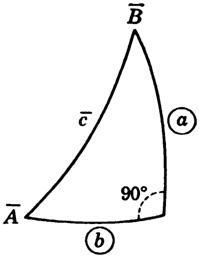

Fig. 9.

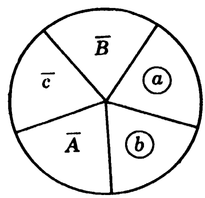

Fig. 10.

Solution. Figures 9 and 10 display the circular parts of a right spherical triangle, the known parts a, b being encircled. Using Napier's rules, in connection with Fig. 10, we write

$$\begin{array}{llll} \sin b = \tan a \cot A, & \text{or} & \cot A = \sin b \cot a, & (a) \\ \sin a = \tan b \cot B, & \text{or} & \cot B = \sin a \cot b, & (b) \\ \cos c = \cos a \cos b, & & & (c) \\ \cos c = \cot A \cot B. & & & (d) \end{array}$$

The symbols l sin, l cot, etc., written in any line of a form mean log sine of the angle at the left of the line, log cotangent of that angle, etc. For convenience the negative part -10 of the characteristic will be omitted in the forms.

The symbol $(-)$ written before a logarithm in any form calls attention to the fact that the antilogarithm of that logarithm is negative. Hence an odd number of symbols $(-)$ appearing in a column used to evaluate a product by logarithms will indicate that the product is negative. An even number of symbols $(-)$ will indicate a positive product.

In the forms of spherical trigonometry we shall omit the expressions $a =$, $b =$, *etc., to save space. The student will understand that each symbol refers to the number at the extreme left of its line.*

The computation of the unknown parts from the formulas (a), (b), (c), and the check by (d) is displayed on page 280.

	(a)		(b)		(c)	
$a = 66°59'31''$	l cot	9.62802	l sin	9.96400	l cos	9.59202
$b = 156°34'19''$	l sin	9.59944	l cot	(−)0.36319	l cos	(−)9.96264
$A = 80°25'01''$	l cot	9.22746				
$B = 154°47'25''$	l cot	(−)0.32719	l cot	(−)0.32719		
$c = 111°1'0''$	l cos	(−)9.55465			l cos	(−)9.55466

Observe that the check obtained by adding log cot A to log cot B to get log cos c checks only the logarithms of the computed parts. Errors made in finding A, B, and c from associated logarithms would not affect the check.

EXERCISES

Solve the following right spherical triangles:

1. $a = 10°32'$, $B = 12°3'$.

2. $c = 46°40'$, $B = 20°50'$.

3. $a = 118°54'$, $B = 12°19'$.

4. $a = 43°27'$, $c = 60°24'$.

5. $b = 48°36'$, $c = 69°42'$.

6. $a = 168°13'45''$, $c = 150°9'20''$.

7. $c = 112°48'$, $B = 56°11'56''$.

8. $c = 32°34'$, $A = 44°44'$.

9. $A = 116°31'25''$, $B = 116°43'12''$.

10. $A = 54°54'42''$, $c = 69°25'11''$.

11. $c = 55°9'32''$, $a = 22°15'7''$.

12. $a = 36°27'$, $b = 43°32'31''$.

13. $a = 29°46'8''$, $B = 137°24'21''$.

14. $a = 144°27'3''$, $b = 32°8'56''$.

15. $b = 36°27'$, $a = 43°32'31''$.

16. $A = 63°15'12''$, $B = 135°33'39''$.

17. $A = 67°54'47''$, $B = 99°57'35''$.

18. $b = 22°15'7''$, $c = 55°9'32''$.

19. $a = 118°30'10''$, $B = 95°36'$.

20. $b = 92°47'32''$, $A = 50°2'1''$.

21. If angle A of a right spherical triangle is 28°, what is the maximum size of angle B?

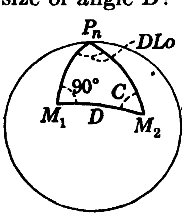

FIG. 11.

22. A ship leaves point M_1 in Fig. 11 sailing due east and follows a great-circle track to a point M_2. If M_1 is in latitude 40°30′ N., longitude 75° W. and if M_2 is in longitude 60° W., find the distance D traveled, the latitude of M_2, and the course angle C at M_2.

Hint. The angle *DLo* at the north pole P_n is the difference in the longitudes of the two points M_1

and M_2. The distances from the points M_1 and M_2 to P_n are respectively the complements of the latitudes of these points.

23. In the spherical triangle ABC (Fig. 12), p is the arc of a great circle perpendicular to side c. Write an expression for B in terms of A, a, and b.

Hint. Find two values of p and equate them.

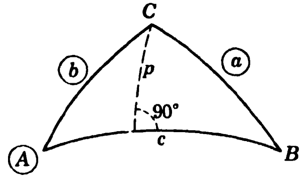

FIG. 12.

24. If in the triangle ABC of Exercise 23, $A = 40°10'$, $a = 46°20'$, and $b = 64°50'$, find B.

25. All lines in Fig. 13 represent arcs of great circles. Find all unknown parts, thus solving a spherical triangle for which two angles and the included side are given.

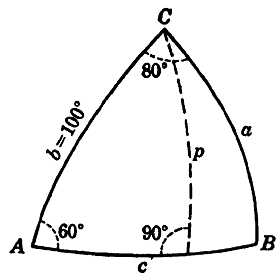

FIG. 13.

138. The ambiguous case. When the given parts are a side and the angle opposite, two solutions are obtained. In such cases each unknown part is found from the sine and hence may be chosen either in the first quadrant or in the second quadrant; that is, in the case of each unknown an angle and its supplement must be written. If A and a represent the given parts and C the right angle, the two triangles will form a lune as indicated in Fig. 14; for in this figure B' appears as $180° - B$, c' as $180° - c$, and b' as $180° - b$.

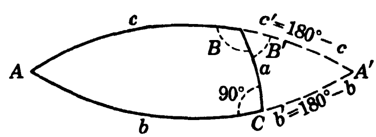

FIG. 14.

The solution of the following example will illustrate the method of finding a double solution when it exists.

Example. Solve the right spherical triangle in which

$$a = 46°45', \qquad A = 59°12'.$$

Solution. Using Napier's rules in connection with Fig. 15 we obtain

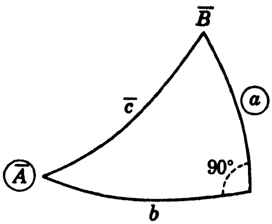

FIG. 15.

$$\sin c = \sin a \csc A, \qquad (a)$$
$$\sin B = \sec a \cos A, \qquad (b)$$
$$\sin b = \tan a \cot A, \qquad (c)$$
$$\sin b = \sin c \sin B. \quad \textit{Check}$$

The solution is displayed below.

	(a) and (*check*)	(b)	(c)
$a = 46°45'$	l sin 9.86235	l sec 0.16419	l tan 0.02655
$A = 59°12'$	l csc 0.06603	l cos 9.70931	l cot 9.77533
$c_1 = \mathbf{57°59'30''}$	l sin 9.92838		
$c_2 = \mathbf{122°0'30''}$			
$B_1 = \mathbf{48°21'27''}$	l sin 9.87350	l sin 9.87350	
$B_2 = \mathbf{131°38'33''}$			
$b_1 = \mathbf{39°19'24''}$	l sin 9.80188		l sin 9.80188
$b_2 = \mathbf{140°40'36''}$			

The six answers were grouped to obtain the solutions b_1, c_1, B_1, and b_2, c_2, B_2 by using the rules (A) and (B) of §136.

EXERCISES

Solve the following right spherical triangles:

1. $b = 35°44'$, $B = 37°28'$.

2. $b = 129°33'$, $B = 104°59'$.

3. $b = 21°39'$, $B = 42°10'10''$.

4. $a = 77°21'50''$, $A = 83°56'40''$.

5. $a = 160°$, $A = 150°$.

6. $b = 42°18'45''$, $B = 46°15'25''$.

7. Apply Napier's rules to Fig. 15 to obtain a formula involving the known parts a, A, and the unknown part b. From this formula show that there may be no solution. Discuss the case that arises when a and A are supplementary.

Solve the following right spherical triangles:

8. $b = 42°18'$, $B = 42°18'$.

9. $a = 20°10'$, $A = 115°20'$.

139. Polar triangles. The poles of a great circle on a sphere are the points where a perpendicular to the plane of the great

circle through its center pierces the surface of the sphere. To obtain the polar triangle of a spherical triangle ABC, construct three great circles on the sphere having their poles at A, B, and C. Two arcs, one having B as pole and the other C as pole, intersect in two points on opposite sides of arc BC. Denote by

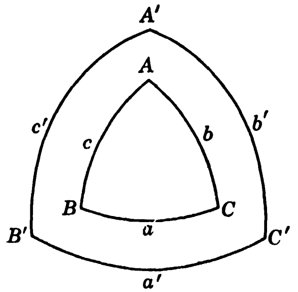

FIG. 16a.

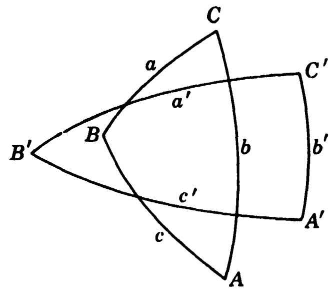

FIG. 16b.

A' the point that lies on the same side of the great circle through BC as A. Locate B' and C' by an analogous procedure. Then triangle $A'B'C'$ is the polar of triangle ABC. Figures 16 (a) and 16 (b) indicate the relations.

The following theorems from solid geometry are important:

1. *If $A'B'C'$ represents the polar triangle of spherical triangle ABC, then ABC is the polar triangle of $A'B'C'$.*

2. *An angle of any spherical triangle is the supplement of the opposite side in the polar triangle.*

In accordance with Theorem 2, we have the following relations between the sides and angles represented in Figs. 16 (a) and (b):

$$\left.\begin{array}{ll} A' = 180° - a, & A = 180° - a', \\ B' = 180° - b, & B = 180° - b', \\ C' = 180° - c, & C = 180° - c'. \end{array}\right\} \qquad (11)$$

If in an equation containing the quantities a, b, c, A, B, C, these quantities be replaced by their values in terms of a', b', c', A', B', C', from (11), a new equation having reference to the polar triangle is obtained. The relations (11) will be used in the next article to solve a spherical triangle having a side equal to 90°.

EXERCISES

1. Use relations (11) to find the parts of the polar triangle of each of the following spherical triangles.

(a) $A = 135°59.1'$, $B = 100°10.1'$, $C = 98°43.3'$, $c = 90°$, $a = 135°20'$, $b = 98°31.5'$.
(b) $a = 54°16.0'$, $b = 114°47.0'$, $C = 70°35.9'$, $c = 90°$, $A = 49°57.9'$, $B = 121°5.5'$.
(c) $a = 116°35.6'$, $b = 105°14.8'$, $c = 43°17.2'$, $A = 112°47.4'$, $B = 84°6.7'$, $C = 44°59.1'$.
(d) $a = 136°19.6'$, $b = 43°18.5'$, $c = 114°43.3'$, $A = 132°15.3'$, $B = 47°19.5'$, $C = 76°48.4'$.

2. For each of the following formulas, write a new formula having reference to the polar triangle:

(a) $\sin a = \sin c \sin A$.
(b) $\tan b = \tan c \cos A$.
(c) $\tan a = \sin b \tan A$.
(d) $\cos c = \cos b \cos a$.
(e) $\sin b = \sin c \sin B$.
(f) $\cos a = \cos b \cos c + \sin b \sin c \cos A$.
(g) $\cos A = -\cos B \cos C + \sin B \sin C \cos a$.
(h) $\dfrac{\cos \frac{1}{2}(A + B)}{\cos \frac{1}{2}(A - B)} = \dfrac{\tan \frac{1}{2}c}{\tan \frac{1}{2}(a + b)}$.
(i) $\dfrac{\sin \frac{1}{2}(A + B)}{\sin \frac{1}{2}(A - B)} = \dfrac{\tan \frac{1}{2}c}{\tan \frac{1}{2}(a - b)}$.

3. For each of the following triangles find the known parts of the polar triangle; solve these polar triangles:

(a) $c = 90°$, $a = 122°48.2'$, $B = 21°35.4'$.
(b) $c = 90°$, $a = 49°30.0'$, $B = 65°36.2'$.

140. Quadrantal triangles. A spherical triangle having a side equal to 90° is called a *quadrantal triangle*. Evidently the polar triangle of a quadrantal triangle is a right spherical triangle. Hence this polar triangle can be solved in the usual way, and the unknown parts of the quadrantal triangle can then be obtained by using relations (11).

Example. Solve the spherical triangle in which $c = 90°$, $A = 115°38'$, $b = 139°58'$.

Solution. Using (11) of §139 we obtain for the polar triangle $C' = 180° - c = 90°$, $a' = 180° - A = 64°22'$, $B' = 180° - b = 40°2'$. The solution of the polar triangle follows:

$a' = 64°22'$	l cot 9.68109	l sin 9.95500	l cos 9.63610
$B' = 40°2'$	l cos 9.88404	l tan 9.92433	l sin 9.80837
$c' = 69°49'37''$	l cot 9.56513		
$b' = 37°8'25''$	l tan 9.87933	l tan 9.87933	
$A' = 73°50'34''$	l cos 9.44446		l cos 9.44447

Using equations (11) again, we obtain $C = 180° - c' =$ **110°10′23″**, $B = 180° - b' =$ **142°51′35″**, $a = 180° - A' =$ **106°9′26″**.

EXERCISES

Solve the following right spherical triangles and then use (11) to obtain the solution of the polar triangle of each:

1. $a = 115°6'$,
$b = 123°14'$.

2. $a = 112°43'30''$,
$c = 85°10'10''$.

Solve the following quadrantal triangles:

3. $B = 117°54'30''$,
$a = 95°42'20''$,
$c = 90°$.

4. $B = 69°45'$,
$A = 94°40'$,
$c = 90°$.

5. $A = 153°16'$,
$b = 19°3'$,
$c = 90°$.

6. $b = 159°33'40''$,
$a = 95°18'20''$,
$c = 90°$.

7. In Fig. 17 $a = 18°12'$, $B = 74°45'$, $c = 90°$. Solve the right triangle ACD, and from it deduce the solution of the quadrantal triangle ABC.

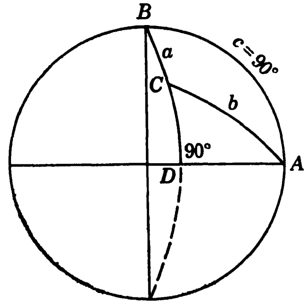

FIG. 17.

141. MISCELLANEOUS EXERCISES

1. Solve the following spherical triangles:

(a) $a = 37°48'12''$,
$b = 59°44'16''$,
$C = 90°$.

(b) $A = 110°47'50''$,
$B = 135°35'34''$,
$c = 90°$.

(c) $A = 55°32'45''$,
$B = 101°47'56''$,
$C = 90°$.

(d) $b = 132°25'$,
$B = 107°30'$,
$C = 90°$.

(e) $B = 74°45'$, $a = 18°12'$, $c = 90°$.

(f) $a = 25°18'45''$, $A = 15°58'15''$, $C = 90°$.

2. Solve the following isosceles spherical triangles:

(a) $c = 51°8'$, $A = B = 41°57'$.

(b) $C = 50°19'40''$, $A = B = 100°12'30''$.

Hint. Draw the arc of a great circle through the vertex perpendicular to the opposite side. This perpendicular bisects the base and the angle at the vertex.

3. Two great circles on a sphere intersect at 35°. A point A on one circle is 65° from their intersection. Find the distance from the intersection to the point nearest to A on the other circle.

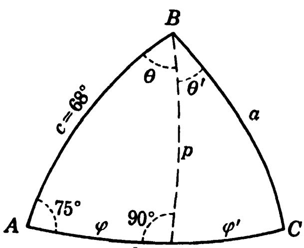

FIG. 18.

4. All lines in Fig. 18 represent arcs of great circles. Find all unknown parts, thus solving a spherical triangle for which two sides and the included angle are given.

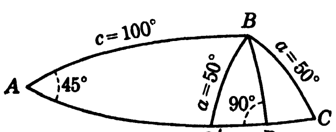

FIG. 19.

5. All lines in Fig. 19 represent arcs of great circles. Find all unknown parts, thus solving a spherical triangle for which two sides and an angle opposite one of them are given.

In Exercises 6 to 15 the terms latitude and longitude will be used extensively. The student should refer to the definitions of these quantities in §162.

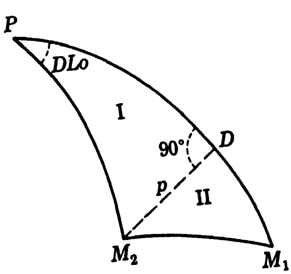

FIG. 20.

6. Figure 20 represents a spherical triangle, with the North Pole at P, Panama in latitude 8°57′ N. at M_1, and Honolulu in latitude 21°18′ N. at M_2. M_2D is the arc of a great circle perpendicular to PM_1 and DLo is 78°20′. Solve the right triangle I completely and afterward triangle II. From the results find the distance M_1M_2 and the course angle at M_1.

7. The northern vertex V (see Fig. 21), or point of highest latitude reached on the great-circle track from M_1 to M_2, is in latitude L_v = 68°27′ N., and longitude λ_v = 20°23′ W. A ship sails on the great-circle track M_1M_2, starting from M_1 in longitude λ_1 = 37°18′ W. to M_2 in longitude λ_2 = 26°28′ W. Find the distance M_1M_2.

Hint. $DLo_1 = \lambda_1 - \lambda_v$, $DLo_2 = \lambda_2 - \lambda_v$, and V is a right angle.

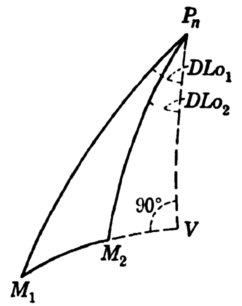

FIG. 21.

8. (*a*) If the difference of longitude of two places A and B on the earth is 50° and their latitudes are 30°, find the distance AB measured on the equal latitude circle.

(*b*) What is the distance AB measured on a great circle? The radius of the earth is approximately 3960 land miles.

9. Two points A and B are the ends of a 500-land-mile arc of a small circle in latitude 36° N. Find the difference in their longitudes. If A_1 and B_1 are both in latitude 36° N. and the arc of a great circle connecting them is 500 land miles long, what is the difference in their longitudes? Assume the radius of the earth is 3960 land miles.

10. The initial course of a certain ship sailing from New York (latitude L = 40°40′ N., long. λ = 73°58′30″ W.) is due east. After she has sailed 600 nautical miles on a great circle, find her latitude, longitude, and course.

11. Find the latitude and distance from New York of the ship in Exercise 10 when her longitude is 15°25′ W.

12. Find the latitude and longitude of the northernmost point on a great circle track sailed by a ship leaving San Francisco (latitude L = 38°28′ N., long. λ = 123°23′ W.) on a course of 310°.

13. What is the shortest distance from New York to the great circle that passes through San Francisco and the nearest point to San Francisco on the 180° meridian?

14. Find the point on the 180° meridian that is nearest San Francisco (latitude L = 38°28′ N., long. λ = 123°23′ W.)?

15. A ship sails from a place in longitude 33°14′25″ W. 2000 nautical miles on a great circle. If the initial course is due east and if the change in longitude is 53°14′25″, find the latitude of departure and the course of arrival.

16. In the case of a right spherical triangle, show that the following relations hold true:

(*a*) $\sin (c - b) \sin (c + b) = \cos^2 B \sin^2 c$.
(*b*) $\sin a \cos b = \cos c \tan a = \sin b \cot B = \sin c \cos B$.
(*c*) $\cos^2 A + \cos^2 B + \sin^2 a \sin^2 B = 1$.
(*d*) $2 \sin c \cos b = \sin (c + b) \sec^2 \frac{1}{2}A$.
(*e*) $2 \sin c \cos b = \sin (c - b) \csc^2 \frac{1}{2}A$.
(*f*) $\cos A + \cos B = \sin (a + b) \csc c$.
(*g*) $\cos B - \cos A = \sin (a - b) \csc c$.
(*h*) $\cos B \sin (c + b) \sec^2 \frac{1}{2}A = \tan a \cot c \sin (c - b) \csc^2 \frac{1}{2}A$.
(*i*) $\sin (a + b) \sin c \sin A = \sin^2 a \cos b + \sin a \cos a \sin b$.
(*j*) $\sec c \sec 2A(2 - \sec^2 A) = \sec a \sec b \sec^2 A$.
(*k*) $\tan^2 \frac{1}{2}a = \tan \frac{1}{2}(c + b) \tan \frac{1}{2}(c - b)$

CHAPTER XIV

THE OBLIQUE SPHERICAL TRIANGLE

142. Law of sines. To prepare for solving spherical triangles, we shall develop general formulas analogous to those developed in Chaps VII and VIII for plane triangles.

The law of sines for spherical triangles, analogous to the law of sines for plane triangles, may be stated as follows:

The sines of the sides of a spherical triangle are proportional to the sines of the angles opposite, or in symbols

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}. \tag{1}$$

In Fig. 1 let a, b, c represent the sides of a spherical triangle and let A, B, C represent the opposite angles. Draw an arc

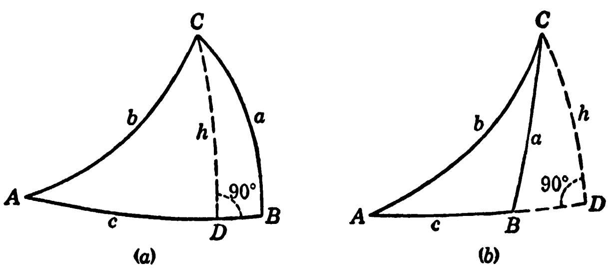

FIG. 1.

$CD(= h)$ of a great circle through the vertex C perpendicular to the side c, or the side c produced, to form the right spherical triangles ACD and BCD. Apply Napier's rules to these right triangles to obtain

$$\sin h = \sin b \sin A, \qquad \sin h = \sin a \sin B.$$

Equating these two values of $\sin h$, we get

$$\sin a \sin B = \sin b \sin A,$$

or, dividing by $\sin A \sin B$,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B}. \tag{2}$$

In like manner, by drawing an arc from A perpendicular to CB and arguing as above, we can show that

$$\frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}. \tag{3}$$

Equations (2) and (3) are together equivalent to (1). The law of sines may be used in the solution of a spherical triangle when a side and the angle opposite are included among the given parts.

When a part of a spherical triangle is found by means of the law of sines, there is often some difficulty in determining whether the part found is of the first quadrant or of the second quadrant; for $\sin A = \sin (180° - A)$. Other formulas must be used in many cases. However, the following theorems from solid geometry will often enable the computer to determine the quadrant.

The order of magnitude of the sides of a spherical triangle is the same as the order of magnitude of the respective opposite angles; or, **in symbols, if**

$$\boldsymbol{a < b < c, \quad \text{then} \quad A < B < C.}$$

The sum of two sides of a spherical triangle is greater than the third side.

EXERCISES

1. Figure 2 represents the spherical triangle ABC with its associated trihedral angle O, the face angles of which are a, b, c. AF is the intersection of two planes, one perpendicular to OB, the other perpendicular to OC. Point F is in plane OCB. Taking $OA = 1$ unit, express the values of all straight-line segments of the figure in terms of a, b, c, B, and C. Derive the law of sines from the result.

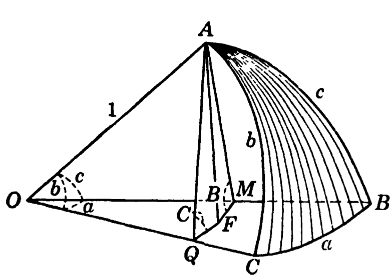

FIG. 2.

2. Check the following data by using the law of sines:

(a) $A = 108°40'$, $B = 134°20'$, $C = 70°18'$, $a = 145°36'$, $b = 154°45'$, $c = 34°9'$.

(b) $A = 47°21'$, $B = 22°20'$, $C = 146°40'$, $a = 117°9'$, $b = 27°22'$, $c = 138°20'$.
(c) $A = 110°10'$, $B = 133°18'$, $C = 70°16'$, $a = 147°6'$, $b = 155°5'$, $c = 32°59'$.

3. Use the law of sines to find the missing parts of the following right spherical triangles:

(a) $a = 58°8'19''$, $b = 32°49'22''$, $B = 37°12'53''$, $c = 63°40'$.
(b) $a = 36°14'6''$, $A = 49°29'56''$, $b = 38°45'$, $c = 51°1'11''$.

4. Use the law of sines to find the missing part of each of the following spherical triangles:

(a) $A = 130°5'22''$, $B = 32°26'6''$, $C = 36°45'26''$, $c = 51°6'12''$, $a = 84°14'29''$.
(b) $A = 70°$, $C = 94°48'12''$, $c = 116°$, $a = 57°56'53''$, $b = 137°20'33''$.

5. Solve the polar triangles of the triangles of Exercise 3.

143. The law of cosines for sides. *The cosine of any side of a spherical triangle is equal to the product of the cosines of the two other sides increased by the product of the sines of the two other sides and the cosine of the angle included between them,* or in symbols

$$\cos a = \cos b \cos c + \sin b \sin c \cos A. \qquad (4)$$

The following proof is analogous to the one given for the law of cosines in plane trigonometry.

In Fig. 1 let arc $AD = \varphi$. Then arc $BD = c - \varphi$. Write these values on the triangle of Fig. 1 (a), and place bars over a, b, A, and B in preparation for using Napier's rules. The result is Fig. 3.

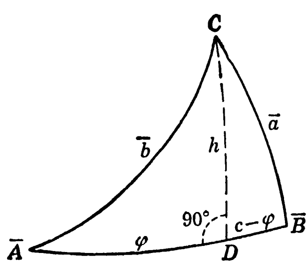

Fig. 3.

Now apply Napier's rules to triangles ACD and BCD to obtain

$$\cos a = \cos h \cos (c - \varphi), \qquad (5)$$
$$\cos b = \cos h \cos \varphi. \qquad (6)$$

Divide (5) by (6) member by member, and transform slightly to get

$$\frac{\cos a}{\cos b} = \frac{\cos h \cos (c - \varphi)}{\cos h \cos \varphi} = \frac{\cos c \cos \varphi + \sin c \sin \varphi}{\cos \varphi}, \qquad (7)$$

or, simplifying further,

$$\cos a = \cos b(\cos c + \sin c \tan \varphi). \tag{8}$$

Again apply Napier's rules, using parts b, A, φ of triangle ACD to obtain

$$\cos A = \cot b \tan \varphi,$$

or

$$\tan \varphi = \cos A \tan b. \tag{9}$$

Replace $\tan \varphi$ in (8) by its value from (9) to get

$$\cos a = \cos b(\cos c + \sin c \cos A \tan b), \tag{10}$$

or, simplifying the right-hand member,

$$\mathbf{\cos a = \cos b \cos c + \sin b \sin c \cos A.} \tag{11}$$

Similarly, we may obtain

$$\cos b = \cos a \cos c + \sin a \sin c \cos B, \tag{12}$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C. \tag{13}$$

An argument differing slightly from the one just used shows that (11) holds for a triangle shaped like the triangle of Fig. 1(b).

The law of cosines applies to the solution of a spherical triangle when two sides and the included angle are given. Although it is not adapted to logarithmic computation, it is used in the derivation of many important formulas of spherical trigonometry.

Example. Find c in the spherical triangle for which $a = 76°24'40''$, $b = 58°18'36''$, $C = 116°30'28''$.

Solution. The law of cosines may be written

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

Here it will be necessary to compute each product in the right-hand member, add the results, and then find c from a table of natural cosines; or find the logarithm of the natural cosine, and then find c from the table giving the logarithms of cosines. The computation is indicated in the following form:

	($\cos a \cos b$)	($\sin a \sin b \cos C$)
$a = 76°24'40''$	l cos 9.37098	l sin 9.98767
$b = 58°18'36''$	l cos 9.72042	l sin 9.92988
$C = 116°30'28''$		l cos(−)9.64965
0.12342	log 9.09140	
−0.36915		log (−)9.56720
−0.24573		

$\therefore c = \cos^{-1}(-0.24573) = \mathbf{104°13'30''}.$

144. The law of cosines for angles. Applying (11) to the polar triangle (see §139) of ABC, we obtain

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'. \tag{14}$$

Using equation (11) of §139 to replace a', b', c', and A' of (14) by $180° - A$, $180° - B$, $180° - C$, and $180° - a$, respectively, we obtain

$$\cos(180° - A) = \cos(180° - B)\cos(180° - C) + \sin(180° - B)\sin(180° - C)\cos(180° - a),$$

or

$$-\cos A = \cos B \cos C - \sin B \sin C \cos a,$$

or

$$\mathbf{\cos A = -\cos B \cos C + \sin B \sin C \cos a.} \tag{15}$$

Similarly, we obtain from (12) and (13)

$$\cos B = -\cos A \cos C + \sin A \sin C \cos b, \tag{16}$$
$$\cos C = -\cos A \cos B + \sin A \sin B \cos c. \tag{17}$$

Evidently this process of applying known formulas to the polar triangle of a given one is very important. It furnishes a method of deriving from every equation applying to a general spherical triangle another equation that may be called the *dual* of the first one. The role played by the sides in the given equation is played by the angles in the dual equation, and the role played by the angles in the given equation is played by the sides in the other. A similar statement applies to theorems relating to a spherical triangle. This principle of duality will come to our attention again and again in the discussion that follows.

Example. In a certain spherical triangle, $A = 60°$, $B = 60°$, and $c = 60°$. Find C.

Solution. Substituting 60° for each of the letters A, B, and c in (17), we obtain

$$\begin{aligned}\cos C &= -\cos 60° \cos 60° + \sin 60° \sin 60° \cos 60°\\ &= -\tfrac{1}{4} + \tfrac{3}{8} = \tfrac{1}{8}.\end{aligned}$$

Hence

$$C = \cos^{-1} \tfrac{1}{8} = \mathbf{82°49'9''}.$$

EXERCISES

1. Use the law of cosines to find a for each of the following spherical triangles:

(a) $b = 60°$, $c = 30°$, $A = 45°$.	(b) $b = 45°$, $c = 30°$, $A = 120°$.	(c) $b = 45°$, $c = 60°$, $A = 150°$.

2. Use the law of cosines for angles to find A for each of the following triangles:

(a) $B = 120°$, $C = 150°$, $a = 135°$.	(b) $B = 135°$, $C = 120°$, $a = 30°$.

3. In a spherical triangle, given $a = 30°$, $b = 45°$, $c = 60°$, find A.

4. Derive the law of sines algebraically from the law of cosines.

Hint. Solve (11) for $\cos A$, form $\sin^2 A$, and reduce the numerator to a form involving cosines only. Then show that $\sin^2 A/\sin^2 a$ is symmetrical in a, b, c.

5. In Fig. 4, ABC represents a spherical triangle with its associated trihedral angle O. BLM is a plane through B perpendicular to OB, intersecting OA produced, in M and OC produced, in L. Taking $OB = 1$ unit, express the values of the line segments OL, OM, BL, BM in terms of a, b, c, then apply the law of cosines of plane trigonometry to the triangles BLM, and OLM, and equate two values of $\overline{LM}^2$ to obtain after slight transformation

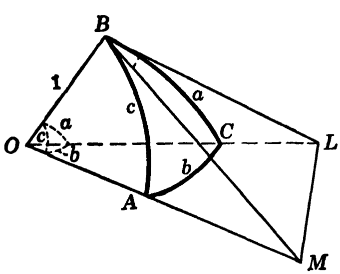

FIG. 4.

$$\cos b = \cos a \cos c + \sin a \sin c \cos B.$$

6. From formula (15) show that

$$\operatorname{hav}(180^\circ - A) = \operatorname{hav}(B + C) - \sin B \sin C \operatorname{hav} a,$$

remembering that $\operatorname{hav} A = \frac{1}{2}(1 - \cos A)$.

7. In each of the triangles of Exercise 1 complete the solution by means of the law of sines.

8. Solve the polar triangles of the triangles of Exercises 1 and 3.

9. Using the law of cosines, prove that in a spherical triangle having three sides of the second quadrant the angles opposite are of the second quadrant.

10. What equations are dual to those expressing the law of sines?

11. Find the equation dual to the one written in Exercise 6.

12. Replace C by 90° in (1), (13), (15), and (17), and then obtain the resulting formulas by applying Napier's rules to the parts of a right spherical triangle.

145. The six cases. When three parts of a spherical triangle are given, the other three parts can be computed. Accordingly a classification of spherical triangles is made on the basis of given parts. Six cases are referred to as follows:

I. Given the three sides.
II. Given the three angles.
III. Given two sides and the included angle.
IV. Given two angles and the included side.
V. Given two sides and an angle opposite one of them.
VI. Given two angles and a side opposite one of them.

For purposes of solution, there are, in a sense, only three cases. If a method of solution for Case I is known, this same method may be applied to solve the polar of a triangle classified under Case II. The solution of a quadrantal triangle in §140 by the method of solving a right spherical triangle illustrates the process. Similarly, the formulas used to solve a triangle classified under Case III may be used to solve the polar of a triangle classified under Case IV; also, the same formulas may be used to solve a triangle coming under Case V and the polar of a triangle classified under Case VI.

146. The half-angle formulas. This article is devoted to the derivation of formulas that may be used to solve triangles for

which the given parts are three sides or three angles. Solving (11) for $\cos A$, we have

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}. \tag{18}$$

Equating 1 minus the left-hand member to 1 minus the right-hand member and simplifying slightly, we get

$$1 - \cos A = \frac{\sin b \sin c + \cos b \cos c - \cos a}{\sin b \sin c},$$

or, replacing $\sin b \sin c + \cos b \cos c$ by $\cos (b - c)$,

$$1 - \cos A = \frac{\cos (b - c) - \cos a}{\sin b \sin c}.$$

Now, replacing $1 - \cos A$ by $2 \sin^2 \frac{1}{2}A$ and changing the right-hand member by using (36) of §57 and the fact that $\sin (-\theta) = -\sin \theta$, we get

$$2 \sin^2 \tfrac{1}{2}A = \frac{2 \sin \frac{1}{2}(a + b - c) \sin \frac{1}{2}(a - b + c)}{\sin b \sin c}. \tag{19}$$

Denote half the sum of the sides by s and write

$$s = \tfrac{1}{2}(a + b + c). \tag{20}$$

Subtracting in succession a, b, and c from both members of (20), we obtain

$$\left.\begin{aligned} s - a = \tfrac{1}{2}(-a + b + c), \qquad s - b &= \tfrac{1}{2}(a - b + c), \\ s - c &= \tfrac{1}{2}(a + b - c). \end{aligned}\right\} \tag{21}$$

Substituting from (21) in (19) and taking the square root of both members, we obtain

$$\sin \tfrac{1}{2}A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}}. \tag{22}$$

Considerations of symmetry show that

$$\sin \tfrac{1}{2}B = \sqrt{\frac{\sin (s - a) \sin (s - c)}{\sin a \sin c}}, \tag{23}$$

$$\sin \tfrac{1}{2}C = \sqrt{\frac{\sin (s - a) \sin (s - b)}{\sin a \sin b}}. \tag{24}$$

Similarly, proceeding as above, we obtain

$$
\begin{aligned}
1 + \cos A &= 1 + \frac{\cos a - \cos b \cos c}{\sin b \sin c}, \\
&= \frac{\cos a - (\cos b \cos c - \sin b \sin c)}{\sin b \sin c}, \\
&= \frac{\cos a - \cos (b + c)}{\sin b \sin c},
\end{aligned}
$$

$$1 + \cos A = \frac{2 \sin \frac{1}{2}(a + b + c) \sin \frac{1}{2}(-a + b + c)}{\sin b \sin c}. \tag{25}$$

Replacing in (25) $1 + \cos A$ by $2 \cos^2 \frac{1}{2}A$, using (20) and (21) and extracting the square root of both members, we get

$$\cos \tfrac{1}{2}A = \sqrt{\frac{\sin s \sin (s - a)}{\sin b \sin c}}. \tag{26}$$

Considerations of symmetry show that

$$\cos \tfrac{1}{2}B = \sqrt{\frac{\sin s \sin (s - b)}{\sin a \sin c}}, \tag{27}$$

$$\cos \tfrac{1}{2}C = \sqrt{\frac{\sin s \sin (s - c)}{\sin a \sin b}}. \tag{28}$$

Dividing (22) by (26), member by member, and replacing $\sin \frac{1}{2}A \div \cos \frac{1}{2}A$ by $\tan \frac{1}{2}A$, we obtain

$$\tan \tfrac{1}{2}A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}}. \tag{29}$$

Multiplying numerator and denominator under the radical by $\sin (s - a)$ and removing $1/\sin^2 (s - a)$ from the radical, we have

$$\tan \tfrac{1}{2}A = \frac{1}{\sin (s - a)} \sqrt{\frac{\sin (s - a) \sin (s - b) \sin (s - c)}{\sin s}}, \tag{30}$$

or

$$\tan \tfrac{1}{2}A = \frac{r}{\sin (s - a)}, \tag{31}$$

where

$$r = \sqrt{\frac{\sin (s - a) \sin (s - b) \sin (s - c)}{\sin s}}. \tag{32}$$

Similarly,

$$\tan \tfrac{1}{2}B = \frac{r}{\sin (s - b)}, \tag{33}$$

$$\tan \tfrac{1}{2}C = \frac{r}{\sin (s - c)}. \tag{34}$$

Since hav $A = \sin^2 \frac{1}{2}A$, formula (22) may be written

$$\text{hav } A = \sin (s - b) \sin (s - c) \csc b \csc c. \tag{35}$$

Similar formulas for hav B and hav C may be obtained from (23) and (24). Formula (35) is often used when haversine tables are available.

147. Cases I and II. Given three sides or given three angles. Evidently formulas (31), (33), and (34) are adapted to solve a spherical triangle when three sides are given. To solve a spherical triangle when the three angles are given, we find the sides of the polar triangle by subtracting each of the given angles from 180° and then applying equations (31), (33), and (34) to find the angles of the polar triangle; subtraction of each of these angles from 180° gives the sides of the original triangle. Also, the formulas of Exercise 1 on page 2[illegible]9 may be used.

Example. Find A, B, and C for a spherical triangle in which $a = 70°14'20''$, $b = 49°24'10''$, $c = 38°46'10''$.

Solution. $s = \frac{1}{2}(a + b + c) = 79°12'20''$. The solution by means of formulas (32), (31), (33), and (34) and the check by the law of sines follows. The number in parenthesis above each column refers to the formula associated with the column.

	(32)	(31)	(33)	(34)
$s - a = 8°58'00''$	l sin 9 19273	l csc 0 80727		
$s - b = 29°48'10''$	l sin 9 69637		l csc 0 30363	
$s - c = 40°26'10''$	l sin 9 81197			l csc 0 18803
$s = 79°12'20''$	l csc 0 00775			
	2)log 8 70882			
r	log 9 35441	log 9 35441	log 9 35441	log 9 35441
$\frac{1}{2}A = 55°25'38''$, **$A = 110°51'16''$**		l tan 0 16168		
$\frac{1}{2}B = 24°28'2''$, **$B = 48°56'4''$**			l tan 9 65804	
$\frac{1}{2}C = 19°13'23''$, **$C = 38°26'46''$**				l tan 9 54244

Check.

a	l sin 9.97364	b	l sin 9 88042	c	l sin 9 79671
A	l sin 9 97058	B	l sin 9.87735	C	l sin 9 79364
	0 00306		0.00307		0 00307

EXERCISES

1. Write $\sigma = \dfrac{A + B + C}{2}$, and use equations (11) of §139 to derive

$$s' = \frac{a' + b' + c'}{2} = 270° - \frac{A + B + C}{2} = 270° - \sigma,$$

$$s' - a' = 90° - (\sigma - A), \qquad s' - b' = 90° - (\sigma - B),$$
$$s' - c' = 90° - (\sigma - C).$$

Then apply equations (22), (26), and (29) to the polar triangle to obtain

$$\cos \tfrac{1}{2}a = \sqrt{\frac{\cos (\sigma - B) \cos (\sigma - C)}{\sin B \sin C}},$$

$$\sin \tfrac{1}{2}a = \sqrt{\frac{-\cos \sigma \cos (\sigma - A)}{\sin B \sin C}},$$

$$\tan \tfrac{1}{2}a = \sqrt{\frac{-\cos \sigma \cos (\sigma - A)}{\cos (\sigma - B) \cos (\sigma - C)}}.$$

2. Solve the following spherical triangles:

(*a*) $a = 30°$, $b = 45°$, $c = 60°$,

(*b*) $a = 30°$, $b = 60°$, $c = 60°$.

(*c*) $a = 150°$, $b = 120°$, $c = 60°$.

(*d*) $A = 60°$, $B = 135°$, $C = 60°$.

(*e*) $A = 60°$, $B = 30°$, $C = 120°$.

(*f*) $A = 150°$, $B = 120°$, $C = 135°$.

3. Solve the following spherical triangles:

(*a*) $a = 110°$, $b = 32°$, $c = 96°$.

(*b*) $a = 108°14'$, $b = 75°29'$, $c = 56°37'$.

(*c*) $a = 78°15'12''$, $b = 101°20'18''$, $c = 112°38'42''$.

(*d*) $a = 70°0'37''$, $b = 125°30'52''$, $c = 63°47'55''$.

(*e*) $A = 80°$, $B = 110°$, $C = 130°$.

(*f*) $A = 59°55'10''$, $B = 85°36'50''$, $C = 59°55'10''$.

(*g*) $A = 89°5'46''$, $B = 54°32'24''$, $C = 102°14'12''$.

(*h*) $A = 172°17'56''$, $B = 8°28'20''$, $C = 4°23'35''$.

4. Solve the polar triangles of the triangles of Exercise 2.

5. Derive the following equations from (22) to (34):

$$\frac{\cos \frac{1}{2}A \cos \frac{1}{2}B}{\sin \frac{1}{2}C} = \frac{\sin s}{\sin c},$$
$$\frac{\cos \frac{1}{2}A \sin \frac{1}{2}B}{\cos \frac{1}{2}C} = \frac{\sin (s - a)}{\sin c},$$
$$\frac{\sin \frac{1}{2}A \cos \frac{1}{2}B}{\cos \frac{1}{2}C} = \frac{\sin (s - b)}{\sin c},$$
$$\frac{\sin \frac{1}{2}A \sin \frac{1}{2}B}{\sin \frac{1}{2}C} = \frac{\sin (s - c)}{\sin c}.$$

6. Prove that the following relation holds true for a right spherical triangle:

$$\tan^2 \tfrac{1}{2}A = \sin (c - b) \csc (c + b).$$

148. Napier's analogies. This article is devoted to deriving formulas that may be used to solve triangles for which the given parts are two sides and the included angle or two angles and the included side. Substituting $A = \frac{1}{2}A$ and $B = \frac{1}{2}B$ in (7) and (10) of §53, we get

$$\sin \tfrac{1}{2}(A + B) = \sin \tfrac{1}{2}A \cos \tfrac{1}{2}B + \cos \tfrac{1}{2}A \sin \tfrac{1}{2}B, \quad (36)$$
$$\sin \tfrac{1}{2}(A - B) = \sin \tfrac{1}{2}A \cos \tfrac{1}{2}B - \cos \tfrac{1}{2}A \sin \tfrac{1}{2}B. \quad (37)$$

Dividing (37) by (36) member by member, we get

$$\frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} = \frac{\sin \frac{1}{2}A \cos \frac{1}{2}B - \cos \frac{1}{2}A \sin \frac{1}{2}B}{\sin \frac{1}{2}A \cos \frac{1}{2}B + \cos \frac{1}{2}A \sin \frac{1}{2}B}. \quad (38)$$

Or, dividing both numerator and denominator of the right-hand member of (38) by $\sin \frac{1}{2}A \sin \frac{1}{2}B$,

$$\frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} = -\frac{\cot \frac{1}{2}A - \cot \frac{1}{2}B}{\cot \frac{1}{2}A + \cot \frac{1}{2}B}. \quad (39)$$

From (31) and (33) we find $\cot \frac{1}{2}A = \dfrac{\sin (s - a)}{r}$ and $\cot \frac{1}{2}B = \dfrac{\sin (s - b)}{r}$. Substituting these values in (39) and canceling r, we obtain

$$\frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} = -\frac{\sin (s - a) - \sin (s - b)}{\sin (s - a) + \sin (s - b)}. \quad (40)$$

Using (34) and (33) of §57 to transform the right-hand member of (40), we get

$$\frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} = -\frac{2 \cos \frac{1}{2}(2s - a - b) \sin \frac{1}{2}(b - a)}{2 \sin \frac{1}{2}(2s - a - b) \cos \frac{1}{2}(b - a)}. \tag{41}$$

Replacing $(2s - a - b)$ by c in (41) and simplifying slightly, we get

$$\frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} = \frac{\tan \frac{1}{2}(a - b)}{\tan \frac{1}{2}c}. \tag{42}$$

Again, using (11) and (8) of §53 with $A = \frac{1}{2}A$ and $B = \frac{1}{2}B$, we get

$$\cos \tfrac{1}{2}(A - B) = \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B + \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B, \tag{43}$$
$$\cos \tfrac{1}{2}(A + B) = \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B - \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B. \tag{44}$$

Dividing (43) by (44) member by member, then dividing numerator and denominator of the right-hand member of the resulting equation by $\sin \frac{1}{2}A \sin \frac{1}{2}B$ and finally replacing $\cot \frac{1}{2}A$ by $\frac{\sin (s - a)}{r}$ and $\cot \frac{1}{2}B$ by $\frac{\sin (s - b)}{r}$, we have

$$\frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} = \frac{\dfrac{\sin (s - a) \sin (s - b)}{r^2} + 1}{\dfrac{\sin (s - a) \sin (s - b)}{r^2} - 1}. \tag{45}$$

Replacing r^2 by its value from (32) and simplifying slightly, we obtain

$$\frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} = \frac{\sin s + \sin (s - c)}{\sin s - \sin (s - c)}. \tag{46}$$

Treating the right-hand member of this equation in a manner similar to that employed in transforming (40), we get

$$\frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} = \frac{\tan \frac{1}{2}(a + b)}{\tan \frac{1}{2}c}. \tag{47}$$

Applying (42) and (47) to the polar triangle, we obtain

$$\frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} = \frac{\tan \frac{1}{2}(A-B)}{\cot \frac{1}{2}C}, \tag{48}$$

$$\frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} = \frac{\tan \frac{1}{2}(A+B)}{\cot \frac{1}{2}C}. \tag{49}$$

The formulas (42), (47), (48), and (49) are known as Napier's analogies. These formulas are analogous to the law of tangents in plane trigonometry.

EXERCISES

1. Apply (42) and (47) to the polar triangle, then proceed in a manner analogous to that pursued in this article and obtain formulas (48) and (49).

2. Use formulas (42), (47), (48), and (49) to prove the following formulas known as Gauss's equations or Delambre's analogies.

$$\sin \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}c} \cos \tfrac{1}{2}C,$$

$$\sin \tfrac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}c} \cos \tfrac{1}{2}C,$$

$$\cos \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}c} \sin \tfrac{1}{2}C,$$

$$\cos \tfrac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}c} \sin \tfrac{1}{2}C.$$

3. Show that the second of Gauss's equations can be written

$$\text{hav}\,(A-B) = \frac{\text{hav}\,(a-b)}{\text{hav}\,c}\,\text{hav}\,(180^\circ - C).$$

4. From formula (47), show that in any spherical triangle one-half the sum of two angles is in the same quadrant as one-half the sum of the opposite sides; that is, $\frac{1}{2}(a+b)$ and $\frac{1}{2}(A+B)$ are in the same quadrant.

5. (*a*) Divide $\sin \frac{1}{2}(A-B) = \sin \frac{1}{2}A \cos \frac{1}{2}B - \cos \frac{1}{2}A \sin \frac{1}{2}B$ by $\cos \frac{1}{2}(A-B) = \cos \frac{1}{2}A \cos \frac{1}{2}B + \sin \frac{1}{2}A \sin \frac{1}{2}B$, member by member, then proceed in a manner similar to that employed in this article in deriving (42) and thus deduce formula (48).

(*b*) Derive formula (19) by dividing $\sin \frac{1}{2}(A+B)$ by $\cos \frac{1}{2}(A+B)$.

6. (*a*) Divide $\sin \frac{1}{2}(A-B)$ by $\cos \frac{1}{2}(A+B)$ and proceed in a manner similar to that outlined in 5 (*a*) and derive the formula

$$\frac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} = \frac{\sin \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \tfrac{1}{2}c \cot \tfrac{1}{2}C.$$

149. Cases III and IV. Given two sides and the included angle or given two angles and the included side. The four formulas (42), (47), (48), and (49) are used to solve a triangle when the given parts are two sides and the included angle, or two angles and the side common to them. If the law of sines is used to find the last unknown after two unknowns have been found, often the ambiguity arising may be removed by using the theorem that states that the order of magnitude of the sides of a spherical triangle is the same as that of their respective opposite angles.

Other sets of formulas may be obtained from (42) and (47) to (49) by the interchange of letters. For example, another set would result from replacing a by c, c by a, A by C, and C by A in (42) and (47) to (49).

Example. Find A, B, and c for a spherical triangle in which $a = 57°56'53''$, $b = 137°20'33''$, $C = 94°48'6''$.

Solution. In this example $\frac{1}{2}(b - a) = 39°41'50''$, $\frac{1}{2}(b + a) = 97°38'43''$, $\frac{1}{2}C = 47°24'3''$. Formulas (48), (49), (42), and (47) may be written in the respective forms

$$\tan \tfrac{1}{2}(B - A) = \sin \tfrac{1}{2}(b - a) \csc \tfrac{1}{2}(b + a) \cot \tfrac{1}{2}C, \qquad (48')$$
$$\tan \tfrac{1}{2}(A + B) = \cos \tfrac{1}{2}(b - a) \sec \tfrac{1}{2}(b + a) \cot \tfrac{1}{2}C, \qquad (49')$$
$$\tan \tfrac{1}{2}c = \tan \tfrac{1}{2}(b - a) \sin \tfrac{1}{2}(B + A) \csc \tfrac{1}{2}(B - A), \qquad (42')$$
$$\tan \tfrac{1}{2}c = \tan \tfrac{1}{2}(b + a) \sec \tfrac{1}{2}(B - A) \cos \tfrac{1}{2}(B + A). \qquad (47')$$

The following form indicates the computation. The number in parenthesis above each column refers to the formula associated with the column.

	(48′)	(49′)	(42′)	check (47′)
$\frac{1}{2}(b - a) = 39°41'50''$	l sin 9 80531	l cos 9 88617	l tan 9 91915	
$\frac{1}{2}(b + a) = 97°38'43''$	l csc 0 00388	l sec (−)0 87602		l tan (−)0 87214
$\frac{1}{2}C = 47°24'3''$	l cot 9 96356	l cot 9 96356		
$\frac{1}{2}B - A) = 30°39'2''$	l tan 9.77275		l csc 0.29260	l sec 0 06535
$\frac{1}{2}(B + A) = 100°38'58''$		l tan (−)0 72575	l sin 9 99245	l cos (−)9 26670
$\frac{1}{2}c = 57°59'56''$			l tan 0 20420	l tan 0 20419

$A = 69°59'56''$ $B = 131°18'0''$ $c = 115°59'52''$.

These results could have been checked by the law of sines.

EXERCISES

1. Solve the following spherical triangles:

(a) $a = 30°$, $B = 45°$, $c = 60°$.

(b) $b = 135°$, $A = 45°$, $c = 60°$.

(c) $a = 30°$, $C = 150°$, $b = 135°$.

(d) $A = 150°$, $c = 30°$, $B = 120°$.

(e) $B = 30°$, $a = 45°$, $C = 60°$.

(f) $A = 60°$, $b = 120°$, $C = 150°$.

2. In the following triangles where two values for a part are given, select the proper value.

(a) $A = 65°13'$, $B = 49°28'$, $130°33'$, $C = 128°16'$, $a = 88°24'$, $b = 56°48'$, $c = 120°11'$.

(b) $A = 50°10'$, $B = 135°5'$, $C = 50°30'$, $a = 69°35'$, $110°25'$, $b = 120°30'$, $c = 70°20'$.

(c) $A = 127°40'$, $B = 45°15'$, $C = 124°42'$, $15°20'$, $a = 68°53'$, $b = 56°50'$, $c = 18°10'$.

(d) $A = 52°20'$, $B = 45°15'$, $C = 124°42'$, $a = 68°53'$, $b = 56°50'$, $c = 104°19'$, $18°10'$.

3. Using Napier's analogies, solve the following spherical triangles:

(a) $c = 116°0'0''$, $A = 70°0'0''$, $B = 131°18'0''$.

(b) $a = 88°37'40''$, $c = 125°18'20''$, $B = 102°16'36''$.

(c) $a = 76°24'0''$, $b = 58°19'0''$, $C = 116°30'0''$.

(d) $a = 86°18'40''$, $b = 45°36'20''$, $C = 120°46'30''$.

(e) $a = 41°6'0''$, $b = 119°24'0''$, $C = 162°22'30''$.

(f) $c = 120°18'33''$, $A = 27°22'34''$, $B = 91°26'44''$.

4. In the following spherical triangles, find the angles by means of Napier's analogies and the required side by using the law of sines.

(a) $a = 42°45'0''$, $b = 47°15'0''$, $C = 11°11'41''$.

(b) $a = 131°15'0''$, $b = 129°20'0''$, $C = 103°37'23''$.

150. Cases V and VI. *Two of the given parts are opposites. Double solutions.* For convenience of reference, a theorem from solid geometry is repeated here.

Theorem. The order of magnitude of the sides of a spherical triangle is the same as that of their respective opposite angles. Or if a and b are a pair of sides of a spherical triangle and A and B the respective opposite angles, we know that if

$$a < b, \qquad \text{then} \qquad A < B. \tag{50}$$

When the given parts of a spherical triangle are two sides and an angle opposite one of them, say, a, b, and A, the angle B may be found by using the law of sines,

$$\sin B = \frac{\sin b}{\sin a} \sin A. \tag{51}$$

Since $\sin B$ does not exceed 1 in magnitude, $\log \sin B$ does not exceed zero. Hence no solution will exist when $\log \sin B > 0$.

When $\log \sin B < 0$, a positive acute angle and its supplement must be considered for B. Each value of B must be consistent with (50). Hence, there will be no solution, one solution, or two solutions according as (50) is satisfied by neither, by one and only one, or by both of the values of B obtained from (51). If $b = a$, then $B = A$, and there is one solution.

Accordingly, begin the solution of a spherical triangle in which a, b, and A are the given parts by using (51) to find $\log \sin B$. If $\log \sin B > 0$, there is no solution. If $\log \sin B < 0$, find two values of B, one a positive acute angle and the other its supplement. Then, to find c and C, use the given parts with each value of B that satisfies (50) in

$$\left.\begin{aligned} \tan \tfrac{1}{2}c &= \frac{\sin \frac{1}{2}(A + B)}{\sin \frac{1}{2}(A - B)} \tan \tfrac{1}{2}(a - b), \\ \cot \tfrac{1}{2}C &= \frac{\sin \frac{1}{2}(a + b)}{\sin \frac{1}{2}(a - b)} \tan \tfrac{1}{2}(A - B). \end{aligned}\right\} \tag{52}$$

These formulas were obtained by solving Napier's analogies (42) and (48) for $\tan \frac{1}{2}c$ and $\cot \frac{1}{2}C$, respectively.

A similar discussion, with the roles of sides and angles interchanged, applies when the given parts are two angles and a side opposite one of them; (51) solved for $\sin b$ would first be used and then (52).

Example. Given $a = 52°45'20''$, $b = 71°12'40''$, $A = 46°22'10''$, find c, B, C.

Solution. Two solutions are to be expected. First using

$$\sin B = \sin b \sin A \csc a \qquad (1')$$

to find B, and afterwards using (42′) and (49) to find c_1, C_1, c_2, and C_2, we obtain the solution indicated below.

	(1′)
$a = 52°45'20''$	l csc 0.09906
$b = 71°12'40''$	l sin 9.97622
$A = 46°22'10''$	l sin 9.85962
$\mathbf{B_1 = 59°24'22''}$	l sin 9.93490
$\mathbf{B_2 = 120°35'38''}$	

	(42′)	(49)
$\frac{1}{2}(B_1 - A) = 6°31'6''$	l csc 0.94492	
$\frac{1}{2}(B_1 + A) = 52°53'16''$	l sin 9.90171	l tan 0.12112
$\frac{1}{2}(b - a) = 9°13'40''$	l tan 9.21075	l sec 0.00565
$\frac{1}{2}(b + a) = 61°59'0''$		l cos 9.67185
$\frac{1}{2}c_1 = 48°46'26''$	l tan 0.05738	
$\mathbf{c_1 = 97°32'52''}$		
$\frac{1}{2}C_1 = 57°49'56''$		l cot 9.79862
$\mathbf{C_1 = 115°39'52''}$		

	(42′)	(49)
$\frac{1}{2}(b - a) = 9°13'40''$	l tan 9.21075	l sec 0.00565
$\frac{1}{2}(b + a) = 61°59'0''$		l cos 9.67185
$\frac{1}{2}(B_2 - A) = 37°6'44''$	l csc 0.21941	
$\frac{1}{2}(B_2 + A) = 83°28'54''$	l sin 9.99718	l tan 0.94211
$\frac{1}{2}c_2 = 14°58'35''$	l tan 9.42734	
$\mathbf{c_2 = 29°57'10''}$		
$\frac{1}{2}C_2 = 13°30'4''$		l cot 0.61961
$\mathbf{C_2 = 27°0'8''}$		

This solution may be checked by the law of sines.

EXERCISES

Solve the following spherical triangles:

1. $a = 68°52'48''$, $b = 56°49'46''$, $B = 45°15'12''$.

2. $a = 34°0'30''$, $A = 61°29'30''$, $B = 24°30'30''$.

3. $a = 42°15'20''$, $A = 36°20'20''$, $B = 46°30'40''$.

4. $a = 59°28'27''$, $A = 52°50'20''$, $B = 66°7'20''$.

5. $b = 80°$, $A = 70°$, $B = 120°$.

6. $a = 63°29'56''$, $b = 132°14'23''$, $C = 61°18'27''$.

151. MISCELLANEOUS EXERCISES

Solve the following spherical triangles:

1. $a = 120°22'40''$, $b = 111°34'27''$, $c = 96°28'35''$.

2. $a = 41°6'0''$, $b = 119°24'0''$, $C = 48°54'38''$.

3. $A = 121°32'41''$, $B = 82°52'53''$, $C = 98°51'55''$.

4. $c = 86°15'15''$, $A = 153°17'6''$, $B = 78°43'32''$.

5. $b = 84°21'56''$, $A = 115°36'45''$, $B = 80°19'12''$.

6. $a = 40°5'26''$, $b = 118°22'7''$, $C = 160°1'23''$.

7. $b = 150°17'26''$, $A = 61°37'53''$, $B = 139°54'34''$.

8. $a = 31°11'7''$, $b = 32°19'18''$, $c = 33°15'21''$.

9. $A = 63°57'39''$, $B = 35°4'3''$, $c = 132°44'8''$.

10. $A = 59°55'10''$, $B = 85°36'50''$, $C = 59°55'10''$.

11. In a spherical triangle given c, A, $a + b$, derive

$$\tan \tfrac{1}{2}A \tan \tfrac{1}{2}B = \frac{\sin (s - c)}{\sin s}.$$

12. Given two sides and the sum of the opposite angles of a spherical triangle derive a formula from Gauss's equations (Exercise 2, §148) for computing the remaining angle.

13. Prove the relation

$$\cot a \sin b = \cot A \sin C + \cos C \cos b.$$

Hint. Multiply equation (13) by $\cos b$, substitute in (11), and then divide by $\sin b \sin a$, etc.

14. If c_1 and c_2 be the two values of the third side when A, a, b are given and the triangle comes under Case V, show that

$$\tan \tfrac{1}{2}c_1 \tan \tfrac{1}{2}c_2 = \tan \tfrac{1}{2}(b - a) \tan \tfrac{1}{2}(b + a).$$

15. If b is the base of an isosceles spherical triangle and if the equal sides a, c be bisected by the arc h of a great circle, show that

$$\sin \tfrac{1}{2}h = \tfrac{1}{2} \sin \tfrac{1}{2}b \sec \tfrac{1}{2}a.$$

16. Prove that

$$\sin (s - a) + \sin (s - b) + \sin (s - c) - \sin s = 4 \sin \tfrac{1}{2}a \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c.$$

17. In a spherical triangle $A = B = 2C$, show that

$$8 \sin^2 \tfrac{1}{2}C(\cos s + \sin \tfrac{1}{2}C) \cos \tfrac{1}{2}c = \cos a.$$

18. Show that

$$\text{hav } a = \frac{\sin \tfrac{1}{2}E \sin (A - \tfrac{1}{2}E)}{\sin B \sin C}$$

where $E = (2\sigma - 180°)$ and $\sigma = \tfrac{1}{2}(A + B + C)$.

19. In an equilateral spherical triangle, show that $2 \cos \tfrac{1}{2}a \sin \tfrac{1}{2}A = 1$.

20. If in a spherical triangle $C = A + B$, show that

$$\cos C = -\tan \tfrac{1}{2}a \tan \tfrac{1}{2}b.$$

21. If the sum of the angles of a spherical triangle is 360°, show that

$$\cos^2 \tfrac{1}{2}a + \cos^2 \tfrac{1}{2}b + \cos^2 \tfrac{1}{2}c = 1.$$

CHAPTER XV

VARIOUS METHODS OF SOLVING OBLIQUE SPHERICAL TRIANGLES

152. Introduction. In this chapter we shall again consider methods of solving triangles coming under the six-case classification of §145. The principal method of this chapter will consist in dividing the given triangle into two right triangles and applying Napier's rules to the parts.

153. Cases III and IV. Consider the solution of the spherical triangle in which the given parts are a, b, and C, that is, two sides and the included angle. Figure 1 represents the spherical

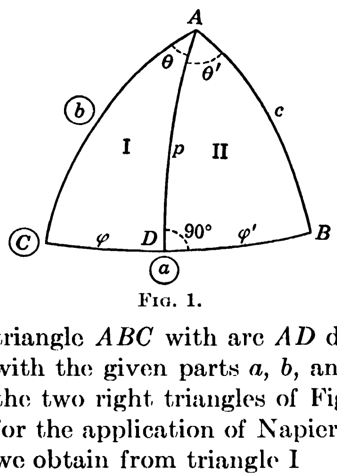

FIG. 1.

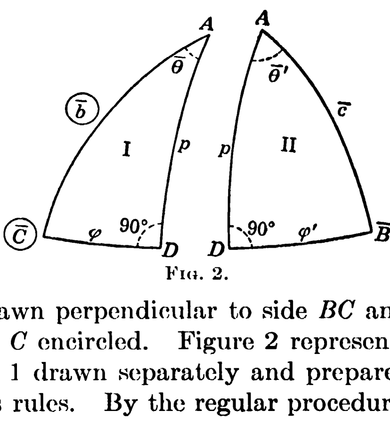

FIG. 2.

triangle ABC with arc AD drawn perpendicular to side BC and with the given parts a, b, and C encircled. Figure 2 represents the two right triangles of Fig. 1 drawn separately and prepared for the application of Napier's rules. By the regular procedure, we obtain from triangle I

$$\tan \varphi = \tan b \cos C, \tag{1}$$
$$\cot \theta = \cos b \tan C, \tag{2}$$
$$\sin p = \sin b \sin C, \tag{3}$$
$$\sin p = \cot \theta \tan \varphi. \quad (Check) \tag{4}$$

After φ, θ and p have been found by means of (1), (2), and (3), the parts p and $\varphi' = a - \varphi$ in triangle II will be known. Now apply Napier's rules to obtain the following formulas for solving triangle II:

$$
\begin{aligned}
\varphi' &= a - \varphi, && (5)\\
\cot B &= \cot p \sin \varphi', && (6)\\
\cot \theta' &= \sin p \cot \varphi', && (7)\\
\cos c &= \cos p \cos \varphi', && (8)\\
\cos c &= \cot \theta' \cot B, \quad (\textit{Check}) && (9)\\
A &= \theta + \theta'. && (10)
\end{aligned}
$$

If the given parts are not named a, b, and C, the computer may derive a new set of formulas, or he may obtain the desired set by interchanging letters in (1) to (10). For example, if the given parts are a, c, and B, get the appropriate formulas by replacing b by c, and c by b, B by C, C by B in (1) to (10). Thus, from (1), (2), and (3), we get

$$
\begin{aligned}
\tan \varphi &= \tan c \cos B,\\
\cot \theta &= \cos c \tan B,\\
\sin p &= \sin c \sin B.
\end{aligned}
$$

To solve a triangle when the two angles and the side common to them are known, use (11) of §139 to find two sides and the included angle of the polar triangle, solve the polar triangle by formulas (1) to (10), and from the result get the desired solution by again using (11) of §139. Also, one may drop a perpendicular from the vertex of one of the given angles to the opposite side and solve the two resulting right triangles by the methods of Chap. XIII.

154. Observations and illustrative example. One can usually draw a rough sketch representing the spherical triangle under consideration and showing its associated pair of right triangles in their proper relative positions. He can then solve the two right triangles and assemble the desired solution from the computed parts.

However, by keeping in mind the following observations, he may use formulas (1) to (10) without reference to a figure.

(*A*) *Each of the parts* a, b, c, A, B, C *of a spherical triangle is positive and less than* 180°.

(*B*) *When* $\tan \varphi$ *is positive,* φ *should be chosen positive and acute. When* $\tan \varphi$ *is negative,* φ *should be chosen in the second quadrant.**

* φ might be taken negative. The remaining part of the solution would have to be carried out in harmony with this choice.

(*C*) *In accordance with Rule A, §136, p and C are of the same quadrant if φ is positive.*

(*D*) *Each of the pairs φ and θ, φ' and θ', must be of the same quadrant and have the same sign. Thus, if φ' is negative and acute, θ' must be negative and acute; if φ is positive and of the second quadrant, θ must be positive and of the second quadrant.*

(*E*) *Angle B obtained from* (6) *is of the first or second quadrant according as cot B is positive or negative. It is not necessarily of the same quadrant as p.*

The following solution will illustrate the application of these observations and the general method of procedure.

Example. Solve the spherical triangle in which $a = 78°43'$, $b = 118°12'$, $C = 59°27'$.

Solution. The following form, showing the solution by means of formulas (1) to (10) of §153, is self-explanatory.

	(1) and (*check*)	(2)	(3)
$a = 78°43'$			
$b = 118°12'$	l tan $(-)0$ 27068	l cos $(-)9$ 67445	l sin 9.94513
$C = 59°27'$	l cos 9 70611	l tan 0 22899	l sin 9 93510
$\varphi = 136°31'48''$	l tan $(-)9$ 97679		
$\theta = 128°40'55''$	l cot $(-)9$ 90344	l cot $(-)9$ 90344	
$p = 49°22'27''$	l sin 9 88023		l sin 9 88023
	(6) and (*check*)	(7)	(8)
$\varphi' = -(57°48'48'')$	l sin $(-)9$ 92753	l cot $(-)9$ 79894	l cos 9 72647
$p = 49°22'27''$	l cot 9 93343	l sin 9 88023	l cos 9 81366
$B =$ **125°58′51″**	l cot $(-)9$ 86096		
$\theta' = -(64°27'56'')$	l cot $(-)9$ 67917	l cot $(-)9$ 67917	
$c =$ **69°42′21″**	l cos 9.54013		l cos 9 54013

$A = \theta + \theta' =$ **64°12′59″**.

Figure 3 shows the right triangles *CAD* and *DBA* in their proper relative positions.

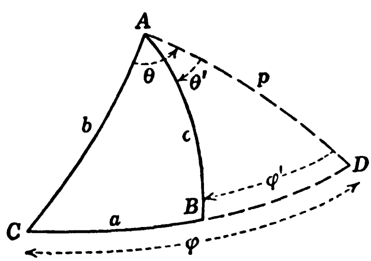

FIG. 3.

EXERCISES

Solve each of the following triangles by solving the two auxiliary right triangles:

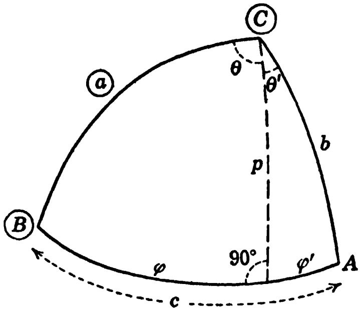

FIG. 4.

1. $C = 129°5'28''$,
   $B = 142°12'42''$,
   $a = 60°4'54''$.
   See Fig. 4.

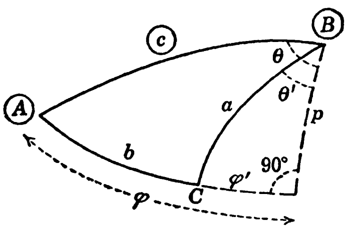

FIG. 5.

2. $A = 31°34'26''$,
   $B = 30°28'12''$,
   $c = 70°2'3''$.
   See Fig. 5.

Solve the following spherical triangles by the method of this article:

3. $a = 88°24'0''$,
   $b = 56°48'0''$,
   $C = 128°16'0''$.
4. $b = 120°30'0''$,
   $c = 70°20'0''$,
   $A = 50°10'0''$.
5. $a = 76°24'0''$,
   $b = 58°19'0''$,
   $C = 116°30'0''$.
6. $a = 88°37'40''$,
   $c = 125°18'20''$,
   $B = 102°16'36''$.
7. $a = 86°18'40''$,
   $b = 45°36'20''$,
   $C = 120°46'30''$.
8. $b = 132°17'30''$,
   $c = 78°15'15''$,
   $A = 40°20'10''$.

Solve the following triangles by solving the polar triangle.

9. $A = 120°10'0''$,
   $B = 100°20'0''$,
   $c = 30°5'0''$.
10. $A = 27°22'34''$,
    $C = 91°26'44''$,
    $b = 120°18'33''$.

155. Case III. Alternate method. Another set of formulas sufficient to solve the spherical triangle for which two sides and

the included angle are known do not contain p. Applying Napier's rule to triangle I of Fig. 6, we obtain

$$\tan \varphi = \tan b \cos C. \quad (11)$$

Also

$$\varphi' = a - \varphi. \quad (12)$$

Again, by using Napier's rules, we obtain from triangles II and I

$$\sin \varphi' = \cot B \tan p,$$
$$\sin \varphi = \cot C \tan p. \quad (a)$$

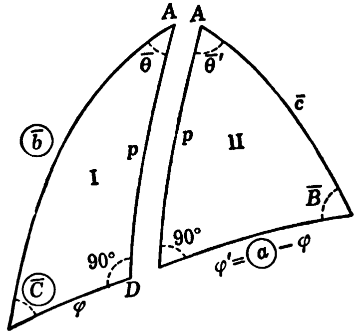

Fig. 6.

Dividing the first of these equations by the second, member by member, and solving the result for cot B, we get

$$\cot B = \cot C \sin \varphi' \csc \varphi. \quad (13)$$

Note that the equations (a) were found by using φ', p, and B in triangle II and the homologous parts φ, p, and C in triangle I. The procedure to get (13) will be followed to obtain a formula for cos c. From triangles II and I, we get

$$\cos c = \cos \varphi' \cos p, \qquad \cos b = \cos \varphi \cos p.$$

Dividing the first of these equations by the second, member by member, and solving for cos c, we get

$$\cos c = \cos b \sec \varphi \cos \varphi'. \quad (14)$$

From triangle I

$$\cot \theta = \cos b \tan C; \quad (15)$$

from triangle II

$$\cot \theta' = \cos c \tan B, \quad (16)$$

and

$$A = \theta + \theta'. \quad (17)$$

The law of sines may be used as a check formula.

The observations of §154, except those referring to p, apply also to the solution based on the formulas of this article.

Example. Use formulas (11) to (17) of this article to solve the spherical triangle in which $a = 68°20'25''$, $b = 52°18'15''$, $C = 117°12'20''$.

Solution. The solution and the check by the law of sines are displayed in the following form:

	(11)	(13)	(14)	(15)	(16)
$a = 68°20'25''$					
$b = 52°18'15''$	l tan 0 11194		l cos 9 78638	l cos 9 78638	
$C = 117°12'20''$	l cos (−)9 66009	l cot (−)9 71100		l tan (−)0 28900	
$\varphi = 149°23'29''$	l tan (−)9 77203	l csc 0 29314	l sec (−)0 06517		
$\varphi' = a - \varphi = -81°3'4''$		l sin (−)9 99468	l cos 9 19188		
$B = \mathbf{45°4'41''}$		l cot 9 99882			l tan 0 00118
$c = \mathbf{96°20'43''}$			l cos (−)9 04343		l cos (−)9 04343
$\theta = 139°56'51''$				l cot (−)0 07538	
$\theta' = -83°40'35''$					l cot (−)9 04461
$A = \theta + \theta' = \mathbf{56°16'16''}$					

Check					
a	l sin 9 96820	b	l sin 9 89832	c	l sin 9 99733
A	l sin 9 91995	B	l sin 9 85008	C	l sin 9 94909
	0 04825		0 04824		0 04824

EXERCISES

Solve the following spherical triangles by the method of this article.

1. $a = 88°24'0''$, $b = 56°48'0''$, $C = 128°16'0''$.

2. $b = 120°30'0''$, $c = 70°20'0''$, $A = 50°10'0''$.

3. $a = 76°24'0''$, $b = 58°19'0''$, $C = 116°30'0''$.

4. $a = 88°37'40''$, $c = 125°18'20''$, $B = 102°16'36''$.

5. $a = 86°18'40''$, $b = 45°36'20''$, $C = 120°46'30''$.

6. $b = 132°17'30''$, $c = 78°15'15''$, $A = 40°20'10''$.

Solve the following triangles by solving the polar triangle.

7. $A = 120°10'0''$, $B = 100°20'0''$, $c = 30°5'0''$.

8. $A = 27°22'34''$, $C = 91°26'44''$, $b = 120°18'33''$.

9. Using Fig. 7, derive formulas (a) to (g).

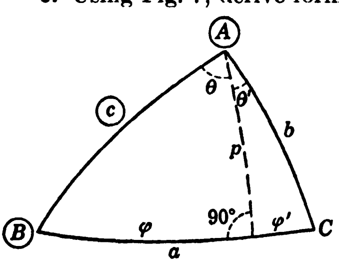

FIG. 7.

$$\cot\theta = \cos c \tan B, \qquad (a)$$
$$\theta' = A - \theta, \qquad (b)$$
$$\tan b = \tan c \cos\theta \sec\theta', \qquad (c)$$
$$\cos C = \cos B \csc\theta \sin\theta', \qquad (d)$$
$$\tan\varphi = \cos B \tan c, \qquad (e)$$
$$\tan\varphi' = \cos C \tan b, \qquad (f)$$
$$a = \varphi + \varphi'. \qquad (g)$$

Using the formulas of Exercise 9, solve each of the following triangles:

10. $a = 129°5'28''$,
$B = 142°12'42''$,
$C = 60°4'54''$.

11. $A = 31°34'26''$,
$B = 30°28'12''$,
$c = 70°2'3''$.

156. Haversine solution of Case III. Evidently the law of cosines could be used to find a when b, c, and A are given. This would not, however, be convenient for logarithmic computation. A formula for finding a directly by using a table of haversines will be developed from the law of cosines.

The law of cosines may be written

$$\cos a = \cos b \cos c + \sin b \sin c \cos A. \tag{18}$$

By definition $\text{hav } \theta = \frac{1}{2}(1 - \cos \theta)$. Solving this for $\cos \theta$, we get $\cos \theta = 1 - 2 \text{ hav } \theta$. Hence

$$\cos a = 1 - 2 \text{ hav } a, \qquad \cos A = 1 - 2 \text{ hav } A. \tag{19}$$

Substituting the expressions for $\cos a$ and $\cos A$ from (19) in (18), we obtain after slight simplification

$$1 - 2 \text{ hav } a = \cos b \cos c + \sin b \sin c - 2 \sin b \sin c \text{ hav } A. \tag{20}$$

Now $\cos b \cos c + \sin b \sin c = \cos (b - c) = 1 - 2 \text{ hav } (b - c)$. Replacing $\cos b \cos c + \sin b \sin c$ by $1 - 2 \text{ hav } (b - c)$ in (20) and solving for hav a, we obtain

$$\textbf{hav } \boldsymbol{a} = \textbf{hav } (\boldsymbol{b} - \boldsymbol{c}) + \textbf{sin } \boldsymbol{b} \textbf{ sin } \boldsymbol{c} \textbf{ hav } \boldsymbol{A}. \tag{21}$$

Similarly,

$$\text{hav } b = \text{hav } (a - c) + \sin a \sin c \text{ hav } B, \tag{22}$$
$$\text{hav } c = \text{hav } (a - b) + \sin a \sin b \text{ hav } C. \tag{23}$$

After a side has been computed by the haversine formula, three sides and an angle will be known. The other two angles may then be obtained by using the law of sines. The facts that when $a < b < c$ then $A < B < C$ and that the sum of two sides is greater than the third side will often serve to determine the quadrant of each angle thus found. Also a rough sketch will sometimes serve the same purpose. When the quadrants of the angles cannot be determined by the methods suggested, other formulas should be used. For this purpose, the result of solving (21) for hav A,

$$\text{hav } A = \frac{\text{hav } a - \text{hav } (b - c)}{\sin b \sin c}, \tag{24}$$

and the corresponding formulas for hav B and hav C are useful.

Example. Use (21) to find the side a of a spherical triangle in which $b = 59°29'30''$, $c = 109°39'40''$, $A = 50°10'10''$; then find B and C by the law of sines.

Solution. The formulas to be used are

$$\text{hav } a = \text{hav } (b - c) + \sin b \sin c \text{ hav } A, \tag{a}$$
$$\sin B = \sin b \sin A \csc a, \tag{b}$$
$$\sin C = \sin c \sin A \csc a. \tag{c}$$

The solution is displayed in the following form:

	(a)	(a)	(b)	(c)
$b = 59°29'30''$	l sin 9 93529		l sin 9.93529	
$c = 109°39'40''$	l sin 9.97391			l sin 9.97391
$A = 50°10'10''$	l hav 9.25465		l sin 9 88533	l sin 9.88533
$c - b = 50°10'10''$		n hav 0 17974		
	log 9.16385	n 0 14583		
a = **69°34'56''**		n hav 0 32557	l csc 0 02818	l csc 0 02818
B = **44°54'35''**			l sin 9 84880	
C = **129°29'54''**				l sin 9.88742

EXERCISES

Using the haversine formula, find the unknown side in the following spherical triangles:

1. $b = 125°8'$,
$c = 64°26'$,
$A = 100°4'$.

2. $a = 131°15'$,
$b = 129°20'$,
$C = 103°37'20''$.

3. $a = 63°29'56''$,
$b = 132°14'23''$,
$C = 61°18'27''$.

4. $C = 48°20'$,
$b = 52°10'$,
$a = 49°20'$.

5. Solve Exercise 3 for B and A by using the law of sines.

6. Using the relation $\cos \theta = 1 - 2 \text{ hav } \theta$, derive from the cosine law
hav c = hav $(a - b)$ hav $(180° - C)$ + hav $(a + b)$ hav C.

157. Cases V and VI. Consider the solution of the spherical triangle in which the given parts are a, b, and A. In this case there may be two solutions. Figure 8 represents the spherical triangle ABC with arc CD drawn perpendicular to side AB and with the given parts A, a, and b encircled. The dotted line indicates a second position that arc CB may assume.

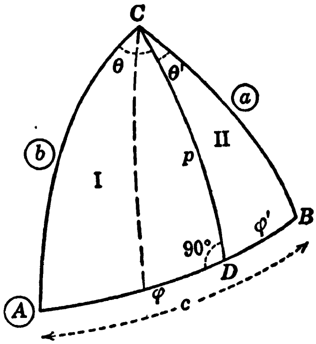

FIG. 8.

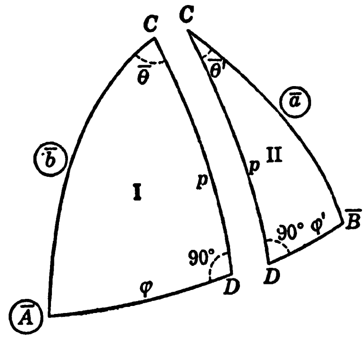

FIG. 9.

To obtain the formulas for solving a spherical triangle in which a, b, and A are the given parts, apply Napier's rules to triangle I in Fig. 9 to obtain

$$\tan \varphi = \tan b \cos A, \tag{25}$$
$$\cot \theta = \cos b \tan A, \tag{26}$$
$$\sin p = \sin b \sin A, \tag{27}$$
$$\sin p = \tan \varphi \cot \theta. \quad (\textit{Check}) \tag{28}$$

Since p is found from (27), p and a will be known in triangle II after triangle I has been solved. Hence apply Napier's rules to triangle II to get

$$\cos \varphi' = \cos a \sec p, \tag{29}$$
$$\sin B = \csc a \sin p, \tag{30}$$
$$\cos \theta' = \cot a \tan p, \tag{31}$$
$$\cos \theta' = \cos \varphi' \sin B. \quad (\textit{Check}) \tag{32}$$

Also it appears from Fig. 8 that

$$c = \varphi + \varphi', \tag{33}$$
$$C = \theta + \theta'. \tag{34}$$

The interchange of certain letter pairs in formulas (25) to (34) will give a new set of formulas applicable to a triangle for which the given parts are denoted by other letters than a, b, and A. A spherical triangle for which two angles and a side opposite one of them are given can be solved by applying formulas (25) to (34) to its polar triangle. Also a perpendicular may be drawn from the vertex of the unknown angle to the opposite side and special formulas derived by means of Napier's rules.

158. Observations and illustrative example. Slight modifications of the observations made in §154 apply to the solution under consideration. Since the cosine of a negative angle is the same as the cosine of an equal positive angle, *two values of φ', one the negative of the other, are chosen,* and the solution corresponding to each value is formed.

Since B is found from its sine, an angle and its supplement are written. From triangle II, $\cot B = \cot p \sin \varphi'$. Therefore B is of the same quadrant as p when φ' is positive. If φ' is negative, B is of the first or second quadrant according as p is of the second or first quadrant.

If $\cos \varphi' = 1$, $\varphi' = 0$, and there is only one solution. If $\log \cos \varphi' > 0$, there is no solution. Also each of the quantities b and B found from (33) and (34) must not be negative nor greater than 180°. Hence no solution corresponds to a value of φ' if either of the quantities $\varphi + \varphi'$ or $\theta + \theta'$ is greater than 180°.

The following solution will illustrate the method of procedure.

Example. Solve the spherical triangle in which $A = 115°12'$, $b = 73°10'$, $a = 110°35'$.

Solution.

	(25) and (*check*)	(26)	(27)
$b = 73°10'$	l tan 0 51920	l cos 9 46178	l sin 9 98098
$A = 115°12'$	l cos (−)9 62918	l tan (−)10 32738	l sin 9 95657
$\varphi = 125°23'51''$	l tan (−)0 14838		
$\theta = 121°36'30''$	l cot (−)9 78916	l cot (−) 9.78916	
$p = 119°59'43''$*	l sin 9 93754		l sin 9 93755

	(29) and (*check*)	(30)	(31)
$p = 119°59'43''$	l sec (−)0 30109	l sin 9 93755	l tan (−)0 23864
$a = 110°35'$	l cos (−)9 54601	l csc 0 02865	l cot (−)9 57466
$\varphi' = \pm(45°18'46'')$	l cos 9 84710		
$B = 112°18'48''$, $67°41'12''$	l sin 9 96620	l sin 9.96620	
$\theta' = \pm(49°24'52'')$	l cos 9 81330		l cos 9.81330

$c = \varphi \pm \varphi' = 170°42'37''$ and $80°5'5''$

$C = \theta \pm \theta' = 171°1'22''$ and $72°11'38''$

Therefore the two solutions are

$$c_1 = \mathbf{170°42'37''}, \quad C_1 = \mathbf{171°1'22''}, \quad B_1 = \mathbf{112°18'48''},\dagger$$
$$c_2 = \mathbf{80°5'5''}, \quad C_2 = \mathbf{72°11'38''}, \quad B_2 = \mathbf{67°41'12''}.$$

* p was chosen in the second quadrant in accordance with Rule A of §136.

† $B = 112°18'48''$ was placed in the solution associated with the positive value of φ' and θ' in accordance with the observation in the second paragraph of this article.

EXERCISES

Solve the following spherical triangles by the method of this article.

1. $a = 40°6'0''$,
$b = 118°22'0''$,
$A = 29°43'0''$.

2. $a = 128°15'0''$,
$b = 129°20'0''$,
$A = 130°25'0''$.

3. $a = 150°57'5''$,
$b = 134°15'54''$,
$A = 144°22'42''$.

4. $a = 52°45'20''$,
$c = 71°12'40''$,
$A = 46°22'10''$.

5. Solve each of the following triangles by solving its polar triangle.

(*a*) $c = 80°13'0''$,
$C = 78°15'0''$,
$B = 75°17'0''$.

(*b*) $a = 115°13'4''$,
$A = 120°43'0''$,
$B = 116°38'0''$.

6. Solve each of the following triangles by dropping a perpendicular from the unknown angle to the opposite side and solving the right triangles formed.

(*a*) $a = 150°42'40''$,
$A = 145°52'10''$,
$C = 79°37'20''$.

(*b*) $a = 147°12'40''$,
$A = 142°12'10''$,
$B = 75°57'20''$.

7. Using Fig. 10, derive formulas (*a*) to (*g*) of this exercise.

$$\begin{aligned}
\tan\varphi &= \cos A \tan b, && (a)\\
\cos\varphi' &= \cos\varphi \cos a \sec b, && (b)\\
c &= \varphi + \varphi', && (c)\\
\tan B &= \tan A \csc\varphi' \sin\varphi, && (d)\\
\cot\theta &= \cos b \tan A, && (e)\\
\cot\theta' &= \cos a \tan B, && (f)\\
C &= \theta + \theta'. && (g)
\end{aligned}$$

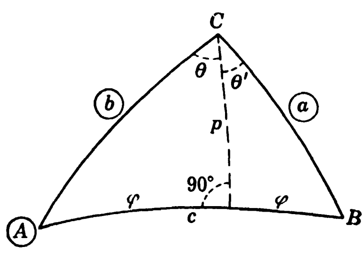

FIG. 10.

8. Using the formulas of Exercise 7, solve Exercises 1 to 3.

159. Cases I and II. The most expeditious method of solving a spherical triangle in which three sides are given employs formulas (31) to (34) of §146. However, one angle may be found by using

$$\cos A = (\cos a - \cos b \cos c) \csc b \csc c,$$

a formula obtained from the law of cosines, or by using (24) of §156, namely

$$\text{hav } A = [\text{hav } a - \text{hav } (b - c)] \csc b \csc c.$$

Two sides and the included angle will then be known, and the method of §153 may be employed. The spherical triangle for which three angles are given may be solved by means of its polar triangle.

EXERCISES

Solve the following spherical triangles:

1. $a = 57°$, $b = 137°$, $c = 116°$.

2. $A = 150°$, $B = 131°$, $C = 115°$.

3. $a = 149°30'$, $b = 131°0'$, $c = 119°20'$.

4. $A = 116°35'36''$, $B = 105°14'48''$, $C = 43°17'12''$.

5. $a = 77°36'12''$, $b = 63°16'48''$, $c = 107°23'12''$.

6. $A = 136°19'36''$, $B = 43°18'30''$, $C = 114°43'18''$.

160. MISCELLANEOUS EXERCISES

Solve the following spherical triangles:

1. $a = 76°24'40''$, $b = 58°18'36''$, $C = 116°30'28''$.

2. $b = 99°40'48''$, $c = 100°49'30''$, $A = 65°33'10''$.

3. $A = 31°34'26''$, $B = 30°28'12''$, $c = 70°2'3''$.

4. $a = 40°5'26''$, $b = 118°22'7''$, $A = 29°42'34''$.

5. $a = 99°40'48''$, $b = 64°23'15''$, $A = 95°38'4''$.

6. $A = 73°11'18''$, $B = 61°18'12''$, $a = 46°45'30''$.

7. $a = 57°17'$, $b = 20°39'$, $c = 76°22'$.

8. $A = 86°20'$, $B = 76°30'$, $C = 94°40'$.

9. A ship sailing on a great circle crosses the equator in longitude 78°26′ W. with course 43°32′. Find its latitude when its longitude is 10° W.

10. A ship sails 5400 nautical miles from San Francisco along a great circle with initial course of 240°25′. Find the position reached. (For San Francisco, longitude $\lambda = 123°23'$ W; latitude $L = 38°28'$ N.)

11. Find the pole (L, λ) of the great circle of Exercise 10.

12. An airplane flies 7000 nautical miles along a great circle. If the initial course is $25°32'$ and if it reaches a point in latitude $18°15'$ N. and longitude $12°15'$ W., find the position of departure.

13. Using (21) and (24), find the initial course and distance for a voyage along a great circle from Los Angeles (latitude $L = 34°03'$ N., longitude $\lambda = 118°15'$ W.) to Auckland (latitude $L = 41°18'$ S., longitude $\lambda = 174°51'$ E.).

14. Using (24) find the three angles of the spherical triangle in which $a = 70°14'20''$, $b = 49°24'10''$, $c = 38°46'10''$.

CHAPTER XVI

APPLICATIONS

161. Nature of applications. Many applications of spherical trigonometry deal with time and with angular distances. These considerations of time and distance may have reference to bodies far removed from the earth (celestial) or to bodies on the earth (terrestrial).

The shape of the earth is approximately that of a sphere having a diameter of 7917 miles. In what follows we shall consider it as a sphere. Hence the problem of finding the great-circle distance between two points on the earth or of locating a point on it is a problem that may be solved by the use of spherical trigonometry. Time enters our considerations because the rotation of the earth about its axis once every day furnishes the basic unit of time.

162. Definitions and notations. The earth revolves about a diameter called its *axis*. One point where the axis cuts the surface of the earth is called the *north pole*, P_n; the other is called the *south pole*, P_s.

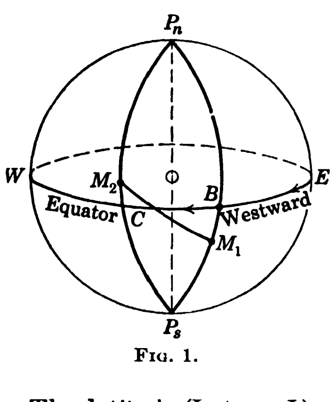

FIG. 1.

The *equator* is the great circle on the earth whose plane is perpendicular to the axis of the earth.

A *meridian* is a great circle on the earth passing through the north pole and the south pole. In Fig. 1, P_nBP_s and P_nCP_s represent meridians. Since meridians cut the equator at right angles, angular distances of points on the earth from the equator are measured along meridians.

The *latitude* (Lat. or L) of a point on the earth is the angular distance of the point from the equator. It is measured along a

meridian north or south of the equator from 0° to 90°. In Fig. 1, CM_2 represents the latitude of M_2. In general, north latitude is considered positive, south latitude negative.

Because of the great importance of triangle $M_1P_nM_2$ in connection with problems relating to distances and angles on the earth, it is called the *terrestrial triangle.* Arc M_1M_2 represents the distance along the great-circle track from M_1 to M_2, and the angle $M_2M_1P_n$ gives the initial direction of the track. The angle of departure $P_nM_1M_2$ measured from the north around through the east from 0° to 360° is called the initial course C_n. For a person situated on the northern hemisphere of the earth at a point such as z in Fig. 2, north is along the tangent to the meridian away from the equator; for a person standing at z facing north, east is on his right, west is on his left, and south is opposite to the direction in which he is facing.

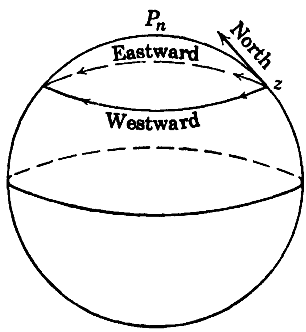

FIG. 2.

Figure 3 indicates directions at four positions on the earth.

The *longitude* (Long. or λ) of a point on the earth is the angle at either pole between the meridian passing through the point and some fixed meridian known as the *prime meridian.* It is measured east or west of the prime meridian from 0° to 180°. The meridian of Greenwich, England, is the prime meridian, not only for English and American navigators but also for those of many other nations.

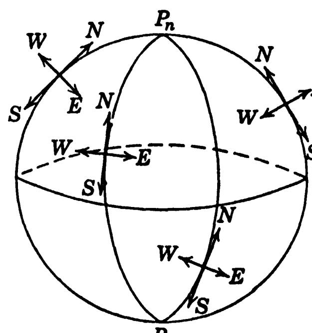

FIG. 3.

The latitude and longitude of a point give its position on the earth just as the two coordinates of a point give its position relative to a set of rectangular axes.

163. Course and distance. In general, the procedure of applying spherical trigonometry to solve problems relating to the earth consists in finding three parts of the terrestrial triangle, solving

for one or more of the other three parts, and interpreting the results. Consider, for example, the problem of finding the great-circle distance between two points M_1 and M_2 when the latitude and the longitude of each point are known. In Fig. 4, P_n represents the north pole, QK_1K_2Q' the equator, P_nGQP_s the

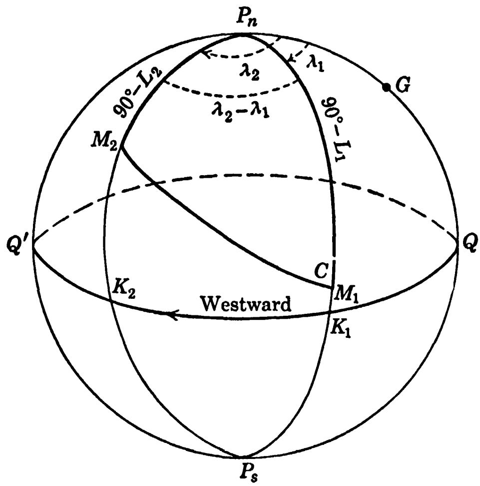

FIG. 4.

meridian of Greenwich, and M_1 and M_2 two places on the earth. The longitudes λ_1 of M_1 and λ_2 of M_2 are known; hence angle

$$M_1P_nM_2 = \lambda_2 - \lambda_1$$

is known. Also, the latitudes $L_1 = K_1M_1$ of M_1 and $L_2 = K_2M_2$ of M_2 are known; hence the arcs $M_1P_n = 90° - L_1 = co\text{-}L_1$ and $M_2P_n = 90° - L_2 = co\text{-}L_2$ are known. Thus, in triangle $M_1P_nM_2$, two sides $M_1P_n = co\text{-}L_1$ and $M_2P_n = co\text{-}L_2$ and the included angle $M_1P_nM_2 = \lambda_2 - \lambda_1$ are known. Consequently, we can solve this triangle by Napier's analogies, by the method of §153 or by that of §156.

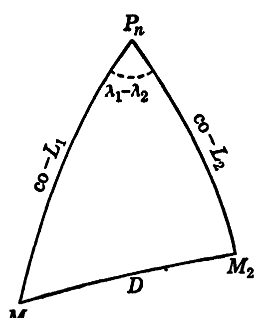

FIG. 5.

Example. Compute the initial great-circle course and the distance for a trip from St. Augustine lighthouse $L_1 = 30°$ N., $\lambda_1 = 76°$ W. to the Strait of Gibraltar $L_2 = 36°$ N., $\lambda_2 = 5°30'$ W.

Solution. Substituting from Fig. 5, $90° - L_1$ for a, $90° - L_2$ for b, $\lambda_1 - \lambda_2$ for C, M_1 for B, and D for c in formulas (11), (12), (13), and (14) of §155, we obtain

$$\tan\varphi = \cos(\lambda_1 - \lambda_2)\tan(co\text{-}L_2) = \cos(\lambda_1 - \lambda_2)\cot L_2, \quad (a)$$

$$\varphi' = 90° - L_1 - \varphi = 90° - (L_1 + \varphi), \quad (b)$$

$$\cot M_1 = \cot(\lambda_1 - \lambda_2)\sin\varphi' \csc\varphi$$

or

$$\cot M_1 = \cot(\lambda_1 - \lambda_2)\cos(L_1 + \varphi)\csc\varphi, \quad (c)$$

$$\cos D = \cos\varphi' \sec\varphi \cos(co\text{-}L_2) = \sin(L_1 + \varphi)\sec\varphi \sin L_2. \quad (d)$$

Substituting the given values in formulas (a), (b), (c), and (d) and evaluating φ, M_1, and D from the results, we obtain the following solution:

	(*a*)	(*c*)	(*d*)	(*Check*)†
$\lambda_1 - \lambda_2 = 70°30'$	$l \cos 9\ 52350$	$l \cot 9\ 54915$		
$L_2 = 36°$	$l \cot 0\ 13874$		$l \sin 9\ 76922$	
$\varphi = 24°40'35''$	$l \tan 9\ 66224$	$l \csc 0\ 37935$	$l \sec 0\ 04159$	
$L_1 + \varphi = 54°40'35''$		$l \cos 9\ 76208$	$l \sin 9\ 91163$	$l \tan 0.14956$
M_1 = **N.63°52'30" E.**		$l \cot 9\ 69058$		$l \cos 9\ 64378$
$D = 58°8'43''$ = **3488.7 miles***			$l \cos 9.72244$	$l \tan 0\ 20666$
1				$\log 0\ 00000$

The problem of finding course and distance is conveniently solved by using formula (23) §156 to find distance D and then using the law of sines to find the course angle. To apply (23), §156, to Fig. 5, replace c by D, a by $90° - L_1$, b by $90° - L_2$, and C by $\lambda_1 - \lambda_2$ to obtain

$$\text{hav } D = \text{hav }(L_2 - L_1) + \cos L_1 \cos L_2 \text{ hav }(\lambda_1 - \lambda_2). \quad (1)$$

The law of sines applied to Fig. 5 gives

$$\sin M_1 = \cos L_2 \sin(\lambda_1 - \lambda_2)\csc D. \quad (2)$$

So far as formula (2) is concerned the angle M_1 may be of the first quadrant or of the second. A navigator usually knows the course approximately and thus knows the quadrant to be expected. Very often the quadrant of M_1 can be determined by considering that the order of magnitude of the sides of a spherical

* 1′ of angle at the center of the earth subtends 1 nautical mile = 6080 ft. on a great circle of the earth. Hence, when an arc of a great circle on the earth is expressed in minutes, it is also expressed in nautical miles.

† The check formula was obtained by drawing a perpendicular from M_1 to P_nM_2 in Fig. 5 and applying Napier's rules.

triangle is the same as that of the opposite angles or by a rough sketch. When the suggested methods fail, the law of sines should not be employed. In such cases, the following formula may be used:

$$\text{hav } A = [\text{hav } a - \text{hav } (b - c)] \csc b \csc c.$$

EXERCISES

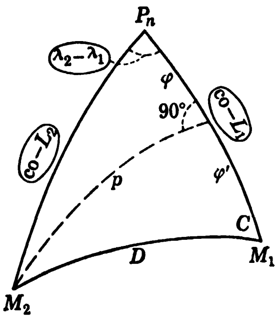

FIG. 6.

1. Figure 6 represents the terrestrial triangle with the arc of a great circle drawn through M_2 perpendicular to P_nM_1. Apply Napier's rules to the figure to obtain

$$\begin{aligned}
\tan \varphi &= \cos (\lambda_2 - \lambda_1) \cot L_2,\\
\varphi' &= 90° - (L_1 + \varphi),\\
\cos D &= \sin L_2 \sec \varphi \sin (L_1 + \varphi),\\
\cot C &= \cot (\lambda_2 - \lambda_1) \csc \varphi \cos (L_1 + \varphi).
\end{aligned}$$

2. In formulas (11) to (14) of §155 substitute $90° - L_1$ for a, $90° - L_2$ for b, $\lambda_2 - \lambda_1$, for C, M_1 for B, and D for c to obtain the formulas of Exercise 1.

3. Substitute for a, b, c, and C of formula (23) of §156 appropriate values from Fig. 6 to obtain

$$\text{hav } D = \text{hav } (L_1 - L_2) + \cos L_1 \cos L_2 \text{ hav } (\lambda_2 - \lambda_1).$$

Then write a formula from the law of sines for finding the course angle M_1.

4. Substitute for a, b, c, A, B, and C appropriate values from Fig. 6 in formulas (42), (47), (48), (49) of §148 to obtain formulas for solving the triangle of Fig. 6 completely.

5. Find the initial compass course and distance in nautical miles for a great-circle voyage from San Diego ($L_1 = 32°43'$ N., $\lambda_1 = 117°10'$ W.) to Hong Kong ($L_2 = 22°9'$ N., $\lambda_2 = 114°10'$ E.). Use the formulas of Exercise 1.

6. The great-circle distance from Cape Flattery, 48°24′ N., 124°44′ W., to Tutuila, 14°18′ S., 170°42′ E., is 5084.75 miles. Find the course of the ship on arrival at Tutuila if it follows a great-circle track from Cape Flattery to Tutuila.

7. Find the distance by great circle from New York, $L_1 = 40°40'$ N., $\lambda_1 = 4^h\ 55^m\ 54^s$ W., to Cape of Good Hope, $L_2 = 33°56'$ S., $\lambda_2 = 1^h\ 13^m\ 55^s$ E.

8. The distance from Cape Flattery, 48°24′ N., 124°44′ W., to Tutuila, 14°18′ S., 170°42′ E., is 5085 miles. Find the initial course for a trip from Cape Flattery to Tutuila, by great circle.

9. Find the initial course and the distance for a great-circle voyage from Cape of Good Hope 34°22′ S., 18°30′ E. to Singapore 1°17′30″ N., 103°51′ E. Also find the latitude and longitude of the northern vertex* (the most northerly point) of this great-circle track. Use the formulas of Exercise 3.

10. Find the initial course and the distance for a voyage along a great circle from Los Angeles $L = 34°03'$ N., $\lambda = 118°15'$ W. to Auckland $L = 41°18'$ S., $\lambda = 174°51'$ E.

11. The northern vertex of the great-circle track from San Francisco, Lat. 38°28′ N., Long. 123°23′ W., to Manila, Lat. 14°35′ N., Long. 120°57′ E., has Lat. 46°07′ N., Long. 163°33′36″ W. Find the latitude reached when the longitude is 180°.

12. The northern vertex of a great-circle track is in $L = 60°50'26''$ N., $\lambda = 60°29'37''$ E. Given the following positions:

Rio de Janeiro:	$L = 22°55'$ S., $\lambda = 43°09'$ W.,
Strait of Gibraltar:	$L = 35°53'$ N., $\lambda = 5°42'$ W.,
Cape St. Roque:	$L = 5°29'$ S., $\lambda = 35°15'$ W.,
Cape Manuel:	$L = 14°39'$ N., $\lambda = 17°27'$ W.

When following this track, what will be the
(*a*) Longitude when in the latitude of Rio de Janeiro?
(*b*) Latitude when in the longitude of Gibraltar?
(*c*) Longitude when in the latitude of Cape St. Roque?
(*d*) Latitude when in the longitude of Cape Manuel?
(*e*) Course and distance when in the latitude of Rio de Janeiro?
(*f*) Distance from vertex when in the longitude of Gibraltar?

13. A ship sails from San Francisco $L = 38°28'24''$ N., $\lambda = 123°22'54''$ W., to Manila $L = 14°35'48''$ N., $\lambda = 120°57'18''$ E., following a great-circle track. Find the course angle at departure, the course angle at arrival, and the distance traveled.

14. Substitute $90° - L_1$ for a, $90° - L_2$ for b, $\lambda_1 - \lambda_2$ for C, M_1 for B, M_2 for A, D for C, in (42), (47), (48), (49) to obtain:

$$\frac{\sin \frac{1}{2}(M_2 - M_1)}{\sin \frac{1}{2}(M_2 + M_1)} = \frac{\tan \frac{1}{2}(L_2 - L_1)}{\tan \frac{1}{2}D}$$

* A meridian passing through the vertex of a great-circle track is perpendicular to the track.

$$\frac{\cos \frac{1}{2}(M_2 - M_1)}{\cos \frac{1}{2}(M_2 + M_1)} = \frac{\cot \frac{1}{2}(L_1 + L_2)}{\tan \frac{1}{2}D}$$

$$\frac{\sin \frac{1}{2}(L_2 - L_1)}{\cos \frac{1}{2}(L_2 + L_1)} = \frac{\tan \frac{1}{2}(M_2 - M_1)}{\cot \frac{1}{2}(\lambda_1 - \lambda_2)}$$

$$\frac{\cos \frac{1}{2}(L_2 - L_1)}{\sin \frac{1}{2}(L_2 + L_1)} = \frac{\tan \frac{1}{2}(M_1 + M_2)}{\cot \frac{1}{2}(\lambda_1 - \lambda_2)}$$

Using these formulas, solve Exercise 8.

164. The celestial sphere. Consider a fixed star so far away from our solar system that the light rays coming to us from this star appear to follow parallel lines independent of our position; for example, light rays coming from this star to us at one position of the earth's orbit appear to have the same direction as light rays coming from the star to us 6 months later when we are on the other side of the orbit of the earth or approximately 186 million miles from the first position. Since, to us, light rays from this star seem to travel in parallel lines, we naturally associate a fixed direction with it.

We shall speak of the *celestial sphere* as a sphere concentric with the earth and having a radius of unlimited length; by this we shall understand that any two parallel lines cut this sphere in the same point, and any two parallel planes cut it in the same great circle. With any point on this sphere is associated a fixed direction; the angular distance between two points on it may be considered, but not an actual distance in miles.

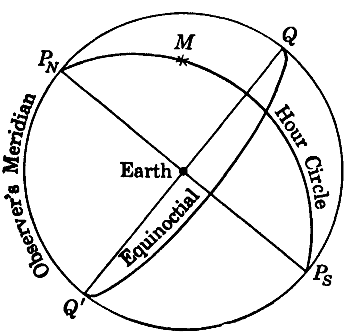

FIG. 7.

Figure 7 represents the celestial sphere with the earth at its center.

The point P_N on the celestial sphere where a line connecting the center of the earth to its north pole cuts the celestial sphere is called the *north celestial pole;* the point P_S diametrically opposite is called the *south celestial pole.*

The plane of the equator of the earth cuts the celestial sphere in the *equinoctial* or *celestial equator.* The *celestial poles* are the poles of the celestial equator.

The great circles such as P_NMP_S in Fig. 7, passing through the celestial poles, are called *hour circles* or *celestial meridians.*

The point Z (see Fig. 8) directly above an observer, that is, the point where a line connecting the center of the earth to an

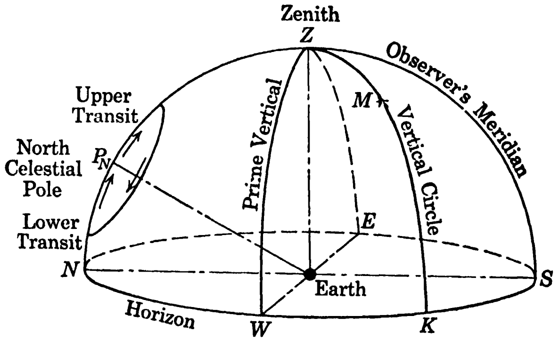

FIG. 8.

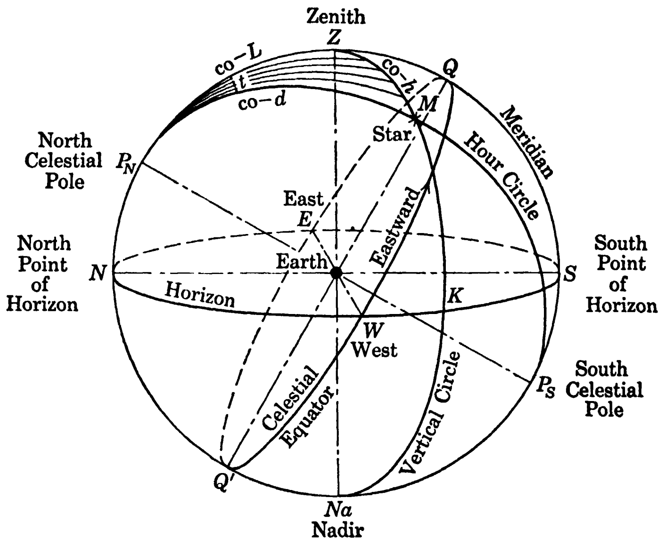

FIG. 9.

observer on it would intersect the celestial sphere, is called the *zenith.* The point on the celestial sphere diametrically opposite the zenith is called the *nadir* Na (see Fig. 9).

The *horizon* $NWSE$ of an observer is the great circle on the celestial sphere having the zenith and nadir as poles. A plane

tangent to the earth at a point on it intersects the celestial sphere in the celestial horizon associated with the point.

The point on the horizon directly below the north celestial pole is called the *north point* of the horizon. The *south point*, the *east point*, and the *west point* of the horizon are then determined in the usual way.

The great circles, such as ZMK of the celestial sphere, which pass through the zenith, are called *vertical circles*. Evidently they are all perpendicular to the horizon. The *prime vertical* is the vertical circle EZW (see Fig. 8) passing through the zenith and the east and west points of the horizon.

Figure 9 exhibits both the equinoctial system and the horizon system.

165. The astronomical triangle. *The spherical triangle* (*see* Fig. 10) *whose vertices are the north celestial pole, the zenith, and the projection of a heavenly body on the celestial sphere is called the astronomical triangle.* The solution of many of the problems of astronomy and of navigation requires the solution of this triangle.

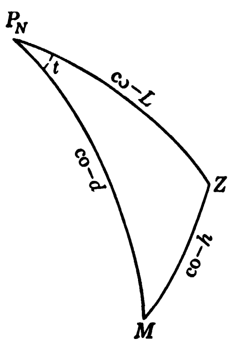

FIG 10.

The great-circle distance of a point on the celestial sphere from the celestial equator is called the *declination* d of the point. This corresponds to the latitude of a point on the earth. Inspection of Fig. 9 shows that the arc P_NM of the astronomical triangle is 90° minus declination, or *co-d*.

The *hour angle* t of a point on the celestial sphere is the angle between the hour circle passing through the zenith of the observer and the hour circle passing through the point.* As the earth turns on its axis, the heavenly bodies appear to move on the celestial sphere. Thus the angle through which the earth must turn to bring the celestial meridian of an observer into coincidence with the hour circle of a point on the celestial sphere appears as the hour angle of the point relative to the observer. The significance of the word hour angle appears when we consider

* Hour angle is often expressed as so many degrees east or west, according as the body observed is in the eastern sky or in the western sky. *It is often measured toward the west from* 0^h *to* 24^h (360°).

that the earth turns on its axis and moves in its orbit in such a way that the sun crosses the meridian of a place once every 24 hours.

The *altitude h* of a point on the celestial sphere is its great-circle distance from the horizon. Inspection of Fig. 9 shows that the side MZ of the astronomical triangle is 90° minus altitude or *co-h.*

The *azimuth* Z_n of a point on the celestial sphere is the angle at the zenith between the vertical circle of the point and the celestial meridian of the observer. It is usually measured from the *north point around through the east from* 0° *to* 360°. It is easy to write the azimuth Z_n when the angle Z of the astronomical triangle has been found.

Evidently the length P_NZ of the astronomical triangle is 90° minus the latitude of the observer, or $90° - L$.

166. Given t, d, L; to find h and Z.* Figure 11 represents the astronomical triangle with the given parts encircled. Since two sides and the included angle are given, we may adapt formulas (11) to (14) of §155 to the triangle of Fig. 11, or we may con-

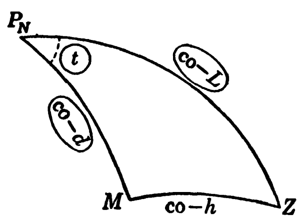

FIG. 11.

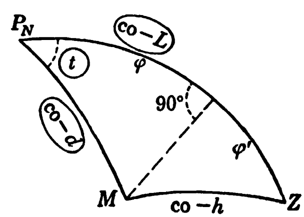

FIG. 12.

struct an arc of a great circle through M perpendicular to P_NZ, letter the triangle as shown in Fig. 12, and then apply Napier's rules to obtain

* If a navigator wishes to observe a number of stars at a particular time, say near sunset, he knows the time and from that can find the angle t; he knows approximately what his latitude will be, and he can find the declination of convenient stars in the Nautical Almanac. Hence he can compute the approximate positions, altitude, and azimuth of several stars in advance and thus expedite the process of locating, identifying, and observing them. Instead of computing h and Z, he can find these quantities in tables when such are available.

$$
\begin{aligned}
\tan \varphi &= \cos t \cot d, && (3)\\
\varphi' &= 90^\circ - L - \varphi = 90^\circ - (L + \varphi), && (4)\\
\cot Z &= \cot t \sin \varphi' \csc \varphi = \cot t \cos (L + \varphi) \csc \varphi, && (5)\\
\sin h &= \cos \varphi' \sec \varphi \sin d = \sin (L + \varphi) \sec \varphi \sin d, && (6)\\
\sin t \cos d \csc Z \sec h &= 1. \quad (\textit{Check}) && (7)
\end{aligned}
$$

If L represents the latitude of a place north of the equator, d should be taken positive for a body having north declination and negative for one having south declination, or vice versa.

Example. Use formulas (3) to (7) to find the altitude h and the azimuth Z_n of a star having $d = 1°9'15''$ S., $t = 45°10'30''$ east, if it is viewed by an observer in latitude 37°30′ N.

Solution. The solution found from the formulas (3), (4), (5), (6), and (7) appears below.

	(3)	(5)	(6)	(7)
$t = 45°10'30''$ E.	l cos 9 84816	l cot 9 99735		l sin 9 85080
$d = -1°9'15''$	l cot (−)1 69580		l sin (−)8 30411	l cos 9 99991
$L = 37°30'0''$ N.				
$\varphi = 91°38'13''$	l tan (−)1 54396	l csc 0 00018	l sec (−)1 54414	
$L + \varphi = 129°8'13''$		l cos (−)9 80015	l sin 9 88966	
Z = N.122°6′43″ E. $= Z_n$		l cot (−)9 79768		l csc 0 07211
h = 33°9′18″			l sin 9 73791	l sec 0 07717
1				log 9 99999

Evidently we could have used Napier's analogies to solve the triangle of the illustrative example, or we could have adapted formula (21) of §156 to the triangle and have used the result to find h.

EXERCISES

1. From Napier's analogies (§148) derive the formulas

$$
\begin{aligned}
\tan \tfrac{1}{2}(Z - M) &= \cot \tfrac{1}{2}t \sin \tfrac{1}{2}(L - d) \sec \tfrac{1}{2}(L + d),\\
\tan \tfrac{1}{2}(Z + M) &= \cot \tfrac{1}{2}t \cos \tfrac{1}{2}(L - d) \csc \tfrac{1}{2}(L + d).
\end{aligned}
$$

2. From formula (21) of §156, derive the formula*

$$\text{hav co-}h = \text{hav } (L - d) + \cos L \cos d \text{ hav } t.$$

* In the practice of navigation the method of Saint Hilaire is frequently used to determine the observer's position. In this method the value of Z is taken from azimuth tables, and h is computed by the formula of Exercise 2. The navigator then compares the computed value of h with the observed value and uses the difference between the two in determining the correction to be applied to the assumed position of his ship.

From the data of Exercises 3 to 10, compute h and Z_n.

3. $d = 6°15'$ S., $t = 14°6'$ W., $L = 21°18'$ N.

4. $d = 38°17'24''$ S., $t = 28°30'29''$ W., $L = 24°32'58''$ N.

5. $d = 59°56'$ N., $t = 60°32'$ E., $L = 44°45'$ N.

6. $d = 10°$ S., $t = 25°$ E., $L = 18°57'16''$ S.

7. $d = 10°$ N., $t = 40°$ W., $L = 35°$ S.

8. $d = 7°$ S., $t = 28°$ E., $L = 41°$ N.

9. $d = 8°$ N., $t = 35°$ E., $L = 39°$ N.

10. $d = 22°30'$ S., $t = 60°$ E., $L = 45°$ S.

From the data of Exercises 11 to 16, compute h.

11. $t = 3^h$ P.M., $d = 5°$ S., $L = 50°$ N.

12. $t = 25°$ E., $d = 10°$ S., $L = 18°57'16''$ S.

13. $t = 2^h\ 40^m$ P.M., $d = 10°$ N., $L = 35°$ S.

14. $t = 1^h\ 13^m\ 12^s$ P.M., $d = 13°21'$ N., $L = 15°54'$ S.

15. $t = 4^h\ 2^m\ 8^s$ P.M., $d = 59°56'$ N., $L = 44°45'$ N.

16. $t = 0^h\ 56^m\ 24^s$ P.M., $d = 6°15'$ S., $L = 21°18'$ N.

17. Check the answers of Exercises 3 to 10 using the formulas of Exercise 1.

18. If the observer's latitude is $29°17'24''$ N., and a star, in declination $30°21'14''$ S., has the hour angle $4^h\ 30^m\ 48^s$ W., find the altitude of the star. Use hav $(90° - h) =$ hav $(L - d) + \cos L \cos d$ hav t.

167. To find the time and amplitude of sunrise. Figure 13 represents a stereographic projection of the astronomical triangle P_NZM when the body M is the sun on the horizon. The dotted line indicates the path of the sun across the sky as a small circle each of whose points is distant co-d from the pole. When the sun crosses the meridian at K, it is noon. Hence t represents the angle through which the earth must turn during the time interval from sunrise to noon. Since the earth turns through 15° per hour, $t/15$ will be the number of hours from sunrise to noon if t is expressed in degrees. The declination of the sun can be found

from the Nautical Almanac,* and the latitude of the observer is supposed known. Therefore, to find a formula for t, apply Napier's rules to right spherical triangle NMP_N (Fig. 14), and write $\cos(180° - t) = \tan d \tan L$, or

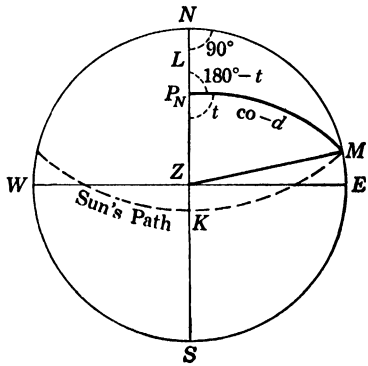

FIG. 13.

$$\cos t = -\tan d \tan L. \quad (8)$$

The angular distance from the east point of the horizon to

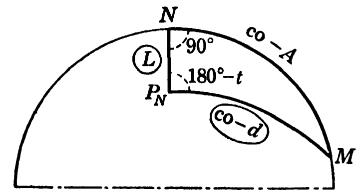

FIG. 14.

the sun at sunrise is called the *amplitude of sunrise*. From right spherical triangle NP_NM of Fig. 14 we find, by using Napier's rules, $\sin d = \cos L \sin A$, or

$$\sin A = \sin d \sec L. \quad (9)$$

From Fig. 14 we obtain the check formula

$$-\cot A \cot t \csc L = 1. \quad (10)$$

Example. Find the amplitude and the time of sunrise at *Annapolis*, $L = 38°59'$ N., at a time when the declination of the sun is 20° S.

Solution. The solution found from formulas (8), (9), and (10) appears below

	(8)	(9)	(10)
$L = 38°59'0''$	l tan 9.90811	l sec 0.10940	l csc 0.20128
$d = -20°0'0''$	l tan (−)9.56107	l sin (−)9.53405	
$t = 72°52'7''$	l cos 9.46918		l cos 9.48889
$A =$ **−26°6′14″**		l sin (−)9.64345	l cot 0.30983
1			log 0.00000

* Owing to refraction of the sunbeams by the earth's atmosphere, the sun will appear to be on the horizon considerably earlier than the results of this computation would indicate. In practice, corrections must be made on this account.

Since 15° indicates a time of 1^h, 72°52′7″ will indicate $4^h\ 51^m\ 28^s$. As t is the time from sunrise till noon, we obtain

$$12^h - (4^h\ 51^m\ 28^s) = \mathbf{7^h\ 8^m\ 32^s}$$

as the local apparent time* of sunrise. The negative sign before the amplitude indicates that the sun appeared on the horizon *south* of the east point.

EXERCISES

1. Find the amplitude of sunrise in latitude 38°58′53″ N. when the declination of the sun is 22°29′00″ S.

2. At Annapolis, Lat. 38°59′ N., the sun in declination 23°27′ N. has the altitude 0°, bearing easterly. Find the local apparent time.

3. Find the amplitude and the local apparent time of sunrise and sunset for Annapolis, Md., L = 38°58′53″ N., at summer and winter solstice ($d = \pm 23°27'7''$).

4. (*a*) Find the local apparent time of sunrise and sunset at Cape Nome, L = 64°23′ N. on Mar. 21, $d = 0°0'0''$, Dec. 21, d = 23°27′ S., and June 21, d = 23°27′ N. (*b*) Find the amplitude of the sun at each occurrence. (*c*) Find the length of the longest day and of the shortest day at Cape Nome.

5. Assuming that the declination of the sun ranges between 23°27′ S. to 23°27′ N., show that a place where the sun rises at midnight must lie within 23°27′ of a pole of the earth.

Hint. In the formula $\cos t = -\tan L \tan d$, let $t = 180° \ (= 12^h)$.

6. For a point on the earth having latitude 80° N. find (*a*) the declination of the sun when the time of daylight is just 24 hr.; (*b*) the declination of the sun when the night lasts just 24 hr.; (*c*) the least altitude and the greatest altitude of the sun during the day when the declination of the sun is 23°27′ N.; (*d*) the declination of the sun when continuous night begins; (*e*) the length of the shortest possible shadow cast by a vertical pole 20 ft. long.

168. To find the time of day. The declination of the sun can be found from the Nautical Almanac for a given time, and the altitude of the sun can be measured with a sextant. Hence, if the latitude of the place is known, the three sides of the astro-

* The noon of local apparent time occurs when the sun is on the meridian of the observer, and the time of day is expressed in terms of the hour angle of the sun. Owing to the fact that the sunbeams are refracted by the earth's atmosphere, the sun appears to be on the horizon slightly earlier than is indicated by the solution given.

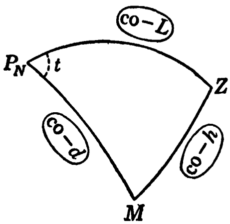

FIG. 15.

nomical triangle are known, and t can be found. Since t represents the angle through which the earth must turn before noon if the sun is in the eastern sky, and since the earth turns through 15° per hour, $t/15$ will be the interval of time before noon if t is expressed in degrees. If the sun is in the western sky, $t/15$ is the time since noon.

To obtain formulas adapted to this case, substitute from Fig. 15

$$a = 90° - h, \qquad b = p = (90° - d), \qquad c = 90° - L,$$
$$A = t, \qquad B = Z, \qquad S = \tfrac{1}{2}(h + p + L)$$

in (22) and (23) of §146, and simplify to obtain

$$\sin^2 \tfrac{1}{2}t = \text{hav } t = \cos S \sin (S - h) \sec L \csc p, \qquad (11)$$
$$\sin^2 \tfrac{1}{2}Z = \text{hav } Z = \sin (S - h) \sin (S - L) \sec h \sec L. \qquad (12)$$

The law of sines may be used to obtain the check formula

$$\sin Z \csc p \csc t \cos h = 1. \qquad (13)$$

Formula (11) gives the time of day, and formula (12) the angle from which the azimuth Z_n of the sun at the time of the observation may be determined.

Example. Find the azimuth Z_n of the sun and the local apparent time in New York, $L = 40°43'$ N., at the instant when the altitude of the sun is 30°10′ bearing west and its declination is 10° N.

Solution. The solution obtained by using formulas (11), (12), and (13) appears below.

	(11)	(12)	(13)
$L = 40°43'$	l sec 0 12036	l sec 0.12036	
$h = 30°10'$		l sec 0 06320	l cos 9.93680
$p = 90° - d = 80°$	l csc 0 00665		l csc 0.00665
$S = 75°26'30''$	l cos 9.40031		
$S - h = 45°16'30''$	l sin 9.85156	l sin 9.85156	
$S - L = 34°43'30''$		l sin 9.75560	
$t = 58°34'9''$	l hav 9.37888*		l csc 0.06891
$= \mathbf{3^h54^m17^s}$			
$Z =$ N. 103°36′20″ W.		l hav 9.79072*	l sin 9.98764
$Z_n = \mathbf{256°23'40''}$			
1			log 0.00000

* Those who do not use haversine tables may divide log hav t and

Since 58°34′9″ is equivalent to $3^h\ 54^m\ 17^s$ and the sun is in the western sky, the time is $3^h\ 54^m\ 17^s.7$ P.M.

EXERCISES

1. In formulas (22) and (23) of §146, substitute $a = 90° - h$, $b = p = (90° - d)$, $c = 90° - L$, $A = t$, $B = Z$, $S = \frac{1}{2}(h + p + L)$, and simplify to obtain formulas (11) and (12).

2. An observation of the altitude of the sun was made in each of the following cities. Find the azimuth of the sun and the local apparent time of observation in each case.

(*a*) Pensacola, Fla., $L = 30°21'$ N., sun's altitude $h = 24°30'$ bearing east, declination 20°42′ N.

(*b*) Philadelphia, Pa., $L = 40°0'$ N., $h = 20°0'$ E., $d = 20°0'$ N.

(*c*) Annapolis, Md., $L = 39°0'$ N., $h = 22°0'$ E., $d = 20°0'$ N.

Given the following data, find t and Z.

3. $L = 42°45'0''$ N., $d = 18°27'0''$ N., $h = 38°36'0''$ E.

4. $L = 25°35'0''$ N., $d = 10°24'0''$ S., $h = 35°19'0''$ E.

5. $L = 45°0'0''$ N., $d = 22°30'0''$ N., $h = 30°0'0''$ W.

6. $L = 30°0'0''$ N., $d = 15°0'0''$ N., $h = 45°0'0''$ W.

169. Ecliptic. Equinoxes. Right ascension. Sidereal time. The earth rotates about its axis once a day, and it also moves around the sun once a year. To an observer on the earth, the sun seems to move about the earth, describing a great circle on the celestial sphere called the *ecliptic*. The plane of the ecliptic is inclined at an angle of approximately 23°27′* to the plane of the celestial equator (see Fig. 16).

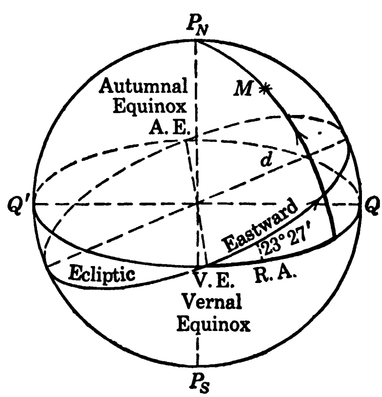

FIG. 16.

To an observer on the earth the sun appears to move eastward on the ecliptic, crossing the celestial equator while moving

log hav Z by 2 to obtain log sin $t/2$ and log sin $Z/2$, respectively, and then find $t/2$ and $Z/2$ from the table of logarithms of trigonometric functions.

* This angle 23°27′ is called *the obliquity of the ecliptic.*

northward at the *vernal equinox* V.E. and while moving southward at the *autumnal equinox* A.E.

The *right ascension RA* of a body on the celestial sphere is the angle *measured eastward* from the hour circle of the vernal equinox to the hour circle of the body; thus the right ascension of the sun varies from 0° to 360°. Evidently a point is located on the celestial sphere by its right ascension and its declination just as a point on the earth is located by its longitude and its latitude.

Relative to the stars, the earth turns about its axis once in approximately $23^h\ 56^m$ mean solar time. This period of time, called the *sidereal day,** is divided into 24 equal parts called sidereal hours, and the sidereal hours are divided into 60 equal sidereal minutes of 60 equal sidereal seconds each. Relative to the stars, the earth rotates through 15° each sidereal hour. The sidereal time of a place is measured from the time when the vernal equinox crosses the meridian of the place. *Hence the right ascension of the zenith of a place when expressed in hours, minutes, and seconds in the usual way is the sidereal time at that place.* From this it follows that the difference in the sidereal times of two points on the earth measures the hour angle between their celestial meridians; hence *the difference in the sidereal times of two points measures the difference in their longitudes.* A corollary to this may be stated: *the difference in sidereal time of Greenwich and that of a second place measures the longitude of the second place relative to Greenwich as prime meridian.*

Example. At a certain instant the sidereal time at one place is 2^h, and at a second place it is $4^h\ 30^m$. Find the longitude of the second place if that of the first place is (*a*) 0°, (*b*) 60° E., (*c*) 60° W.

* Besides sidereal time, we shall consider two other kinds, namely, *local apparent time* and *mean solar time.* The noon of local apparent time occurs when the sun is on the meridian of the observer, and the time of day is expressed in terms of the hour angle of the sun. Mean solar time is defined in terms of a fictitious sun that travels along the celestial equator at a uniform rate and makes a complete circuit in the same time as the actual sun. It is mean solar noon when the fictitious sun is on the meridian, and the mean solar time at any instant is the hour angle of the fictitious sun. This fictitious sun is used in order that we may have a day of uniform length throughout the year.

Solution. In Fig. 17 the circle represents the equator. V.E. represents the position of the vernal equinox, and A, B, and G represent, respectively, the points on the equator where the meridian of the first place, that of second place, and that of Greenwich meet the celestial equator. Since the sidereal time of A is 2^h, arc $VE\,A$ is $2 \times 15° = 30°$. Similarly, $VE\,B$ is $67\frac{1}{2}°$ and $AB = 37\frac{1}{2}°$. In case (a), Greenwich and A have the same meridian; hence the longitude of B is **$37\frac{1}{2}°$ E.**

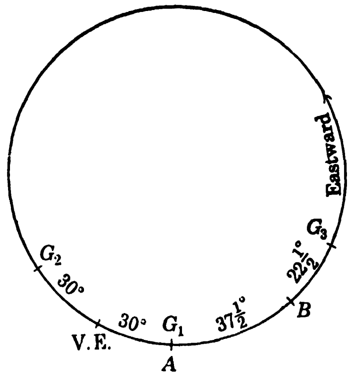

FIG. 17.

In Case (b), the meridian of Greenwich must be represented at G_2 in Fig. 17, since A is in longitude 60° E. Hence the longitude of B in this case is $60° + 37\frac{1}{2}° =$ **$97\frac{1}{2}°$ E.**

In Case (c), Greenwich must have the position G_3 in Fig. 17, since A is 60° west of Greenwich. Hence the longitude of B is $60° - 37\frac{1}{2}° =$ **$22\frac{1}{2}°$ W.**

EXERCISES

1. When it is 0^h (sidereal time) in Greenwich, it is 4^h at a certain place; find the longitude of this place.

2. At a place in longitude 81°15′ W. the sideral time is $10^h\ 17^m\ 30^s$. Find the sidereal time at Greenwich.

3. The longitude of a first place differs from that of a second place by 95°30′. When the sidereal time of the first place is 10^h, find the sidereal time of the second place if it is (a) east of the first place; (b) west of the first place.

4. An observer in longitude 24°30′ W. observes a star whose RA is $12^h\ 31^m\ 10^s$. A radio signal gives Greenwich sidereal time at the instant of the observation as $4^h\ 20^m\ 30^s$. Find the hour angle of the star.

5. If ST_1 is the sidereal time at a first place in longitude λ_1 west of Greenwich and ST_2 the sidereal time of a second place farther west, find the longitude of the second place.

6. On Jan. 13, 1932, the RA of the star Vega was $18^h\ 34^m\ 36^s$. What was the hour angle of Vega at the instant when the local sidereal time was $12^h\ 54^m\ 16^s$?

7. At a certain time, the Greenwich hour angle for the Star Rigel was 279°42′ W. Find the local hour angle of Rigel for an observer in Long. 76°38′30″ E.

170. The time sight. The data and formulas considered in §168 may be used to find the longitude of an observer whose latitude is known. This method of determining longitude at sea is called *the time sight.* In Fig. 18, P_NG represents the celestial meridian of Greenwich, P_NO the celestial meridian of the observer and P_NM the celestial meridian of the sun. The angle t found by the method of §168 determines the local apparent time at O; the angle GP_NM determines the local apparent time of Greenwich. Hence the longitude in degrees

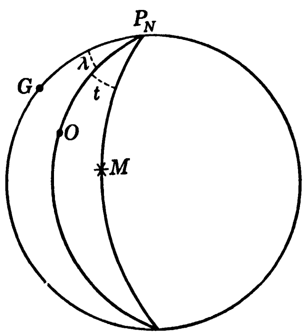

FIG. 18.

$$\lambda = \text{angle } GP_NO = \text{angle } GP_NM - t$$

of O is obtained by multiplying by 15 the difference in hours between the local apparent time of Greenwich and that of O. Sometimes it will be necessary to add angle GP_NM and angle t and sometimes to subtract them, depending on their relative positions. The local apparent time of Greenwich is obtained by radio, by telegraph, or by computing it from Greenwich mean time shown by a chronometer. The longitude is east or west according as the local time is later or earlier than Greenwich local time.

If the object M is a star, we still have

$$\lambda = \text{angle } GP_NM - t,$$

where t is computed as in §168, and the angle GP_NM is obtained by subtracting Greenwich sidereal time (computed from Greenwich mean time as given by a chronometer) from the right ascension of the star (obtained from a Nautical Almanac).

EXERCISES

In each of the following sets of data, ST refers to sidereal time of Greenwich, RA to the right ascension of an observed star, d to its declination, h to its altitude, and L to the latitude of the observer. Find the longitude of the observer for each situation.

1. $L = 30°0'0''$ N.,
   $d = 22°30'0''$ N.,
   $h = 45°0'0''$ W.,
   $ST = 4^h\ 10^m$,
   $RA = 13^h\ 5^m$.
2. $L = 12°0'0''$ S.,
   $d = 5°0'0''$ N.,
   $h = 45°0'0''$ W.,
   $ST = 10^h\ 6^m$,
   $RA = 8^h\ 7^m$.
3. $L = 39°0'0''$ N.,
   $d = 20°0'0''$ N.,
   $h = 22°0'0''$ E.,
   $ST = 5^h\ 8^m$,
   $RA = 2^h\ 0^m$.
4. $L = 30°30'0''$ N.,
   $d = 15°30'0''$ N.,
   $h = 44°30'0''$ W.,
   $ST = 17^h\ 15^m\ 24^s$,
   $RA = 10^h\ 5^m\ 6^s$.
5. $L = 40°0'0''$ N.,
   $d = 8°0'0''$ N.,
   $h = 20°0'0''$ E.,
   $ST' = 0^h\ 47^m\ 24^s$,
   $RA = 1^h\ 5^m\ 7^s$.
6. $L = 43°30'0''$ N.,
   $d = 15°0'0''$ N.,
   $h = 20°0'0''$ W.,
   $ST = 13^h\ 5^m\ 15^s$,
   $RA = 0^h\ 15^m\ 20^s$.

171. Meridian altitude. To find the latitude of a place on the earth. Figure 19 represents the cross section of the earth and of the surrounding celestial sphere by the plane of the meridian of an observer. qq' represents the equator of the earth; z, the position of the observer; and P_nP_s, the axis of the earth. QQ', Z, P_NP_S, N, and S represent, respectively, the celestial equator, the zenith, axis of celestial sphere, north point of the horizon, and south point of the horizon. Since qz represents the latitude of the observer and since arc qz = arc QZ = arc NP_N, it appears that the *latitude of an observer on the earth is equal to the declination of his zenith and to the altitude of the pole elevated above his horizon.*

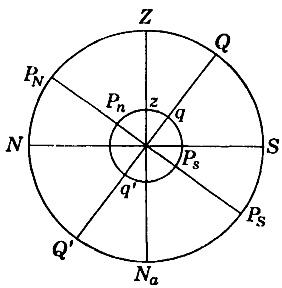

FIG. 19.

If, then, an observer knows the declination d of* a star M (see Fig. 20) and observes its altitude h† just as it crosses his meridian above the pole, he can find his latitude by writing

$$L = NP_N = h - (90° - d).$$

* The declination of a star can be found from the Nautical Almanac.

† Various corrections to the observed altitude are generally necessary to obtain the true altitude.

The student should draw a figure for each case. First, a figure like Fig. 20 should be drawn showing the circle, Z, N, and S. Then the star M should be located on the figure so that arc $NM = h$ if the star bears north or so that $SM = h$ if it bears south.

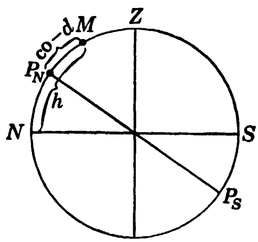

FIG. 20.

Next, the pole should be located so that arc

$$MP_N (\text{or } MP_S) = 90° - d.$$

Finally, the altitude of the pole elevated above the horizon should be computed from the figure.

Example. Find L if the declination of a star is 62° S. and if its altitude as it crosses the meridian at upper culmination* is 50° bearing south.

Solution. Since the star bears south and since it appears in the sky 50° above the horizon, it is represented in Fig. 21 on the right side of the circle so that arc $SM = 50°$. Next

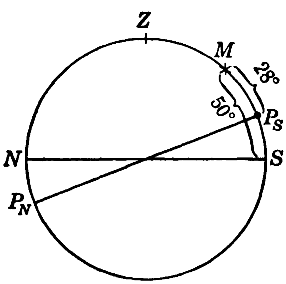

FIG. 21.

$$MP_S = 90° - d = 90° - 62° = 28°$$

is laid off to locate P_S. Hence the latitude is

$$L = 50° - 28° = 22° \text{ S.}$$

The observer must have been in south latitude since the south pole was elevated above the horizon.

EXERCISES

From the meridian altitude h, the declination d, and the bearing of the observed body as indicated, find the latitude of the observer in each of the following cases:

* The stars appear to move through the sky, each describing a small circle, one of whose poles is the celestial north pole, the other, the celestial south pole. Thus each star crosses the plane of the meridian of a place twice every 24 hr., the first time on one side of the pole and the second time on the opposite side. The greater of the two altitudes of meridian transit is the altitude of upper culmination; the lesser is the altitude of lower culmination.

Assume in each of the Exercises 1 to 12 that the body is in upper culmination.

	d	h		d	h
1.	50° N.	40° N.	**7.**	41°39′ N.	82°11′ N.
2.	40° S.	20° S.	**8.**	37°15′ N.	40°21′ N.
3.	20° N.	60° S.	**9.**	11°0′ N.	70°19′ N.
4.	50°25′ S.	35°29′ S.	**10.**	17°39′ S.	72°21′ S.
5.	30°15′ S.	47°35′ N.	**11.**	47°23′ S.	35°26′ S.
6.	28°10′ N.	71°12′ S.	**12.**	23°13′ N.	75°40′ S.

Assume in each of the Exercises 13 to 16 that the body is in lower culmination.

13.	59°49′ N.	44°11′ N.	**15.**	73°16′ N.	28°48′ N.
14.	77°54′ S.	25°18′ S.	**16.**	42°29′ N.	25°23′ S.

17. Two observers, A and B, are at different places on the same meridian. At the same instant each observer measured the meridian altitude of a star having declination 26°16′ S. A observed the star bearing south at an altitude 30°17′, B observed the star bearing north at an altitude 60°17′. Find the great-circle distance between A and B.

172. Given t, d, h, to find L. This is the double-solution case, since the given parts of the astronomical triangle are two sides and the angle opposite one of them. A method of finding L when t, d, and h are given is obtained by applying Napier's rules to the right triangles in Fig. 22. From triangle I, we have $\cos t = \tan \varphi \tan d$ or

$$\tan \varphi = \cos t \cot d. \tag{14}$$

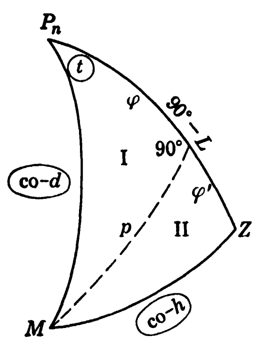

FIG. 22.

From triangles I and II, we get

$$\sin d = \cos p \cos \varphi,$$
$$\sin h = \cos p \cos \varphi'.$$

Dividing the second of these equations by the first, member by member, and solving the result for $\cos \varphi'$, we obtain

$$\cos \varphi' = \cos \varphi \sin h \csc d. \tag{15}$$

Then $90° - L = \varphi + \varphi'$, or

$$L = 90° - (\varphi + \varphi'). \tag{16}$$

Two solutions are obtained by choosing φ' from (15), first positive and then negative. Since approximate position is generally known, only the desired value need be computed. If north declination be considered as negative, the latitude found from (16) will be north if $90° - (\varphi + \varphi')$ is positive and south if $90° - (\varphi + \varphi')$ is negative.

EXERCISES

1. From the following data, compute in each case the latitude.

(*a*) $t = 35°$ W., $d = 0°$ N., $h = 42°$.
(*b*) $t = 29°$ W., $d = 7°$ S., $h = 34°$.

2. From the following data, compute in each case the latitude and azimuth.

(*a*) $t = 30°$ W., $d = 15°$ N., $h = 60°$.
(*b*) $t = 32°$ W., $d = 26°$ N., $h = 40°$.
(*c*) $t = 31°12'13''$ W., $d = 15°12'7''$ N., $h = 59°11'44''$.
(*d*) $t = 10°$ E., $d = 23°$ S., $h = 22°$.

173. MISCELLANEOUS EXERCISES

1. From $\cos x = 1 - 2 \text{ hav } x$ prove

$$\sin x \sin y = \text{hav } (x + y) - \text{hav } (x - y),$$
$$\cos x \cos y = 1 - \text{hav } (x + y) - \text{hav } (x - y),$$

and thence, from the law of cosines:

$$\text{hav } a = \text{hav } (b + c) \text{ hav } A + \text{hav } (b - c) \text{ hav } (180° - A),$$
$$\text{hav } B = \frac{\text{hav } b - \text{hav } (c - a)}{\text{hav } (c + a) - \text{hav } (c - a)},$$

or

$$\text{hav } (180° - B) = \frac{\text{hav } (c + a) - \text{hav } b}{\text{hav } (c + a) - \text{hav } (c - a)}.$$

2. Given $t = 45°10'30''$ W., $d = 1°9'15''$ S., $L = 37°30'$ N., find the azimuth Z_n.

3. Given $t = 55°$ E., $d = 15°$ S., and $L = 42°$ N., find h and Z.

4. Given $t = 30°$ W., $d = 45°$ N., $h = 60°$, find L and Z.

5. Given $t = 30°$ E., $d = 15°$ S., $h = 60°$, find L and Z.

6. From the following data, compute in each case the latitude and azimuth.

(*a*) $h = 68°$,
$t = 10°$ E.,
$d = 23°$ S.

(*b*) $t = 30°11'$ E.,
$d = 22°29'$ N.,
$h = 44°57'$.

7. In each of the following exercises, L represents the latitude of the observer, d the declination of a star, and h its altitude. Find in each case the hour angle t and the azimuth Z_n of the star.

(*a*) $L = 45°$ N., $d = 22°30'$ N., $h = 30°$ W.
(*b*) $L = 30°$ S., $d = 15°$ N., $h = 37°30'$ E.

8. An airplane following a great-circle track travels from a place having $L = 37°50'$ N., $\lambda = 122°20'$ W. (near Oakland, Calif.) to a place having $L = 40°40'$ N., $\lambda = 74°10'$ W. (near Newark, N. J.). How close does it pass to a point for which $L = 41°50'$ N., $\lambda = 87°40'$ W. (near Chicago, Ill.)?

9. Compute the distance and the intial course for a voyage along a great circle from Yokohoma, $L = 35°26'41''$ N., $\lambda = 139°39'0''$ E., to Diamond Head, Hawaii, $L = 21°51'8''$ N., $\lambda = 157°48'44''$ W.

10. Compute the distance and the initial course for a voyage along a great circle from Brisbane, Australia, $L = 27°27'32''$ S., $\lambda = 153°1'48''$ E., to Acapulco, $L = 16°49'10''$ N., $\lambda = 99°55'50''$ W. Also find the latitude and longitude of the southern vertex of the track.

11. Compute the distance and initial course for a great-circle voyage from a point having $L = 37°42'$ N., $\lambda = 123°4'$ W., near Farallon Island Lighthouse, to a point having $L = 34°50'$ N., $\lambda = 139°53'$ E., near the entrance to the Bay of Tokyo.

12. Find distance and the initial course of a great-circle voyage from San Diego, $L = 32°43'$ N., $\lambda = 117°10'$ W., to Cavite, $L = 14°30'$ N., $\lambda = 120°55'$ E.

13. Find where the track of the preceding exercise crosses the meridian of 157°49′ W. and at what distance from the harbor of Honolulu, $L = 21°16'5''$ N., $\lambda = 157°49'$ W., then due south.

14. The initial course by great-circle track from San Francisco, $L = 37°50'$ N., $\lambda = 122°30'$ W., to Yokohama, $L = 35°30'$ N., $\lambda = 140°$ E., is 302°59′05″. Find the longitude of the most northerly point of this path.

15. Find the latitude and longitude of the most northerly point reached by a ship sailing from San Francisco, Lat. 37°48′ N., Long. 122°28′ W., to Calcutta, Lat. 22°53′ N., Long. 88°19′ E.

16. An airplane follows a great-circle track from New York, $L = 40°40'$ N., $\lambda = 74°10'$ W., to $L = 56°30'$ N., $\lambda = 3°0'$ W. (near Edinburgh, Scotland). Where will it make its nearest approach (*a*) to the North Pole? (*b*) To $L = 46°50'$ N., $\lambda = 71°10'$ W. (near Quebec, Canada)?

17. Find the distance in degrees between the sun and the moon when their right ascensions are, respectively, $15^h\ 12^m$, $4^h\ 45^m$ and their respective declinations are $21°30'$ S., $5°30'$ N.

18. Find the distance in degrees between Regulus $RA = 10^h$, $p = 77°19'$ and Antares $RA = 16^h\ 20^m$, $p = 116°06'$.

19. An observer in Lat. $60°23'20''$ S. finds the altitude of a star when crossing the prime vertical* to be $38°23'20''$, bearing east. Find the declination of the star.

20. A star in declination $47°52'15''$ S., bearing east, makes its prime-vertical transit in altitude $58°20'00''$. Find the hour angle of the star.

21. What is the latitude of the place at which the sun rises exactly in the northeast on the longest day of the year?

22. Find the local apparent time of sunrise and sunset at

(*a*) London: $L = 51°29'$ N., if d of sun $= 13°17'$ N.
(*b*) Panama: $L = 8°57'$ N., if d of sun $= 18°29'$ N.
(*c*) New Orleans: $L = 29°58'$ N., if d of sun $= 4°30'$ N.
(*d*) Sydney: $L = 33°52'$ S., if d of sun $= 4°30'$ N.

23. Find the length (*a*) of the longest day; (*b*) of the shortest day at Leningrad $L = 59°56'30''$ N., $\lambda = 30°19'22''$ E.

24. Find the hour angle and amplitude of moonrise at Washington, D. C., $L = 38°59'$ N., on a day when the moon's declination is $25°28'$ N.

25. If twilight continues until the sun is 18° below the horizon, find the length of dawn, dark night, bright day, and twilight in Annapolis, $L = 38°58'53''$ N. (*a*) at summer solstice ($d = 23°27'7''$ N.); (*b*) winter solstice ($d = 23°27'7''$ S.); (*c*) when the sun is at an equinox.

26. The following observations have been made of a heavenly body in upper culmination. Find the latitude in each case.

	Declination	Observed altitude	Bearing
(*a*)	28°10′ N.	71°12′	South
(*b*)	73°02′ N.	58°40′	North
(*c*)	44°17′ S.	65°23′	South
(*d*)	30°15′ S.	47°35′	North
(*e*)	50°25′ S.	35°29′	South
(*f*)	40°16′ N.	40°14′	North

* For definition of prime vertical, see §164.

27. What relations must exist between L and d for a lower culmination to be visible? What relation always exists at a visible lower culmination between h and d?

28. In each of the following observations of a lower culmination, find the latitude:

	Declination	Observed altitude	Bearing
(*a*)	88°50′ N.	37°20′	North
(*b*)	46°22′ S.	32°15′	South
(*c*)	59°49′ N.	44°11′	North
(*d*)	77°54′ S.	25°18′	South

29. The right ascension of the sun is 45°; find (*a*) the length of the night at a point in latitude 60° N.; (*b*) the length of the shadow cast by a vertical stick 10 ft. long at 10 A.M. (local apparent time) at a point in latitude 40° N.; (*c*) the direction of a wall that casts no shadow at 10 A.M. at a place having latitude 40° N.

Hint. Compute the declination of the sun and then draw the astronomical triangle.

30. At a place in Lat. 51°32′ N., the altitude of the sun is 35°15′ bearing west and its declination is 21°27′ N. Find the local apparent time.

31. In London, L = 51°31′ N., for an afternoon observation the altitude of the sun is 15°40′. If its declination is 12° S., find the local apparent time.

32. (*a*) A navigator in latitude 15°23′36″ S. observes a star having $RA = 12^h\ 27^m\ 32^s$, d = 22°16′36″ N., at an altitude h = 17°26′30″ W. If the sidereal time ST of Greenwich at the instant of observation is $10^h\ 27^m\ 34^s$, find the longitude of the navigator.

(*b*) Also find the longitude of a second navigator in latitude 62°21′39″ N. who at the same instant observes a star having $RA = 6^h\ 27^m\ 30^s$, d = 26°55′21″ N. at an altitude h = 33°17′44″ W.

33. Find to the nearest minute the direction of the shadow of a vertical staff in Lat. 38°59′ N. at 6 A.M. local apparent time, when the declination of the sun is 23°27′ N.

34. Find the direction of a wall in Lat. 52°30′ N. that casts no shadow at 6 A.M. on the longest day of the year.

35. An explorer claimed to have reached the north pole. He took the picture of a flagpole 6 ft. high. From measurements made on the photograph it appeared that the 6-ft. pole cast a shadow 10.1 ft. long. Prove that he must have been at least 7° from the pole.

Find the shortest length of shadow that a stick 6 ft. long could possibly cast on level ground when held vertical at the north pole.

36. If the altitude of the north pole is 45° and if the azimuth of a star on the horizon is 135°, find the polar distance of the star.

37. Find the time of day when the sun bears due east and when it bears due west on the longest day of the year at Leningrad (Lat. 59°56′ N.).

38. Two points on the earth are in latitude 40° N. and their difference in longitude $DLo = 70°$. How much does the parallel of latitude joining these points exceed in length the arc of the great circle joining them? How far apart are the mid-points of the two tracks? (Use 3437 nautical miles for the radius of the earth.)

39. Find the altitude of the sun at 6^h A.M. at Munich (Lat. 48°9′ N.) on the longest day of the year.

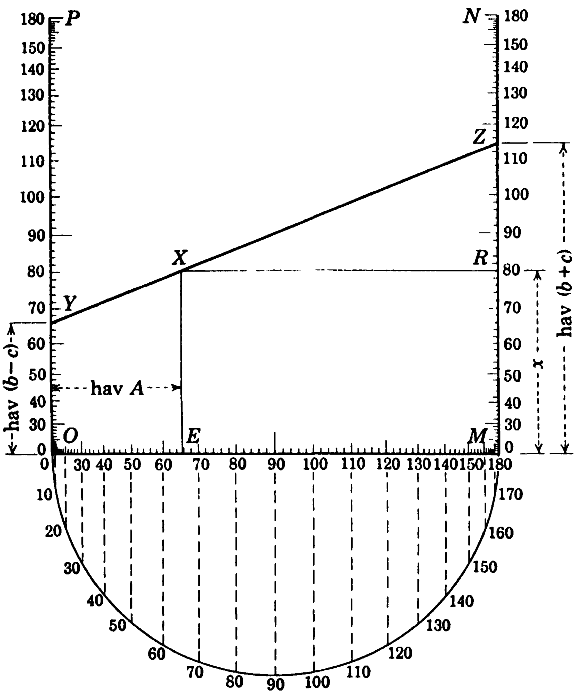

FIG. 23.

40.* If a, b, c, and A refer to a spherical triangle and if in Fig. 23 $OY = \text{hav}\,(b - c)$, $MZ = \text{hav}\,(b + c)$, $OE = \text{hav}\,A$, and $OM = 1$ unit, prove that $x(= EX = MR)$ is equal to hav a.

* This plan was devised by Prof. John Tyler, U. S. Naval Academy.

Hence, if we take $OP = OM = MN = 1$ unit, make a scale on OP by marking angles θ between $0°$ and $180°$ at points on OP distant in each case hav θ from O, a scale on OM by marking angles θ at points on OM distant in each case hav θ from O, and a scale on MN by marking angles θ at points on MN distant in each case hav θ from M, show how we may find the third side of a spherical triangle, when two sides and the included angle are given, by drawing three straight lines and reading the result.

APPENDIX A

1. The mil. The *mil* is an angular unit equal to $\frac{1}{6400}$ of four right angles.

The word mil, meaning one-thousandth, originated from the idea of adopting as a unit the angle that subtends an arc equal to $\frac{1}{1000}$ of the radius. Such an angle subtends 1 ft. at a distance of 1000 ft., 1 yd. at a distance of 1000 yd., etc. This manifestly furnishes a quick method of estimating the distance of an object whose size is known. There would under these circumstances be $\dfrac{2\pi}{0.001}$ or 6283.18+ such units subtended by a circle. This number is too inconvenient to be of practical use in calibrating instruments. The circle is therefore divided into 6400 equal parts, and each of these is called a mil. The arc subtended by a central angle of 1 mil therefore equals $\dfrac{2\pi R}{6400}$ or $(0.00098+)R$, or so nearly $\frac{1}{1000}$ of the radius that it may be so taken for purposes not demanding great accuracy. This property, coupled with the knowledge that in small angles the chord very nearly equals the arc, enables us to say for rapid and rough approximation:

A mil subtends a chord equal to $\frac{1}{1000}$ of the distance to the chord.

With due regard to the degree of approximation, a small number of mils (several hundred) subtends a chord equal to the small number times $\frac{1}{1000}$ of the distance to the chord, or, in symbols

$$s = \frac{r\theta}{1000}$$

where θ is in mils and s and r are expressed in the same unit.

The methods of rapid approximate measurement of angles and distances by the use of the mil system were first developed by the Field Artillery in computing firing data. Their use was extended to mapping, sketching, and reconnaissance. During the World War the Infantry adopted the system, and it has now become general.

The mil as a unit has the advantage that it is convenient in size for certain military measurements.

Example 1. Two points, A and B, are 50 yd. apart and 2000 yd. away. How many mils should they subtend (see Fig. 1)?

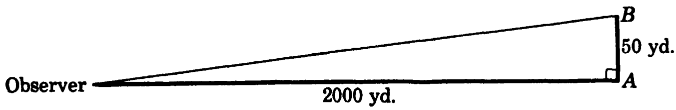

FIG. 1.

Solution. 50 divided by $\frac{2000}{1000} = 25$.

Or, at 2000 yd., 2 yd. corresponds to 1 mil; therefore 50 yd. corresponds to 25 mils.

Example 2. An observer measures the angular distance between two points, A and B, 5000 yd. away, to be 30 mils. How far apart are A and B?

Solution. $\frac{5000}{1000} \times 30 = 150$.

Or, at 5000 yd., 1 mil subtends 5 yd.; therefore 30 mils subtends 150 yd.

Example 3. The angular distance between A and B is observed to be 40 mils. They are 100 yd. apart. How far away are they?

Solution. $\frac{100}{40} \times 1000 = 2500$.

Or 40 mils corresponds to 100 yd.; therefore 1 mil corresponds to $2\frac{1}{2}$ yd., but $2\frac{1}{2}$ is $\frac{1}{1000}$ of 2500 yd.

EXERCISES

1. A battery with a front of 60 m. is observed from a point 3000 m. away, measured on a line normal to the battery. What angle does the battery subtend? (Or what is its front in mils?)

2. A four-gun battery 4000 m. away has a front of 15 mils. How many meters between muzzles?

3. The guns in your battery have wheels $1\frac{1}{2}$ m. in diameter. You measure a wheel as 5 mils. How far are you from the battery?

4. An observer measures the front of a target to be 40 mils at a point 6000 m. away. What should a scout (a) 3000 m. in front of the same observer measure it to be? (b) 4000 m. in front of the observer?

5. Two targets, T and t, are 20 m. apart. The range TG, perpendicular to the line of targets, is 5000 m. Two guns, G and g, are also 20 m. apart, the angle TGg being 1500 mils. Take t and g both on the same side of TG.

(*a*) What is angle tgG in order that the gun g may be laid on t?

(*b*) What change in deflection of G must be given to lay it on t?

6. A hostile trench measures 80 mils from your position. A scout 500 meters in front of you measures it 100 mils. What is the distance of the trench from your position?

7. You signal to a man at a distant tree to post himself 20 yd. from the tree (measured perpendicular to the line from the tree to you). The man is now 8 mils from the tree. How far away is the tree?

8. An observer finds that he is on the same level with the top of a distant tower that is 34 yd. high. The angular depression of the base of the tower is 8 mils. How far away is the tower?

9. From D a distant object B appears to the right of an object A, which is 6000 meters away. An observer at D measures the angle ADB to be 35 mils. He moves to C, 180 meters to the right on a line normal to AD, and measures the angle ACB to be 15 mils. How far away is B?

Hint. Sum of angles of a triangle is constant.

10. From Trophy Point, near the U. S. Military Academy, the angular elevation of Fort Putnam is 210 mils, and its distance is 600 yd. Also, the elevation of the top of the West Academic Building is 120 mils, and its distance is 250 yd. The West Academic Building and Fort Putnam are 500 yd. apart. What is the angular elevation of Fort Putnam as measured from the top of the West Academic Building?

APPENDIX B

2. The range finder. A range finder is an instrument designed to obtain the distance of an object from the instrument. Essentially it is a mechanism in a tube by means of which images caught at the ends of the tube can be brought into alignment by turning a thumbscrew.

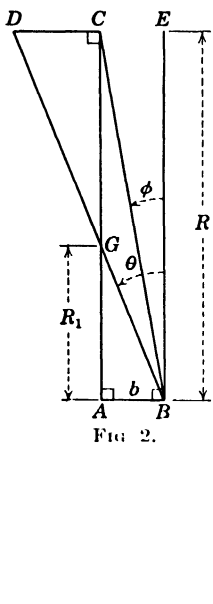

FIG. 2.

In Fig. 2 line AB represents a range finder of length b. AC and BE are lines perpendicular to AB. When the two images of point C caught at the ends A and B are brought into alignment, the distance $AC = R$ can be read on a dial. When the image of point C caught at end A is brought into alignment with the image of point D caught at B, the distance $AG = R_1$ is registered on the dial. The distances R and R_1 in Fig. 2 must be so great as compared with b that the errors in the equations

$$R\phi = b, \qquad R_1\theta = b, \tag{1}$$

$$\phi = \frac{b}{R}, \qquad \theta = \frac{b}{R_1}, \tag{2}$$

are negligible. On the other hand when the range of an object is so great that the angles represented by ϕ and θ in Fig. 2 are small, relative to the errors inherent in the mechanism of the range finder, trustworthy results cannot be obtained. A 12-ft. range finder is effective for distances from 100 to 25,000 yd.; a 26-ft. instrument, for ranges from 1200 to 50,000 yd.; a 30-ft. instrument, from 2400 to 60,000 yd.

The following examples illustrate the principles involved in the use of range finders.

Example 1. Let Fig. 2 represent a range finder of length b set parallel to line CD. If $b = 10$ yd. and if the distance

$R_1 = 2500$ yd. and $R = 10{,}000$ yd. have been found by using the instrument, find the length of CD. Also find CD in terms of R, R_1 and b.

Solution. Denote angle EBC by ϕ and angle EBD by θ. Since these angles are small, use equations (2) to obtain

$$\frac{b}{R} = \frac{10}{10000}, \qquad \frac{b}{R_1} = \frac{10}{2500}.$$

By using (1), we obtain

$$CD = R\theta - R\phi = 10000[\tfrac{10}{2500} - \tfrac{10}{10000}] = 30 \text{ yd. (approx.)}.$$

To find CD in terms of R, R_1, and b, use (2) and (1) to obtain

$$\phi = \frac{b}{R}, \qquad \theta = \frac{b}{R_1}, \qquad CD = R(\theta - \phi), \text{ (approx.)}.$$

Replacing $(\theta - \phi)$ in the last equation by their values from the first two, we obtain

$$CD = R\left(\frac{b}{R_1} - \frac{b}{R}\right) = \frac{bR(R - R_1)}{RR_1} = \frac{b(R - R_1)}{R_1}. \tag{3}$$

Example 2. Figure 3 indicates how a range finder may be used to obtain the direction angle α for an object CD of small known length a by means of the ranges R and R_1 which may be read from the instrument. Find angle α in terms a, b, R, and R_1, assuming that a and b are small as compared with R and R_1. Find α if $a = 50$ yd., $R = 3000$ yd., $R_1 = 1000$ yd., and $b = 10$ yd.

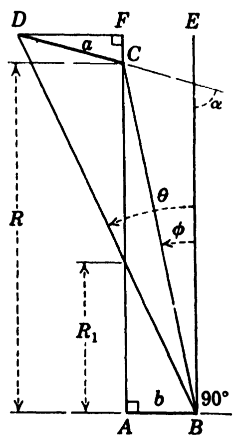

FIG. 3.

Solution. Referring to Fig. 3, observing that CF is small and using (3) in the solution of Example 1, we have

$$FD = \frac{b(R - R_1)}{R} \text{ (approx.)}.$$

Since angle $FCD = \alpha$, $\sin \alpha = \sin (FCD) = FD/a$, or replacing FD by the value just found,

$$\sin \alpha = \frac{b(R - R_1)}{aR}. \tag{4}$$

For the values mentioned in the example,

$$\sin \alpha = \frac{10(3000 - 1000)}{50(3000)} = \frac{2}{15}, \qquad \text{and} \qquad \alpha = \mathbf{7°40'}.$$

Example 3. A range finder is poorly adjusted. Show how the range given by such an instrument may be corrected.

Solution. When a range finder is not well adjusted it will register inaccurate distances. Referring to Fig. 4, we may say in such a case, that the ranges R and R_1 are based on angles $\phi \pm d$ and $\theta \pm d$ where d is the error due to poor adjustment of the instrument. Hence

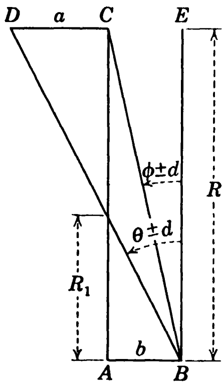

Fig. 4.

$$\phi \pm d = \frac{b}{R}, \qquad \theta \pm d = \frac{b}{R_1} \tag{5}$$

If x is the corrected range, we have $x(\theta - \phi) = a$, since θ and ϕ are the true angles. Then we may write

$$x = \frac{a}{\theta - \phi} = \frac{a}{(\theta \pm d) - (\phi \pm d)}, \tag{6}$$

or, replacing $\theta \pm d$ by b/R_1 and $\phi \pm d$ by $\dfrac{b}{R}$ from (5), we obtain the corrected range

$$x = \frac{a}{\dfrac{b}{R_1} - \dfrac{b}{R}} = \frac{aRR_1}{b(R - R_1)}. \tag{7}$$

For example, if $a = 50$ yd., $R = 12{,}000$ yd., $R_1 = 2100$ yd., and $b = 10$ yd., the corrected range would be

$$x = \frac{50(12{,}000)(2100)}{10(12{,}000 - 2100)} = 12{,}727 \text{ yd.},$$

and the correction increment is **727 yd.**

EXERCISES

1. In Fig. 2 find (*a*) CD if $R = 10{,}000$ yd., $R_1 = 2000$ yd., and $b = 30$ ft. (*b*) R if $R_1 = 1500$ yd., $CD = 180$ ft., $b = 36$ ft. (*c*) CD if $\theta = 990''$, $\phi = 165''$, $b = 36$ ft.

2. In Fig. 3 find (*a*) α if $R = 10{,}000$ yd., $R_1 = 2500$ yd., $a = 180$ ft., $b = 36$ ft. (*b*) find α if $\phi = 188''$, $\theta = 960''$, $a = 165$ ft., $b = 30$ ft. (*c*) find a if $\alpha = 9°30'$, $R = 3500$ yd., $R_1 = 1000$ yd., $b = 30$ ft.

3. In Fig. 4 find the correction increment (*a*) if $R = 15{,}000$ yd., $R_1 = 2800$ yd., $CD = 165$ ft., $b = 36$ ft. (*b*) if $\phi = 185''$, $\theta = 545''$, $b = 48$ ft., $CD = 300$ ft.

4. In Fig. 5 $b = 36$ ft., (*a*) find DT if $R = 14{,}000$ yd., $R_1 = 2000$ yd., $R_2 = 800$ yd. (*b*) find DT and DC if $\alpha = 70°$, $\phi = 155''$, $\theta_1 = 1710''$, $\theta_2 = 4200''$.

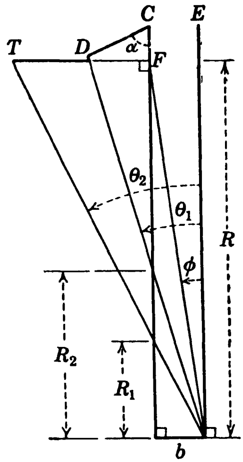

FIG. 5.

5. The captain of a vessel equipped with a coincident range finder of effective length 30 ft. desires to find the distance between two channel buoys C and D. He trains his range finder on buoy C and reads range $R_c = 14{,}000$ yd. He then aligns the image of D with the image of C and reads on the dial $R_1 = 2000$ yd. If the range finder is parallel to CD for the readings, find the distance between the buoys.

6. Two masts on a freighter are 165 ft. apart. The captain of a cruiser wishes to find the distance to the freighter with a range finder that is poorly adjusted. He trains the range finder on the right-hand mast and reads on the dial 15,000 yd. He then aligns the image of the second mast with that of the first and reads on the dial 2800 yd. If the range finder is parallel to the freighter, find the corrected range and the angular error of θ for his instrument.

APPENDIX C

3. Stereographic projections. In the applications of this chapter, the student will frequently find it convenient to draw a figure showing the main features of the problem under consideration. For this reason the following facts relating to stereographic projections are presented.

Consider a plane through the center of the sphere in Fig. 6 and the poles P_n and P_s of the great circle in which the plane intersects the sphere. A straight line connecting any point P on the sphere to P_s cuts the plane in a point called the *stereographic projection* of the point. The stereographic projection of a curve lying on the sphere is the locus of the stereographic projections of its points. The point P_s is called the *center of projection*, the plane is called the *primitive plane*, and the great circle cut out by the primitive plane is called *the primitive circle*. The angular measure of an arc of a great circle that has a given arc as a projection is called the *true length* of the given arc.

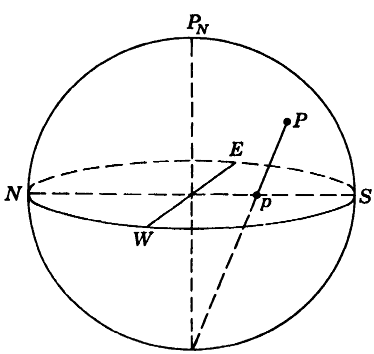

FIG. 6.

Figure 6 represents the sphere with center of projection P_s, with primitive plane $WSEN$, and with p the stereographic projection of P. The truth of the following statements, numbered I, II, III, IV, and V, is easily perceived.

I. The points of the hemisphere on the same side of the primitive plane as P_s project outside the primitive circle, and the points on the other hemisphere project inside the primitive circle.

II. The projection of any great circle through the center of projection P_s is a straight line through the center of the primitive circle.

III. The primitive circle projects into itself.

IV. The projection of any great circle passes through the ends of a diameter of the primitive circle. For the plane of the great circle cuts the primitive circle in a diameter and the ends of this diameter project into themselves.

V. The part of the projection of an arc of a great circle that lies inside the primitive circle has a true length of 180°, and if this arc is bisected each part has a true length of 90°.

The following statements, numbered VI and VII, are of fundamental importance. The proofs are omitted.

VI. The stereographic projection of a circle lying on a sphere is a circle or a straight line.

VII. The angle of intersection of two arcs on a sphere is equal to the angle of intersection of their stereographic projections.

4. Construction of some simple projections. The projection of a great circle can be drawn when the two points where it

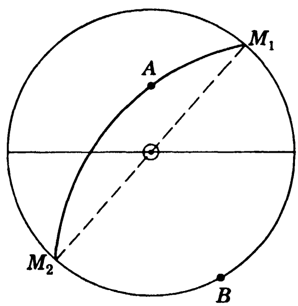

FIG. 7.

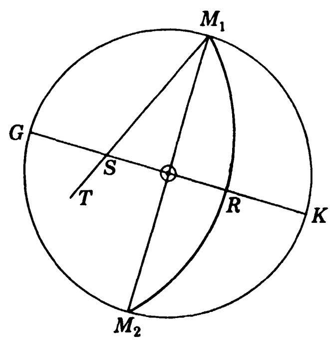

FIG. 8.

crosses the primitive circle at the ends of a diameter and the projection of another point are known. For, by VI, §3, the projection is a circle three points of which are known. For example, suppose that a great circle cuts the primitive circle shown in Fig. 7 at point M_1 and that A is the projection of another of its points. If O is the center of the primitive circle, M_1 lies on the projection by IV, §3. Therefore the circle through M_1, A, and M_2 is the required projection. Only the stereographic projection of one-half of a great circle is shown in Fig. 7.

Again, the projection of a great circle can be drawn when a point where the great circle cuts the primitive circle and the inclination of the plane of the circle to the primitive plane are

known. For, by IV, §3, two points at the ends of a diameter are known, by VI the projection is a circle, and by VII the angle between the primitive circle and the projection arc known.

Suppose that the great circle whose stereographic projection is to be drawn cuts the primitive circle GM_1K shown in Fig. 8, at M_1 and that its plane is inclined 35° to the primitive plane. Draw the mutually perpendicular diameters M_1M_2 and GK, construct with a protractor the line M_1T, making an angle of 35° with OM_1 and meeting GK at S. With S as a center and SM_1 as radius, draw the required circle M_1RM_2. The circle symmetrical over M_1M_2 with the one drawn also satisfies the given conditions.

EXERCISES

1. What great circles project into straight lines?

2. What is the nature of the projection of any circle passing through the center of projection?

3. What is the true length of the arc M_1R in Fig. 3? Give a reason for your answer.

4. Construct the projections of the great circles whose planes are inclined at 30°, 60°, 90°, 120°, and 150°, respectively, with the primitive plane, assuming that each one passes through a point M_1 chosen on the circumference of the primitive circle.

5. Draw a circle to be used as primitive circle. Through the ends of one of its diameters construct a circle. This second circle is the projection of a great circle. Now construct the projections of two other great circles through the ends of the same diameter, each of whose planes is inclined at 30° to the plane of the great circle whose projection is drawn first.

5. To find the true length of a projected arc. The actual magnitude of an arc of a great circle that has a given arc as its projection has been called the *true length* of the given arc. The object of this article is to give, without proof, a method of finding the true length of any arc that is the stereographic projection of a part of a great circle.

Let arc ACB in Fig. 9 represent the projection of a great circle on the primitive plane ABF. It passes through the ends A and B of a diameter and cuts the perpendicular diameter EF at C. Draw line AC and prolong it to meet the primitive circle in D,

lay off arc DG equal to 90° toward the inside of the projected circle, and draw GA meeting EF at X. The true length of arc ST is then obtained by drawing XS and XT to meet the

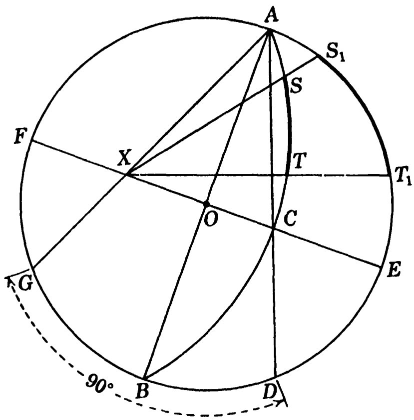

FIG. 9.

primitive circle in S_1 and T_1, respectively, and then using a protractor to find the length in degrees of arc S_1T_1.

If the method just described be applied to find the true length of a part of a diameter, the point X, will be found to fall at the end of the perpendicular diameter. Hence, the true length of OC in Fig. 9 is the arc BD, and the true length of XC is the arc GD or 90°. It now appears that X is the projected pole of the great circle represented by ACB in Fig. 9; consequently we may refer to X as the pole of great circle ACB.

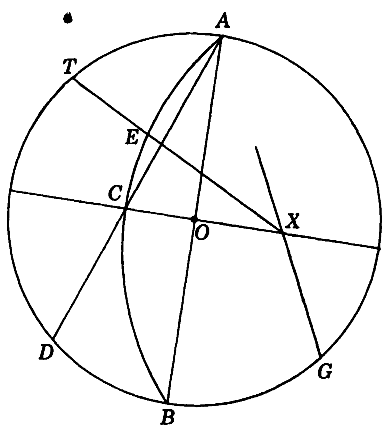

FIG. 10.

Evidently we can now lay off an arc of any desired true length from a given point on a projection of a great circle. Thus, to lay off 50° from A along the arc ACB in Fig. 10, lay off arc AT equal to 50°, locate the pole X of arc ACB, and draw XT meeting arc ACB in E. The arc AE has a true length of 50°.

Note that arc $AC = 90°$, and arc $AO = 90°$. Therefore, in accordance with a theorem from solid geometry, angle OAC is measured by the true length of arc CO, or by arc DB. A little reflection on the processes just illustrated will enable the draftsman to measure with facility angles and arcs defined by projections of great circles.

To measure the angle between two projected arcs of great circles through point A, lay off arc $AD = 90°$ on one circle and arc $AE = 90°$ on the other, draw straight lines AD and AE to meet the primitive circle in D and E, respectively, and measure arc DE with a protractor. Since A is the pole of arc DE and angle A is measured by the true length of arc DE, the reason for the construction is apparent.

Also, *the angle between two arcs may be obtained by measuring the angle between their radii drawn to the point of intersection.*

EXERCISES

1. Draw a primitive circle and the projections of three great circles making 45°, 90°, and 135° angles, respectively, with the primitive and all passing through the ends of the same diameter. Divide each arc inside the primitive circle into six parts, each having a true length of 30°. Also check the angle between the primitive and the projection by finding the true lengths of parts of the diameter perpendicular to the one having its end on the projected circle.

2. Draw the projections of two great circles meeting in a point A inside the primitive circle. Lay off arc $AD = 90°$ on one projection and arc $AE = 90°$ on the other. Now find the true length of arc ED; that is, measure the angle EAD. Perform this operation three or four times, using different great circles in each case.

3. Through the ends A and B of the diameter of a primitive circle draw a projected circle making a 60° angle with the primitive circle. Lay off arc AC equal to 60° on the primitive circle and draw through the ends C and D of a diameter the projection of a great circle making a 45° angle with the primitive. Now measure all arcs and angles formed inside the primitive circle.

6. To measure the parts of a spherical triangle by stereographic projection. A spherical triangle can be solved graphically by drawing its projection and measuring its sides and angles. An example will illustrate the method.

Example. Use stereographic projection to solve the triangle in which side $b = 120°$, side $c = 75°$, and the included angle $A = 60°$.

Solution. The solution will be explained by referring to Fig. 11. Draw the primitive circle ACF. Then draw any diameter AE and the perpendicular diameter DF. Lay off arc $ADC = b = 120°$. Draw AO_1 so that angle $OAO_1 = 60°$. With O_1 as center, draw circular arc ABE. Then angle $DAB =$

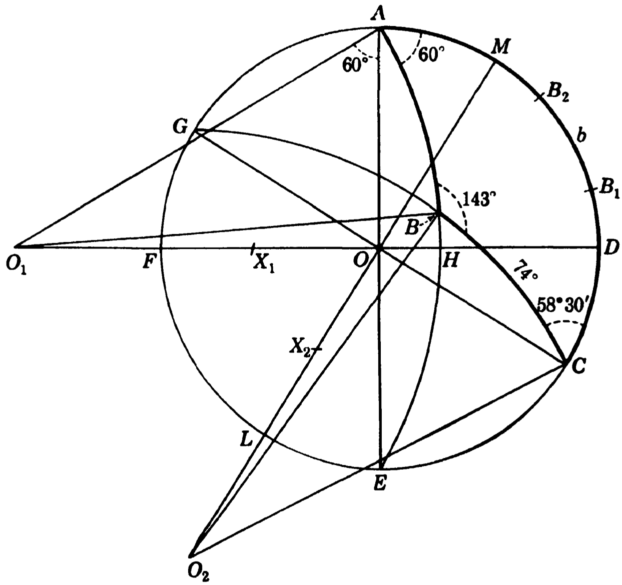

FIG. 11.

$60°$. Find the pole X_1 of arc ABE, lay off arc $AB_1 = 75°$, draw B_1X_1 to meet arc ABE in B. Then arc AB has a true length of $75°$. Now draw diameter CG and construct the circular arc CBG with center O_2. Then triangle ABC is a stereographic projection of the required triangle. To measure the unknown parts, draw diameter LM perpendicular to CG, and locate the pole X_2 of arc CBG. Draw X_2B to meet the primitive circle in B_2. Then the true length of CB is equal to arc CB_2, which is found by means of a protractor to be $74°$. Next draw O_2C. Then angle BCD is equal to angle $GCO_2 =$ **58°30′**. Also, angle CBA is $180° -$ angle O_1BO_2 or **131°30′**.

EXERCISES

1. Draw the stereographic projection of a spherical triangle in which $a = 60°$, $b = 90°$, $C = 60°$, and measure B and c.

2. Draw a stereographic projection of each of the spherical triangles that have the given parts indicated, and measure the unknown parts:

(*a*) $a = 60°$, $b = 60°$, $C = 90°$.

(*b*) $A = 60°$, $B = 60°$, $c = 120°$.

(*c*) $A = 120°$, $b = 75°$, $c = 150°$.

(*d*) $b = 120°$, $c = 120°$, $A = 75°$.

INDEX

D

E

F

G

H

I

L

M

N

O

P

Q

R

ANSWERS

§3. Pages 8, 9

2. $\frac{3}{5}, \frac{4}{5}, \frac{3}{4}; \frac{4}{5}, \frac{3}{5}, \frac{4}{3}; \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}, 3; \frac{1}{\sqrt{101}}, \frac{10}{\sqrt{101}}, \frac{1}{10}$
3. $\cos A = \frac{12}{13}$, $\tan A = \frac{5}{12}$
4. $\sin A = \frac{24}{25}$, $\tan A = \frac{24}{7}$
5. $\sin A = \frac{8}{17}$, $\cos A = \frac{15}{17}$
6. $\cos A = \frac{15}{17}$, $\tan A = \frac{8}{15}$
7. $\sin A = \frac{7}{25}$, $\tan A = \frac{7}{24}$
8. $\sin A = \frac{8}{17}$, $\tan A = \frac{8}{15}$
11. 550 ft.
12. 1120 ft.
13. 9 ft.
14. 198.5 ft.
15. 1500 ft.

§4. Pages 11, 12

1. $\frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}}, \frac{3}{5}$, etc.; $\frac{3}{5}, \frac{4}{5}, \frac{3}{4}$, etc.; $\frac{3}{5}, \frac{4}{5}, \frac{3}{4}$, etc.; $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1$, etc.; $\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, \frac{1}{2}$, etc.; $\frac{21}{29}, \frac{20}{29}, \frac{21}{20}$, etc.
2. $\frac{5}{13}, \frac{12}{13}, \frac{5}{12}$, etc.; $\frac{12}{13}, \frac{5}{13}, \frac{12}{5}$, etc.
3. (*a*) $\cos\theta = \frac{3}{5}$, $\tan\theta = \frac{4}{3}$, etc.; (*b*) $\sin\theta = \frac{8}{17}, \frac{15}{17}$, etc.; (*c*) $\sin\theta = \frac{\sqrt{3}}{2}$, $\tan\theta = \sqrt{3}$, etc.
4. (*a*) 1; (*b*) 1
6. 180 ft.
7. 45.0 ft.
8. 396 ft.

§5. Pages 14, 15

2. 0.000291, 1, 0.000291, etc.; 1, 0.000291, 3436, etc.
3. 0, 1, 0, etc.; 1, 0, ∞, etc.
4. $\frac{9}{41}, \frac{40}{41}, \frac{9}{40}$, etc.; $\frac{40}{41}, \frac{9}{41}, \frac{40}{9}$, etc.
5. $\frac{\sqrt{3}}{2}, \frac{1}{2}, \sqrt{3}$, etc.
6. $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1$, etc.
9. (*a*) $\frac{1}{\sqrt{3}}$, (*b*) $\sqrt{6}$, (*c*) 1, (*d*) $\frac{1}{3\sqrt{2}}$
11. $\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}, \frac{3}{2}$, etc.
12. $\frac{1}{2\sqrt{2}}(\sqrt{3} + 1), \frac{\sqrt{2}}{2(\sqrt{3} - 1)}$
13. 0.577 miles
14. 22.5 ft.
15. 482.8 yd.

§7. Pages 18 to 20

1. $a = 41.80$, $b = 49.79$; $b = 62.92$, $c = 97.88$; $a = 140.8$, $c = 812$; $a = 96.14$, $c = 102.3$

2. (*a*) $a = 48.79$, $b = 69.62$; (*b*) $b = 1134$, $c = 1152$; (*c*) 350, 610.5; (*d*) $b = 19.32$, $a = 5.18$; (*e*) $a = 42.3$, $b = 90.6$; (*f*) $a = 21.84$, $c = 63.84$

3. 738.0, 307.7

4. (*a*) $a = 312$, $b = 416$; (*b*) $b = 469.3$, $c = 997.3$; (*c*) $c = 76.0$, $a = 68$; (*d*) $b = 96.2$, $a = 231.0$; (*e*) $b = 61.2$, $c = 183.5$; (*f*) $a = 803$, $c = 852$

5. 68 ft.

6. 245.7 ft., 172.2 ft.

7. 274.7 ft.

8. 66 ft.

9. 37.17 ft.

10. 58.4 ft.

11. 5590 yd.

12. 105.0 ft.

13. 3.915 cm.

14. 24.28 yd.

§8. Pages 22, 23

1. $x = 13.5$, $y = 19.7$, $z = 22.5$

2. $x = 19.2$, $y = 14.4$, $z = 10$

3. $s = 6$, $t = 5.54$, $w = 2.31$, $x = 8$, $y = 3.08$, $z = 7.38$

4. $x = 150$, $w = 250$, $y = 117.6$, $z = 220.6$

5. $y = 74.27$

6. $BD = 72.14$

7. $v = 2.4$, $w = 3.2$, $q = 5.52$, $R = 2.330$, $s = 2.517$, $t = 3.915$

§9. Pages 23 to 27

1. $\frac{2}{\sqrt{29}}$, $-\frac{5}{\sqrt{29}}$, $\frac{2}{5}$, etc.; $\frac{4}{5}$, $\frac{3}{5}$, $\frac{4}{3}$, etc.

2. $\frac{15}{17}$, $\frac{8}{17}$, $\frac{8}{15}$

4. (*a*) $\cos A = \frac{3}{5}$, $\tan A = \frac{4}{3}$, etc.; (*b*) $\sin A = \frac{8}{17}$, $\cos A = \frac{15}{17}$, etc.; (*c*) $\sin A = \frac{5}{13}$, $\tan A = \frac{5}{12}$, etc.

5. (*a*) $\frac{336}{625}$, (*b*) $-\frac{527}{625}$

6. 1

7. $\frac{1}{8}(3 + \sqrt{21})$

8. 39, 36

9. $b = 65$, $c = 57$, $a = 68$, altitude to $b = 52.62$, altitude to $a = 50.34$

11. $a = 12$, $b = 6\sqrt{3}$, $c = 3\sqrt{6}$

12. $a = 3\sqrt{34}$, $b = 4\sqrt{34}$, $c = 5\sqrt{34}$; $\frac{3}{5}$, $\frac{4}{5}$, $\frac{3}{4}$, etc.

13. $AD = 28, AO = 21, OB = 20, OC = 15, DC = 4\sqrt{130}, OE = \frac{21}{26}\sqrt{130}$
$\sin\beta = \frac{20}{29}$, $\cos\beta = \frac{21}{29}$, $\tan\beta = \frac{20}{21}$, etc.; $\sin\gamma = \frac{3}{5}$, $\cos\gamma = \frac{4}{5}$
$\tan\gamma = \frac{3}{4}$, etc., $\sin\delta = \frac{7}{\sqrt{130}}$, $\cos\delta = \frac{9}{\sqrt{130}}$, $\tan\delta = \frac{7}{9}$, etc.

14. $AO = 57.12$ ft.

15. $CD = 12, AD = 35, AB = 30, AE = EB = 15, CB = 13, CE = 4\sqrt{34}$;
$\sin DEC = \frac{5}{\sqrt{34}}$, $\cos DEC = \frac{3}{\sqrt{34}}$, $\tan DEC = \frac{5}{3}$, etc.

16. $AD = 25$, $DB = 15$, $AE = \frac{80}{3}$, $CE = \frac{64}{3}$, $ED = \frac{5}{3}\sqrt{481}$;
$\sin AED = \frac{15}{\sqrt{481}}$, $\cos AED = \frac{16}{\sqrt{481}}$, $\tan AED = \frac{15}{16}$

17. $DA = 1$, $DC = 1$, $OD = \sqrt{3}$, $DB = 2 - \sqrt{3}$, $AB = 2\sqrt{2 - \sqrt{3}}$;
$\sin 15° = \dfrac{\sqrt{2 - \sqrt{3}}}{2}$, $\cos 15° = \dfrac{1}{2\sqrt{2 - \sqrt{3}}}$, $\tan 15° = 2 - \sqrt{3}$, etc.

18. $\sin 22\frac{1}{2}° = \dfrac{\sqrt{2 - \sqrt{2}}}{2}$, $\cos 22\frac{1}{2}° = \dfrac{\sqrt{2}}{2\sqrt{2 - \sqrt{2}}}$, $\tan 22\frac{1}{2}° = \dfrac{2 - \sqrt{2}}{\sqrt{2}}$

19. 2.757 cm.
21. 5272 ft.
22. 184.6 ft.
23. 16.78 miles

§12. Pages 31, 32

1. (*a*) $\cos 15°$; (*b*) $\sin 3°$; (*c*) $\cot 30'$; (*d*) $\csc 40°40'$; (*e*) $\tan 44°10'$; (*f*) $\sec 19°39'44''$
2. $20°$; $10°$; $5°$; $9°20'$
3. (*a*) $\cos \theta$; (*b*) $\sin \theta$; (*c*) $\csc \theta$; (*d*) 1; (*e*) $\sec \theta$; (*f*) $\cos \theta$; (*g*) 1; (*h*) 1; (*i*) $\tan \theta$
6. $11°51'25\frac{5}{7}''$; $6°28'$; $4°35'20''$; $14°42'$

§13. Pages 33, 34

1. (*a*) $\cos^2 \beta$; (*c*) $\tan^2 \beta$; (*e*) $-\cot^2 \beta$; (*g*) 1; (*h*) $-\sin^2 \theta \tan^2 \theta$
2. (*a*) 1; (*c*) $\cot^2 \varphi$; (*f*) 1
3. (*a*) $2 \sin^3 \theta - 2 \sin^5 \theta$; (*b*) $2 \sin^2 \theta - 1$; (*f*) $\dfrac{\sin^3 \theta}{(1 - \sin^2 \theta)^{\frac{3}{2}}}$

§14. Pages 35 to 37

1. $\sec \theta$
2. $\tan \theta$
3. 1
4. 1
5. -1
6. -1

§15. Pages 39, 40

1. $\cos A = \sqrt{1 - \sin^2 A}$, $\tan A = \sin A/\sqrt{1 - \sin^2 A}$, etc.
2. $\sin A = \sqrt{1 - \cos^2 A}$, $\tan A = \sqrt{1 - \cos^2 A}/\cos A$, etc.
3. (*a*) $\sin A = 1/\sqrt{1 + \cot^2 A}$, $\cos A = \cot A/\sqrt{1 + \cot^2 A}$,
$\tan A = \dfrac{1}{\cot A}$, etc.
(*b*) $\sin A = \sqrt{\sec^2 A - 1}/\sec A$, $\cos A = 1/\sec A$,
$\tan A = \sqrt{\sec^2 A - 1}$, etc.

§16. Pages 42 to 44

1. $(1 + \tan^2 A)^2 = \sec^2 A + \tan^2 A \sec^2 A$
2. $DE = a \cos A \sec B$, $CE = a \sin A + a \cos A \tan B$
3. $a \cos^4 \theta$
4. $a \sin^4 \theta$
5. 43.2, 75.23
6. 71.88, 92.21
7. $\tan \frac{1}{2}\theta = \dfrac{\sin \theta}{1 + \cos \theta}$
8. $AB = a \sin^2 \theta$, $a \cos^2 \theta$

9. $FD = \sin\varphi \sin\theta$, $CD = \cos\varphi \sin\theta$
10. $FD = \sec\theta \tan\varphi \sin\theta = \tan\theta \tan\phi$
11. $\sin 2\theta = 2 \sin\theta \cos\theta$

§17. Pages 44 to 47

1. (a) $\cos 25°$, (b) $\cot 41°$, (c) $\csc 8°$
2. (a) $\cos^2\theta$ (b) 1 (c) 2 (d) $\sec^2\theta$ (e) $\sec^2\theta$ (f) $\sin^2\theta$ (g) 2
4. (a) $\dfrac{1-\sin^2 A}{\sin A}$ (b) $1/\sin A$ (c) $\sin A$ (d) $1 - 2\sin^2 A$
5. (a) $\cos A$ (b) $\cos^2 A$
6. (a) $\tan\theta$ (b) $\tan^2\theta + \tan^4\theta$
7. (a) $1/\sin\theta\cos\theta$ (b) $(1-\cos\theta)/\sin\theta$ (c) $(1+\sin\theta)/\cos\theta$
9. (a) $a\sin\theta$ (b) $b\sin\theta$ (c) $b\tan\theta$ (d) $a\sin^4\theta$ (e) $a\sin^6\theta$ (f) $b\csc\theta$ (g) $b\sin\theta\sec\theta$ (h) $2a\sin^3\theta\sec\theta$ (i) $2a\cos\theta$
38. 12.68
39. 69.14, 107.5
41. $x = 14.0042$, $y = 21.786$
42. $AC = a\sin\theta\cot\phi$, $AB = a\sin\theta\cot\phi\cot\alpha$

§18. Page 49

2. 7
3. $\frac{5}{3}$ right angles clockwise
4. $\frac{1}{15}$
5. 24 right angles
6. (a) 1; (b) $2\frac{1}{3}$; (c) $8\frac{1}{3}$; (d) 8000; (e) $\frac{4}{365}$; (f) $\frac{1}{2190}$

§19. Pages 50, 51

3. (a) $\frac{5}{13}$, $\frac{12}{13}$, $\frac{5}{12}$, etc.; (b) $\dfrac{y}{\sqrt{x^2+y^2}}$, $\dfrac{x}{\sqrt{x^2+y^2}}$, $\dfrac{y}{x}$, etc.
4. (a) On a line parallel to y-axis and 3 units to left of it
5. 0; 0
6. (a) I; (b) II; (c) IV; (d) III
7. (a) pos. I, II; neg. III, IV

§20. Pages 53 to 55

1. (a) $-\frac{4}{5}$, $-\frac{3}{5}$, $\frac{4}{3}$, etc.; (b) $-\frac{4}{5}$, $\frac{3}{5}$, $-\frac{4}{3}$, etc.
3. $-\frac{1}{3}\sqrt{3}$, $-\sqrt{3}$, $\frac{2}{3}\sqrt{3}$, -2
5. (a) $\frac{5}{13}$, $\frac{12}{13}$, $\frac{5}{12}$, etc.; (d) $-\frac{5}{13}$, $\frac{12}{13}$, $-\frac{5}{12}$, etc.
6. (a) $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$, etc.
   (c) $\sin\theta = -\frac{12}{13}$, $\cos\theta = -\frac{5}{13}$, $\tan\theta = \frac{12}{5}$, etc.
   (e) $\sin\theta = -\frac{7}{25}$, $\cos\theta = \frac{24}{25}$, $\tan\theta = -\frac{7}{24}$, etc.
   (g) $\sin\theta = -\dfrac{3}{\sqrt{13}}$, $\cos\theta = \dfrac{2}{\sqrt{13}}$, $\tan\theta = -\frac{3}{2}$, etc.
   (i) $\sin\theta = 0$, $\cos\theta = -1$, $\tan\theta = 0$
7. (a) I, II; (d) II, IV
8. (a) II; (d) III

9. (*a*) $\sin\theta = \frac{3}{5}$, $\tan \frac{3}{4}$, etc. (*c*) $\cos\theta = \frac{15}{17}$, $\tan\theta = -\frac{8}{15}$, etc.
(*e*) $\sin\theta = -\frac{8}{17}$, $\cos\theta = -\frac{15}{17}$, $\tan\theta = \frac{8}{15}$, etc.

(*g*) $\cos\theta = -\dfrac{\sqrt{3}}{2}$, $\tan\theta = -\dfrac{1}{\sqrt{3}}$, etc.

(*i*) $\sin\theta = -\frac{5}{13}$, $\cos\theta = \frac{12}{13}$, etc.
(*k*) $\sin\theta = -\frac{12}{13}$, $\tan\theta = -\frac{12}{5}$, etc.

10. $-\frac{24}{7}$ **11.** 3 **12.** $-\frac{13}{20}, \frac{37}{20}$

§22. Pages 57, 58

1. $-\frac{1}{2}$, $-\frac{1}{2}\sqrt{3}$, $\frac{1}{3}\sqrt{3}$, etc.
3. (*a*) 30°, 150°; (*c*) 30°, 210°; (*e*) 45°, 315°;
(*b*) 210°, 330°; (*d*) 150°, 330°; (*f*) 135°, 225°
4. (*a*) 90°; (*c*) 0°, 180°; (*e*) 0°, 180°; (*g*) 90°, 270°; (*i*) 0°, 180°
(*b*) 180°; (*d*) 90°, 270°; (*f*) 270°; (*h*) 90°, 270°;
5. (*b*) −0.966, 0.259, −3.732, −0.268, 3.864, −1.035
6. 2. **7.** (*a*) $\frac{1}{2}(\sqrt{3}+2)$; (*b*) $\sqrt{2}+\frac{1}{2}$; (*c*) $\frac{5}{2}$; (*d*) $-\frac{5}{2}$. **8.** 1
16. (*a*) 3 (*b*) 4 (*c*) −2 (*d*) 4

§24. Pages 60, 61

1. sin 40°, − cos 40°, − tan 40°, etc.
2. − sin 35°, cos 35°, − tan 35°, etc.
3. (*a*) − sin 63°, − cos 63°, − tan 63°, etc.
(*d*) sin 10°, − cos 10°, − tan 10°, etc.
(*h*) sin 70°, − cos 70°, − tan 70°, etc.

§25. Pages 63, 64

1. (*a*) − sin 85°, − cos 85°, tan 85°, etc.
(*b*) − sin 85°, cos 85°, − tan 85°, etc.
2. cos 5°, − tan 22°, − csc 23°, etc.
3. (*a*) $-\sin\theta$ (*c*) $-\tan\theta$ (*e*) $\csc\theta$ (*g*) $\cot\theta$
(*b*) $\cos 2\theta$ (*d*) $-\sec\theta$ (*f*) $-\sin 2\theta$ (*h*) $\cos\theta$
4. (*a*) sin 160° = − sin 200° = − sin 340° = cos 70°, etc.
(*c*) − tan 105° = cot 15° = tan 255°, etc.
(*f*) cot 67° = − cot 113°, = − cot 293°, etc.
(*i*) cot 10° = tan 80° = − tan 100°, etc.
6. (*a*) $-\sin^2 25° - \cos^2 86°$

§26. Pages 64 to 66

1. $\sin\theta = -\dfrac{2}{\sqrt{13}}$, $\cot\theta = -\frac{3}{2}$, etc. **2.** $\cos\theta = -\frac{4}{5}$, $\tan\theta = \frac{3}{4}$, etc.
3. (*a*) 210°, 330° (*c*) 135°, 315° (*e*) 210°, 330°
(*b*) 60°, 240° (*d*) 45°, 315° (*f*) 120°, 240°
4. (*a*) sin 75° (*c*) sec 20° (*e*) − csc 70°
(*b*) − cos 10° (*d*) cot 62° (*f*) tan 4°
5. (*a*) sin 10° (*c*) − tan 15° (*e*) − csc 10°
(*b*) − cos 15° (*d*) − tan 30° (*f*) − sec 5°

6. (a) $-\dfrac{1}{\sqrt{3}}$ (c) $-\frac{1}{2}\sqrt{3}$ (e) $-\sqrt{2}$

(b) $-\frac{1}{2}\sqrt{3}$ (d) $\sqrt{2}$ (f) $\sqrt{3}$

7. $\frac{1}{4}(1 - \sqrt{2})$ 8. $\dfrac{\sqrt{3} - 2}{3}$ 9. sin 80° cos 80°

14. -1 15. $-\frac{1}{4}(3 + 2\sqrt{2})$

§28. Page 68

1. (a) 6.72, (b) 985, (c) 69,300, (d) 4940
2. 49 ft.

§29. Page 70

1. 0.678 3. 0.407 5. 2.153 7. 42°13′
2. 0.582 4. 2.663 6. 3.563 8. 24°46′
9. 58°28′ 10. 62°37′

§30. Page 72

1. $b = 28.40$, $c = 42.78$, $B = 41°35'$
2. $a = 40.23$, $b = 22.52$, $A = 60°46'$
3. Impossible
4. $A = 50°27'$, $B = 39°33'$, $c = 3.943$
5. $a = 106.2$, $c = 125.6$, $A = 57°45'$
6. $a = 22.20$, $b = 42.10$, $B = 27°48'$
7. $c = 45.61$, $A = 64°0'$, $B = 26°0'$
8. $a = 12.76$, $b = 34.73$, $B = 20°10'$

§31. Pages 74 to 76

1. 8°5′
2. 6.301 miles, 8.044 miles
3. 0.7178 miles
4. 114.3
5. 50°33′
6. 11.48
7. 6821
8. 3214
9. 127.2, 141.2
10. 23.34, 166.1
11. 99.0 ft.
12. 20.90 ft.
13. 0.1299 miles

§32. Page 78

1. $A = 36°52'$, $B = 53°8'$, $b = 80$
2. $B = 51°20'$, $c = 80.9$, $b = 63.2$
3. $A = 21°10'$, $b = 1884$, $c = 2020$
4. $B = 26°$, $a = 410$, $c = 457$
5. $A = 83°48'$, $a = 36.98$, $b = 4.02$
6. $B = 46°30'$, $a = 7.71$, $b = 8.12$
7. $A = 27°4'$, $a = 24.37$, $c = 53.56$
8. $A = 43°18'$, $B = 46°42'$, $b = 0.662$
9. $B = 17°53'$, $b = 26.91$, $c = 87.6$

§33. Page 79

1. $A = 31°20'$, $B = 58°40'$, $c = 23.7$
2. $A = 41°2'$, $B = 48°58'$, $c = 153.8$
3. $A = 65°$, $B = 25°$, $c = 55.2$

4. $A = 33°9'$
$B = 56°51'$
$c = 499$

5. $A = 39°30'$
$B = 50°30'$
$c = 44$

6. $A = 67°23'$
$B = 22°37'$
$c = 13$

7. $A = 45°$
$B = 45°$
$c = 18.67$

8. $A = 30°37'$
$B = 59°23'$
$c = 82.5$

9. $A = 3°42'$
$B = 86°18'$
$c = 4.8$

§35. Page 81

1. 9.80599 − 10
2. 9.93542 − 10
3. 9.17665 − 10
4. 9.73470 − 10
5. 9.93499 − 10
6. 9.95656 − 10
7. 9.56544 − 10
8. 0.55211
9. 0.82153
10. 9.98988 − 10

§36. Page 82

1. 11°54'31''
2. 6°8'9''
3. 44°12'7''
4. 7°43'44''
5. 33°29'52''
6. 80°31'59''
7. 52°16'58''
8. 53°57'31''
9. 6°2'28''
10. 52°8'53''

§37. Pages 83 to 85

1. $a = 9.8030$
$c = 17.091$
$B = 55°$

2. $a = 5.9407$
$b = 2.0205$
$A = 71°13'$

3. $b = 810.80$
$A = 47°31'32''$
$B = 42°28'28''$

4. $A = 74°09'05''$
$B = 15°50'55''$
$c = 9.0220$

5. $a = 388.25$
$b = 548.90$
$B = 54°43'35''$

6. $A = 58°26'54''$
$B = 31°33'06''$
$c = 757.26$

7. $B = 13°23'38''$
$b = 22.757$
$A = 76°36'22''$

8. $b = 18.168$
$c = 39.810$
$A = 62°50'46''$

9. $a = 17.350$
$b = 17.854$
$B = 45°49'22''$

10. $A = 29°38'28''$
$c = 6.6550$
$B = 60°21'32''$

11. $b = 17.595$
$c = 74.247$
$B = 13°42'28''$

12. $a = 193.55$
$b = 1660.9$
$A = 6°38'49''$

13. 30.559 ft.
14. 65.714 miles
15. 2964.2 ft.
16. 0°19'45''
17. 9.8768 ft.
18. 35°15'51''
19. 19.031 in.
20. 10,524 ft.
21. 35°32'16''
22. 957.75 ft.
23. 99.990 ft.
24. 2957.2 miles
25. 1°8'46'', 8100 ft.
26. $r = 7.5492$, $R = 8.1710$
27. $B = 40°47'2''$

§38. Pages 86 to 89

1. 48.798 ft.
2. 14.392 ft.
4. $MN = a \cot \phi \cos^2 \phi$
5. $AOB = 11.964$
6. $x = m \sin (\theta - \varphi) \csc (\alpha - \theta) \cos \alpha$
8. 4470.1 ft.
9. 89.3 ft.
10. 272.40 ft.
11. 864 ft., 708 ft., 246 ft.
12. 69.768 ft.

13. 275.94 ft.

14. (*a*) 20.558 miles
(*b*) 39.847 miles

§39. Pages 89 to 93

1. $A = 34°12'20''$, $b = 153.00$, $B = 55°47'60''$

2. $a = 434.16$, $b = 449.58$, $B = 46°$

3. $a = 58.239$, $c = 75.330$, $A = 50°38'$

4. $b = 96{,}915$, $c = 10{,}904$, $A = 27°16'26''$

5. $a = 2.2883$, $b = 5.4275$, $A = 22°51'40''$

6. $A = 26°47'26''$, $c = 8.8762$, $B = 63°12'34''$

7. 5374.0 yd., 8302.2 yd.

8. 4880 cu. yd.

9. radius $= \frac{9}{32}(3\sqrt{2} - 2\sqrt{3})$

10. 139°10′4″, 80.598 miles

11. 0.71407 miles

12. 24,099

13. 34.151 ft.

14. $h = 142.5$ ft., $d = 128$ ft.

15. (*a*) 3.415 miles; (*b*) 6.830 miles

16. 28°22′52″

17. 10,910 ft.

18. 345.81 ft., 116.75 ft.

19. 284 ft., 291 ft.

20. 7.87 mi.

§41. Pages 95, 96

1. (*a*) $\frac{1}{4}\pi$; (*b*) $\frac{1}{3}\pi$; (*c*) $\frac{1}{2}\pi$; (*d*) π; (*e*) $\frac{2}{3}\pi$; (*f*) $\frac{3}{4}\pi$; (*g*) $\frac{1}{8}\pi$; (*h*) $\frac{10}{9}\pi$; (*i*) $\frac{8}{3}\pi$

2. (*a*) 60°; (*b*) 135°; (*c*) 2.5°; (*d*) 210°; (*e*) 1200°; (*f*) 176.40°

3. (*a*) 0.01745; (*b*) 0.0002909; (*c*) 0.000004848; (*d*) 0.1778; (*e*) 3.152; (*f*) 5.244

4. (*a*) 5°44′; (*b*) 143°14′; (*c*) 91°40′; (*d*) 343°46′

5. (*a*) $\frac{1}{3}\sqrt{3}$ (*b*) $\frac{1}{2}\sqrt{3}$ (*c*) $\frac{1}{2}\sqrt{2}$ (*d*) $\sqrt{3}$ (*e*) 1 (*f*) -1 (*g*) $\frac{1}{3}\sqrt{3}$ (*h*) -2 (*i*) 0

6. (*a*) $\frac{\pi}{6}, \frac{\pi}{72}$ (*b*) $\frac{\pi}{2}, \frac{\pi}{24}$ (*c*) $\frac{3}{2}\pi, \frac{\pi}{8}$ (*d*) $4\pi, \frac{1}{3}\pi$ (*e*) $13\pi, \frac{13\pi}{12}$

7. (*a*) $x = 0, y = 0$
(*b*) $x = 0.36234, y = 1$
(*c*) $x = 0.15642, y = 0.58578$
(*d*) $x = 3.29816, y = 3.41422$
(*e*) $x = 4.23598, y = 3.73206$
(*f*) $x = 8.33030, y = 3.73206$
(*g*) $x = 1.14160, y = 2$
(*h*) $x = 6.28318, y = 4$
(*i*) $x = 11.42476, y = 2$
(*j*) $x = 12.56636, y = 0$
(*k*) $x = 43.98226, y = 4$

8. (*a*) $x = 5, y = 0$
(*b*) $x = 7.03450, y = 1.71215$
(*c*) $x = -13.4930, y = 13.3610$

9. 91°21′

§42. Pages 97 to 99

1. (*a*) 226.20 ft.; (*b*) 358.14 ft.; (*c*) 217.92 ft.; (*d*) 4.2935 ft.; (*e*) 0.13264 ft.; (*f*) $4a$ ft.

2. (*a*) 36°; (*b*) 1°12′; (*c*) 7′12″; (*d*) 1°26′24″; (*e*) 336°50′24″

4. 7.5 ft. **5.** 94°4′ **6.** 75 yd. **7.** $\frac{1}{33}$

8. 247.16 r.p.m., 25.882 radians per second

9. 0.00098175, 1018.1

11. 72 yd.

12. 0.015708

13. 69.088 miles, 932.71 miles

14. 2160 miles

15. 2.2270 ft.

16. 62.857 radians per second

17. 1760 radians per minute

18. 17.045 miles per hour

19. 7.3304 ft. per second

20. 846.40 ft.

21. 222.67 ft., 4583.8 ft.

22. 589.33 ft.

23. 20.944 ft., 200 ft.

24. 294.51 ft.

25. 2.9630 mils

§45. Pages 104 to 106

3. $\sin(A+B) = \sin C$, $\cos(A+B) = -\cos C$, $\tan(A+B) = -\tan C$

4. $1+\cos\theta$, $1+\sin\theta$, hav θ, hav θ, vers θ, covers θ

5. (*a*) $-\cos 10°$ (*b*) $-\tan 70°$ (*c*) $-\cot 20°$ (*d*) $\cos 20°$ (*e*) $-\tan 80°$ (*f*) $-\sin 60°$

6. (*a*) $\cos\theta$ (*b*) $-\tan\theta$ (*c*) $-\tan\theta$ (*d*) $-\cos\theta$ (*e*) $\tan\theta$ (*f*) $-\sec\theta$ (*g*) $\sec\theta$ (*h*) $-\sin\theta$

7. (*a*) 0.984, −0.177, −5.539, −0.180;
(*b*) − 0.582, 0.813, −0.716, −1.397;
(*c*) 0.295, 0.955, 0.309, 3.239

8. (*a*) 3 (*b*) −1 (*c*) $\csc^2\theta$ (*d*) $\cos^2\theta$ (*e*) $-\cot\theta$

9. (*a*) $-\frac{1}{4}(\sqrt{3}+1)$ (*b*) 0 (*c*) 0

11. $\frac{1}{8}(4\sqrt{3}-27)$ **12.** $\frac{1}{4}(2-3\sqrt{3})$

13. $-\cos^2 x - \sin^2 x \tan x$

14. (*a*) $-3\sqrt{3}$; (*b*) $\frac{1}{8}$; (*c*) $\frac{1}{2}$; (*d*) -1; (*e*) $-\sqrt{3}$; (*f*) $-\frac{1}{2}\sqrt{3}$

§50. Pages 115, 116

1. (*a*) $\frac{2}{5}\pi$ (*b*) $\frac{1}{4}\pi$ (*c*) 2π (*d*) $\frac{1}{4}\pi$ (*e*) $\frac{1}{3}\pi$ (*f*) π (*g*) $\frac{\pi}{2}$ (*h*) 1 (*i*) 3π (*j*) $\frac{2\pi}{3}$ (*k*) $\frac{2}{3}\pi$ (*l*) π (*m*) π (*n*) $\frac{2\pi}{277}$

2. (*a*) 1 (*b*) 4 (*c*) $\frac{1}{2}$ (*d*) 8.6 (*e*) 334 (*f*) $\frac{3}{16}$ (*g*) 1 (*h*) 8

10. $\frac{2\pi}{377}$, 110

§51. Pages 116 to 119

1. $\frac{\pi}{18}$, $\frac{1}{6}\pi$, $\frac{1}{4}\pi$, $\frac{3}{4}\pi$, $\frac{5}{4}\pi$, $-\frac{3}{2}\pi$, $-\frac{\pi}{10}$, -0.42324

4. 60°, 180°, 120°, 315°, 114°36′, 286°29′, −171°53′

5. (a) $-\tan 30°$ (b) $\cos 25°43'$ (c) $-\cot 36°$ (d) $-\csc 25°43'$
6. (a) 2.4 (b) 137°30′
7. 3.3510
8. 0.42 radian
9. 18.40 miles per second
10. 30.159 radians per second, 753.98 ft. per minute
11. (a) 0 (b) 2 (c) 1 (d) 0 (e) -3.9793 (f) $-\sqrt{3}$ (g) 8
12. (a) $\cos^2 x - \sin^2 x$ (b) 1 (c) $\cot^2 A$ (d) 1 (e) $-\cos^2 \theta$ (f) 0 (g) 1
18. $\dfrac{2c}{(c^2 - 1)\sqrt{c^2 + 1}}$
19. $\sin(-\theta) = \frac{15}{17}$, $\cos(-\theta) = -\frac{8}{17}$, $\tan(-\theta) = -\frac{15}{8}$, etc.
20. $\sin\theta = \dfrac{1}{\sqrt{5}}$, $\cos\theta = -\dfrac{2}{\sqrt{5}}$, $\tan\theta = -\frac{1}{2}$, etc.
21. $\frac{119}{169}$
22. $-\frac{2}{5}$
27. $(n - 2)\pi$
28. 523.6
29. 92,800,000 miles
30. 830.79 ft.
31. 182.42 ft.
32. 304.10 ft.

§53. Pages 122 to 124

1. $\frac{2}{9}(1 + \sqrt{10})$, $\frac{1}{9}(4\sqrt{2} - \sqrt{5})$.
3. $\frac{1}{4}\sqrt{2}(\sqrt{3} + 1)$, $\frac{1}{4}\sqrt{2}(\sqrt{3} - 1)$, etc.
4. $\frac{1}{4}\sqrt{2}(\sqrt{3} + 1)$, etc.
5. 0
6. (b) 0.0178
7. $\frac{33}{65}$
8. $\frac{4}{5}, \frac{3}{5}$
9. $\sin 2A = 2\sin A\cos A$, $\cos 2A = \cos^2 A - \sin^2 A$
11. (a) $\cos y, -\sin y$
(b) $\sin y, -\cos y$
(c) $-\sin y, -\cos y$
(d) $-\cos y, -\sin y$
(e) $-\cos y, \sin y$
(f) $-\sin y, \cos y$
(g) $\sin y, \cos y$
(h) $-\cos x, \sin x$
(i) $-\sin x, -\cos x$
(j) $\cos x, -\sin x$
(k) $-\sin y, \cos y$
(l) $\dfrac{1}{\sqrt{2}}(\cos y - \sin y)$, $\dfrac{1}{\sqrt{2}}(\cos y + \sin y)$
(m) $\dfrac{1}{\sqrt{2}}(\cos y + \sin y)$, $\dfrac{1}{\sqrt{2}}(\cos y - \sin y)$
(n) $\frac{1}{2}(\cos y + \sqrt{3}\sin y)$, $\frac{1}{2}(\sqrt{3}\cos y - \sin y)$
(o) $\frac{1}{2}(\sqrt{3}\cos y - \sin y)$, $\frac{1}{2}(\cos y + \sqrt{3}\sin y)$
15. $\dfrac{1}{2\sqrt{3}}(\sqrt{3} + \sqrt{2})$
24. $3\sin\theta - 4\sin^3\theta$
25. $4\cos^3\theta - 3\cos\theta$

§55. Pages 126 to 128

3. $-(2+\sqrt{3})$
5. $\sin(\alpha+\beta) = -\frac{33}{65}$; $\cos(\alpha+\beta) = \frac{56}{65}$; $\tan(\alpha+\beta) = -\frac{33}{56}$, etc.
6. $\sin(\alpha-\beta) = -\frac{308}{533}$; $\cos(\alpha-\beta) = -\frac{435}{533}$; $\tan(\alpha-\beta) = +\frac{308}{435}$, etc.
7. $-\frac{1}{2}$
8. 3
14. (a) $\sin 5x$; (b) $\cos x$; (c) $\sin x$; (d) 0; (e) $\cos 2x$; (f) $\sin 2x$
15. (a) $\tan 5x$; (b) $\tan 2x$
20. (a) $4\sin(\theta+30°)$; (b) $\sqrt{2}a\sin(\theta+45°)$; (c) $\sin(\theta+45°)$; (d) $2\sqrt{3}\sin(\theta-30°)$; (e) $5\sin(\theta+53°8')$; (f) $2\cos(\theta+45°)$

§56. Pages 130 to 132

1. $-\frac{24}{25}, \frac{7}{25}, -\frac{24}{7}, \frac{3}{10}\sqrt{10}, \frac{1}{10}\sqrt{10}, 3$
2. $\frac{1}{2}\sqrt{2-\sqrt{2}}, \frac{1}{2}\sqrt{2+\sqrt{2}}$
6. $\pm(4\sin x - 8\sin^3 x)\sqrt{1-\sin^2 x}$, $\dfrac{4\tan x - 4\tan^3 x}{1 - 6\tan^2 x + \tan^4 x}$
8. $\frac{1}{4}(\sqrt{5}-1)$
9. $-\frac{119}{120}, \frac{5}{13}, \frac{120}{169}, -\frac{169}{120}$

§57. Pages 134 to 136

1. (a) $2\sin 30°\cos 5°$; (c) $2\cos 45°\cos 20°$; (e) $2\cos 3x\cos x$; (g) $2\sin 2x\cos x$
2. (a) $\frac{1}{2}(\sin 10x - \sin 4x)$; (b) $\frac{1}{2}(\cos 10x + \cos 4x)$; (c) $\frac{1}{4}(\cos 2x + \cos 4x - \cos 6x - 1)$; (d) $\frac{1}{4}(\sin 15x + \sin 9x + \sin 5x - \sin x)$
26. $2\sin[45° + \frac{1}{2}(x-y)]\cos[-45° + \frac{1}{2}(x+y)]$
27. $2\cos[45° + \frac{1}{2}(x-y)]\sin[-45° + \frac{1}{2}(x+y)]$
29. $4\sin 4\alpha\cos 2\alpha\cos\alpha$

§58. Pages 136 to 139

2. (a) $\frac{56}{65}$
   (c) $\frac{33}{65}$
5. (a) $\frac{6}{7}$
   (b) $\frac{2}{9}$
39. Varies from 0 to 1
45. $1 - 18\sin^2\alpha + 48\sin^4\alpha - 32\sin^6\alpha$

§59. Pages 142 to 144

1. $x = y = 4\sqrt{3}$, $x = 18$, $y = 31.172$
2. Fig. 5 $\begin{cases} x = 35\sin 60°\csc 70°; \\ y = 35\sin 50°\csc 70°; \end{cases}$ Fig. 6: $x = y = 35\sin 70°\csc 40°$ Fig. 7: $x = 40\sin 111°20'\csc 30°$
   Fig. 8: $x = 60\sin 74°25'\csc 40°$, $y = 60\sin 25°35'\csc 40°$
3. $x = \csc 30°\sin 80°$, $y = \csc 30°\sin 50°$, $z = \csc 30°\sin 50°\sin 80°\csc 60°$, $p = \csc 30°\sin 50°\sin 40°\csc 60°$
4. $\sin B = 0.68627$, $x = 624\sin(118° - B)\csc 62°$
5. $[312\sin(118° - B)(\csc 62°)]\,485\sin 62°$
6. $x = a\sin 65°\csc 40°$, $y = a\sin 75°\csc 40°$, $x = a\csc\theta\sin(\theta+\varphi)$, $y = a\csc\theta\sin\varphi$

7. $x = \sin 50° \csc 60°$, $z = \sin 50° \csc 30°$, $w = \sin 50° \csc 70°$, $y = \sin^2 50° \csc 60° \csc 70°$

§61. Pages 148, 149

1. $\sqrt{52}, \dfrac{6 \sin 60°}{\sqrt{52}}, \dfrac{8 \sin 60°}{\sqrt{52}}$ 2. $\tan \frac{1}{2}(A - B) = \dfrac{1 - \sqrt{3}}{1 + \sqrt{3}}$

3. Fig. 23: $x = \sqrt{34 - 15\sqrt{3}}$, $\sin A = \dfrac{3 \sin 30°}{x}$, $\sin B = \dfrac{5 \sin 30°}{x}$

4. $\frac{1}{7} \tan 45°$

§62. Pages 149 to 151

1. $\sqrt{1873 - 924\sqrt{2}}$ 2. $\frac{5}{61} \tan 67\frac{1}{2}°$ 3. $462 \sin 45°$

5. $Area = \dfrac{c^2 \sin A \sin B}{2 \sin (A + B)}$ 6. $\frac{9}{16}$

7. (a) $8 \sin 60° \sin 40° \csc 50° \csc 35°$ (b) 10.136

8. $h = m \sin w \csc (w + z) \sin y \csc (x + y)$

§65. Page 155

1. $b = 4.4217$, $c = 1.7302$, $C = 22°24'$
2. $b = 4382.9$, $c = 6136.0$, $A = 81°47'12''$
3. $a = 895.14$, $b = 728.40$, $C = 67°34'31''$
4. $a = 177.64$, $b = 213.78$, $B = 62°19'53''$
5. $a = 241.18$, $b = 165.68$, $C = 68°12'15''$
6. $b = 695.32$, $c = 345.64$, $C = 21°14'20''$
7. 345.43
8. 73.548 ft.
10. (a) 3.113
11. 26,624 ft., 26,689 ft.
12. 2232 2 ft.
13. 590.43 ft.
14. 192.41 ft

§66. Pages 160, 161

1. $B_1 = 24°57'54''$, $C_1 = 133°47'41''$, $c_1 = 615.67$; $B_2 = 155°2'6''$, $C_2 = 3°43'29''$, $c_2 = 55.410$
2. $A_1 = 134°18'3''$, $C_1 = 24°25'13''$, $a_1 = 623.19$; $A_2 = 3°8'29''$, $C_2 = 155°34'47''$, $a_2 = 47.718$
3. $B_1 = 51°9'6''$, $C_1 = 87°37'54''$, $c_1 = 116.82$; $B_2 = 128°50'54''$, $C_2 = 9°56'6''$, $c_2 = 20.172$
4. $a_1 = 167.64$, $A_1 = 81°39'07''$, $C_1 = 55°09'21''$; $a_2 = 35.124$, $A_2 = 11°57'49''$, $C_2 = 124°50'39''$
5. $B = 36°26'46''$, $C = 76°1'14''$, $c = 308.73$
6. $a = 31.672$, $C = 90°$, $A = 23°47'50''$
7. $B = 26°12'38''$, $C = 117°23'22''$, $c = 72.022$

8. $c_1 = 60.303$, $B_1 = 56°20'08''$, $C_1 = 91°21'22''$; $c_2 = 24.561$, $B_2 = 123°39'52''$, $C_2 = 24°01'38''$

9. $c_1 = 3.7834$, $B_1 = 79°12'00''$, $C_1 = 46°30'00''$; $c_2 = 2.1960$, $B_2 = 100°48'00''$, $C_2 = 24°54'00''$

10. $B_1 = 45°23'28''$, $A_1 = 99°00'12''$, $a_1 = 300.29$; $B_2 = 134°36'32''$, $A_2 = 9°47'08''$, $a_2 = 51.670$

11. Impossible

13. 17,091

14. $47°47'36''$

§67. Pages 163, 164

1. $A = 77°12'53''$, $B = 43°30'7''$, $c = 14.987$

2. $A = 86°23'9''$, $B = 30°1'21''$, $c = 671.27$

3. $B = 67°37'44''$, $C = 51°9'16''$, $a = 220.10$

4. $A = 40°28'17''$, $B = 99°51'43''$, $c = 27.458$

5. $B = 51°57'20''$, $C = 77°22'16''$, $a = 83.732$

6. $A = 92°51'28''$, $B = 22°30'32''$, $c = 0.53660$

7. $A = 52°10'33''$, $B = 17°17'27''$, $c = 7.3962$

8. $A = 46°49'58''$, $B = 22°29'32''$, $c = 45.198$

10. 5119.5 ft.

11. 147.96 ft.

12. 4064.1, $165°53'45''$

14. (*a*) 87.690

15. Not horizontal; 5281.7 ft.

17. 443.19 ft.

§69. Pages 168, 169

1. $A = 106°46'40''$, $B = 46°53'14''$, $C = 26°20'6''$

2. $A = 27°20'32''$, $B = 143°7'48''$, $C = 9°31'40''$

3. $A = 8°20'1''$, $B = 33°40'5''$, $C = 137°59'54''$

4. $A = 44°42'16''$, $B = 49°37'26''$, $C = 85°40'24''$

5. $A = 27°46'44''$, $B = 33°46'52''$, $C = 118°26'20''$

6. $A = 51°53'12''$, $B = 59°31'48''$, $C = 68°35'00''$

7. $A = 28°6'52''$, $B = 115°2'4''$, $C = 36°51'8''$

8. $A = 45°37'18''$, $B = 75°19'32''$, $C = 59°3'10''$

9. $A = 80.4°$, $B = 56.6°$, $C = 43.0°$

10. $A = 46.6°$, $B = 58°$, $C = 75.5°$

11. $A = 106°$, $B = 39.8°$, $C = 34.1°$

13. $72.6°$

14. 495.53 ft.

§71. Pages 170 to 176

1. $A = 40°49'36''$, $B = 23°31'24''$, $c = 58.416$

2. $A = 41°47'45''$, $B = 54°20'09''$, $C = 83°52'05''$

3. $C = 69°13'45''$, $b = 462.76$, $c = 499.00$

4. $A = 52°10'33''$, $B = 17°17'27''$, $c = 0.073964$

5. $A = 46°56'24''$, $B = 57°11'08''$, $C = 75°52'32''$

6. $B_1 = 56°56'56''$ $\quad B_2 = 123°3'4''$
$C_1 = 90°45'4''$ $\quad C_2 = 24°38'56''$
$c_1 = 58.456$ $\quad c_2 = 24.382$

7. $AC = 1474.0$ ft., $BC = 1252.7$ ft.
8. 6328.7 ft. **9.** 84°8′12″ **10.** 722.18
12. 52.431 **16.** 3.1959 miles per hour
15. 373 ft. **17.** 731.13 ft., 50°38′
18. 6463.0 ft. **19.** 88.016 ft. **21.** 8.0126 nautical miles
22. 4°44′25 **32.** 2109.8 yd.
23. 231.94 ft., 328.93 ft. **35.** 509.77 yd.
30. 2554.7 ft. **37.** 107.24
42. $PB = 403.68$, $PA = 140.89$, $PC = 734.98$
45. 79.4 yd., 1°49′1″
46. $\frac{R}{\theta}[\theta - \sin^{-1}(\sin\theta\cos\varphi)]$, $\tan^{-1}(\tan\theta\sin\varphi)$, where $\sin^{-1}$ ($\tan^{-1}$) means angle whose sine (tangent) is

§72. Page 178

1. 30°, 150° **5.** 135°, 315° **9.** 60°, 300°
2. 60°, 120° **6.** 120°, 240° **10.** 210°, 330°
3. 225°, 315° **7.** 135°, 225° **11.** 60°, 120°
4. 60°, 240° **8.** 45°, 315° **12.** 25°36′, 154°24′

§74. Pages 180, 181

1. (*a*) $\frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi$ (*d*) $\frac{4\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi$
(*b*) $\frac{\pi}{3} + 2n\pi, \frac{2\pi}{3} + 2n\pi$ (*e*) $2n\pi, \pi + 2n\pi$; (or $n\pi$)
(*c*) $\frac{\pi}{4} + 2n\pi, \frac{3\pi}{4} + 2n\pi$ (*f*) $\frac{3\pi}{2} + 2n\pi$
(*g*) $19°28' + n360°, 160°32' + n360°$
(*h*) $25°36' + n360°, 154°24' + n360°$
(*i*) $204°37' + n360°, 335°23' + n360°$
(*j*) $\frac{\pi}{4} + 2n\pi, \frac{7\pi}{4} + 2n\pi$ (*n*) $\frac{3\pi}{4} + 2n\pi, \frac{7\pi}{4} + 2n\pi$
(*k*) $\frac{3\pi}{4} + 2n\pi, \frac{5\pi}{4} + 2n\pi$ (*o*) $\frac{\pi}{2} + n\pi$
(*l*) $\frac{5\pi}{6} + 2n\pi, \frac{7\pi}{6} + 2n\pi$ (*p*) $\frac{\pi}{4} + 2n\pi, \frac{5\pi}{4} + 2n\pi$
(*m*) $\frac{7\pi}{6} + 2n\pi, \frac{11\pi}{6} + 2n\pi$ (*q*) $n\pi$
(*r*) $66°38' + n360°, 246°38' + n360°$

2. (*a*) $\frac{11\pi}{6} + 2n\pi$ (*c*) $\frac{3\pi}{4} + 2n\pi$ (*e*) $\frac{5\pi}{6} + 2n\pi$
(*b*) $\frac{7\pi}{6} + 2n\pi$ (*d*) $\frac{5\pi}{4} + 2n\pi$ (*f*) $\frac{5\pi}{3} + 2n\pi$

3. (a) $21°6' + n360°$, $158°54' + n360°$
(b) $53°8' + n360°$, $306°52' + n360°$
(c) $41°59' + n360°$, $221°59' + n360°$
(d) $25°28' + n360°$, $205°28' + n360°$
(e) $73°0' + n360°$, $287°0' + n360°$
(f) $55°44' + n360°$, $124°16' + n360°$
(g) $53°8' + n360°$, $306°52' + n360°$
(h) $41°49' + n360°$, $138°11' + n360°$
(i) $51°20' + n360°$, $231°20' + n360°$
(j) $48°11' + n360°$, $311°49' + n360°$
(k) $48°49' + n360°$, $228°49' + n360°$
(l) $3°49' + n360°$, $176°11' + n360°$

5. (a) $30° + k60°$ (b) $k36°$ (c) $27°22' + k90°$ (d) $20° + k60°$

6. (a) $45° + k180°$ (b) $30° + k180°$ (c) $135° + k180°$ (d) $18°53' + k180°$

§75. Pages 183, 184

1. (a) $\frac{1}{4}\pi$ (b) $\frac{1}{3}\pi$ (c) 0 (d) $\frac{\pi}{4}$ (e) $\frac{1}{3}\pi$ (f) 0 (g) $\frac{1}{4}\pi$ (h) $\frac{1}{3}\pi$ (i) $\frac{1}{4}\pi$ (j) $\frac{1}{2}\pi$ (k) $\frac{1}{6}\pi$ (l) $\frac{1}{3}\pi$ (m) $\frac{1}{2}\pi$ (n) $\frac{1}{6}\pi$ (o) $\frac{1}{3}\pi$ (p) 0 (q) $\frac{1}{6}\pi$ (r) $\frac{1}{3}\pi$

2. (a) $-30°$ (b) $-45°$ (c) $-60°$ (d) $-45°$ (e) $-60°$ (f) $-30°$

3. (a) $135°$ (b) $150°$ (c) $120°$ (d) $135°$ (e) $150°$ (f) $120°$

4. (a) $-\frac{2}{3}\pi$ (b) $-\frac{3}{4}\pi$ (c) $-\pi$ (d) $-\frac{5}{6}\pi$ (e) $-\frac{5}{6}\pi$ (f) $-\frac{3}{4}\pi$ (g) $-\frac{2}{3}\pi$ (h) $-\frac{1}{2}\pi$ (i) $-\frac{1}{2}\pi$

5. (a) $-30°$ (b) $45°$ (c) $150°$ (d) $90°$ (e) $-135°$ (f) $-180°$ (g) $-45°$ (h) $60°$ (i) $60°$ (j) $-135°$ (k) $180°$

6. (a) $-60°$ (b) $114°27'$ (c) $-54°44'$ (d) $-111°29'$ (e) $115°16'$ (f) $-171°1'$ (g) $-4°15'$ (h) $155°55'$ (i) $-85°36'$

7. (a) $\frac{1}{3}\pi$ (b) $-\frac{1}{6}\pi$ (c) $\frac{1}{6}\pi$ (d) $-\frac{1}{3}\pi$ (e) π (f) $-\frac{2}{3}\pi$

§77. Pages 187 to 189

1. $\frac{2}{3}$
2. $\frac{3}{5}$
3. $\frac{1}{12}\sqrt{119}$
4. $\frac{1}{3}\sqrt{5}$
5. $-\sqrt{\frac{8}{7}}$
6. $-\frac{4}{5}$
7. $-\frac{3}{5}$
8. $-\frac{3}{5}$
9. $2/\sqrt{5}$
10. $\frac{1}{2}\sqrt{5}$
11. ± 1
12. $\dfrac{\sqrt{30.16}}{5.4}$
13. 0
14. $\frac{4}{3}$
15. $4/\sqrt{17}$
16. (a) $-\frac{1}{8}$ (b) $2/\sqrt{3}$ (c) 1 (d) -0.993

36. $a\sqrt{2 - 2a^2}\sqrt{1 + b} + (2a^2 - 1)\sqrt{\dfrac{1 - b}{2}}$

§78. Pages 190 to 192

1. (a) 30°, 150°, 210°, 330°
(b) 45°, 135°, 225°, 315°
(c) 60°, 120°, 240°, 300°
(d) 60°, 120°, 240°, 300°
(e) $22\frac{1}{2}$°, $112\frac{1}{2}$°, $202\frac{1}{2}$°, $292\frac{1}{2}$°
(f) 10°, 50°, 130°, 170°, 250°, 290°

2. (a) 120°, 240°
(b) 30°, 150°, 210°, 330°
(c) 60°, 120°
(d) 60°, 300°
(e) 30°, 150°, 210°, 330°
(f) 45°, 225°
(g) 135°, 315°

3. (a) $\frac{1}{3}\pi$, $\frac{3}{4}\pi$, $\frac{4}{3}\pi$, $\frac{7}{4}\pi$
(b) $\frac{1}{2}\pi$, $\frac{2}{3}\pi$, $\frac{4}{3}\pi$
(c) $\frac{1}{6}\pi$, $\frac{5}{6}\pi$, $\frac{7}{6}\pi$, $\frac{11}{6}\pi$
(d) $\frac{1}{2}\pi$, $\frac{7}{6}\pi$, $\frac{11}{6}\pi$, $\frac{3}{2}\pi$

4. (a) $n360°$, $120° + n360°$, $240° + n360°$
(b) $30° + n360°$, $150° + n360°$
(c) $270° + n360°$
(d) $45° + n180°$, $105° + n180°$, $165° + n180°$
(e) $56°19' + n180°$, $135° + n180°$
(f) $33°41' + n180°$, $45° + n180°$
(g) $37°59' + n45°$
(h) $90° + n180°$, $\pm 60° + n180°$, $\pm 120° + n180°$
(i) $51°19' + n360°$, $308°41' + n360°$, $180° + n360°$
(j) $30° + n360°$, $150° + n360°$, $90° + n360°$
(k) $45° + n90°$
(l) $45° + n180°$, $71°34' + n180°$
(m) $120° + n360°$, $240° + n360°$
(n) $9°44' + n360°$, $151°21' + n360°$
(o) $n360°$, $90° + n360°$
(p) $60° + n360°$
(q) $105° + n180°$, $165° + n180°$
(r) $90° + n180°$, $120° + n360°$, $240° + n360°$
(s) $30° + n180°$, $150° + n180°$

5. (a) $n180°$, $\pm 60° + n360°$,
(b) $90° + n180°$, $30° + n360°$, $150° + n360°$
(c) $n180°$, $\pm 60° + n180°$, $\pm 120° + n180°$
(d) $90° + n180°$, $210° + n360°$, $330 + n360°$
(e) $45° + n90°$, $15° + n180°$, $75° + n180°$
(f) $30° + n360°$, $330° + n360°$, $n180°$
(g) $n90°$, $30° + n90°$, $60° + n90°$
(h) $n90°$, $52°14' + n180°$, $127°46' + n180°$
(i) $n180°$, $\pm 60° + n180°$

6. (a) $n\pi$
(b) $2n\pi$, $\frac{2}{3}\pi + 2n\pi$, $\frac{4}{3}\pi + 2n\pi$
(c) $n\pi$
(d) $n\pi$

§79. Pages 194, 195

1. (a) $n60°$, $15° + n30°$
(b) $n45°$
(c) $5° + n20°$, $22\frac{1}{2}° + n90°$
(d) $\dfrac{n180°}{7}$
(e) $9° + n18°$
(f) $45° + n180°$, $5° + n20°$

(g) $-25°20' + n360°$, $131°36' + n360°$
(h) $90° + n360°$, $196°16' + n360°$
(i) $142°37' + n360°$, $262°37' + n360°$
(j) $8°8' + n360°$, $217°6' + n360°$
(k) $135° + n180°$, $161°34' + n180°$
(o) $\theta = n45°$, $\pm 12° + n72°$
(s) $x = \frac{1}{2}\sqrt{10}$, $\theta = 71°34' + n360°$; $x = \frac{-1}{2}\sqrt{10}$, $\theta = 251°34' + k360°$

2. $r = \sqrt{38}\begin{cases} \varphi = 54°12' + n360°, & \theta = 56°19' + n360° \\ \varphi = 125°48' + n360°, & \theta = 236°19' + n360° \end{cases}$

$r = -\sqrt{38}\begin{cases} \varphi = -54°12' + n360°, & \theta = 236°19' + n360° \\ \varphi = -125°48' + n360°, & \theta = 56°19' + n360° \end{cases}$

3. $\tan\left(x + \frac{1}{2}\alpha\right) = \frac{m+1}{m-1}\tan\frac{\alpha}{2}$ which determines $x + \frac{1}{2}\alpha$, and therefore x

4. $x = \tan^{-1}\left[\frac{a\sin\varphi - b\sin\theta}{b\cos\theta - a\cos\varphi}\right]$

$m = [a^2 + b^2 - 2ab\cos(\varphi - \theta)]^{\frac{1}{2}}\csc(\varphi - \theta)$

5. $m\sin x = \frac{b\cos\theta - a\sin\phi}{\cos(\theta - \phi)}$,

$m\cos x = \frac{b\sin\theta + a\cos\phi}{\cos(\theta - \phi)}$

$x = \tan^{-1}\frac{b\cos\theta - a\sin\phi}{b\sin\theta + a\cos\phi}$

$m = [a^2 + b^2 - 2ab\cos(\theta + \phi)]^{\frac{1}{2}}\sec(\theta - \phi)$

6. $m\sin x = \frac{b\cos\theta - a\cos\phi}{\sin(\theta + \phi)}$,

$m\cos x = \frac{b\sin\theta + a\sin\phi}{\sin(\theta + \phi)}$

$x = \tan^{-1}\frac{b\cos\theta - a\cos\phi}{b\sin\theta + a\sin\phi}$

$m = [a^2 + b^2 - 2ab\cos(\theta + \phi)]^{\frac{1}{2}}\csc(\theta + \phi)$

7. $x = m\cos\alpha + n\sin\alpha$ $\qquad y = m\sin\alpha - n\cos\alpha$

§80. Pages 196, 197

2. (a) $y = \frac{1}{5}\sqrt{5}$
(b) 1
(c) $\pm\frac{1}{3}\sqrt{5}$
(d) $\frac{ab + \sqrt{(1 - a^2)(1 - b^2)}}{b\sqrt{1 - a^2} - a\sqrt{1 - b^2}}$
(e) 0, $\pm\frac{1}{3}\sqrt{3}$
(f) No solution
(g) $\frac{1}{2}$
(h) 13
(i) $\sqrt{n^2 + m^2}$, $m > 0$, $n > 0$; $-\sqrt{n^2 + m^2}$, $m < 0$, $n < 0$.
(j) $\frac{1}{2}\sqrt{3}$
(k) ± 1
(l) 0

§81. Pages 197 to 199

1. (*a*) $\pm\frac{5}{13}$. (*c*) $\dfrac{2a}{1 - a^2}$ (*e*) $2a^2 - 1$ (*g*) $n\pi + \dfrac{\pi}{6}$

(*b*) $\pm\dfrac{1}{\sqrt{2}}$ (*d*) $\frac{7}{24}$ (*f*) $\dfrac{1}{\sqrt{a^2 + 1}}$ (*h*) $n\pi \pm \dfrac{\pi}{4}$

3. (*a*) $71°34' + n360°$, $251°34' + n360°$

(*b*) $158°32'\ n360°$, $201°28'\ n360°$

(*c*) $n180°$

4. (*a*) $199°28' + n360°$, $340°32' + n360°$

(*b*) $70°32' + n360°$, $289°28' + n360°$

(*c*) $45° + n180°$, $116°34' + n180°$

(*d*) $210° + n360°$, $330° + n360°$, $41°49' + n360°$, $138°11' + n360°$

(*e*) $90° + n180°$, $210° + n360°$, $330° + n360°$

(*f*) $204°28' + n360°$, $335°32' + n360°$

(*g*) $76°40' + n180°$, $347°3' + n180°$

(*h*) $135° + n180°$

(*i*) $= 270° + n360°$, $126°52' + n360°$

(*j*) $n360°$

(*k*) $60° + n360°$

(*l*) $30° + n90°$, $35°16' + n90°$

5. (*a*) $n90°$ (*c*) $n180°$, $30° + n90°$, $60° + n90°$

(*b*) $\dfrac{\pi}{16} + \dfrac{n\pi}{4}$, $\frac{1}{4}\pi - n\pi$

6. $180° + n360°$, $\dfrac{90° + n360°}{11}$

7. (*a*) $n360°$, $106°16' + n360°$ (*b*) $77°20' + n360°$, $180° + n360°$

8. (*a*) $240° + n360°$, $300° + n360°$

(*b*) $210° + n360°$, $330° + n360°$

(*c*) $\pm 30° - n180°$

(*d*) $49°21' + n360°$, $310°29' + n360$

(*e*) $\pm 60° + n720°$, $\pm 300° + n720°$

9. (*a*) $n90°$, $120° + n360°$, $240° + n360°$

(*b*) $n60°$, $\pm 35°16' + n180°$

(*c*) 30°, 90°, 150°, 210°, 270°, 330° (add $n360°$ to each)

10. (*d*) $\dfrac{(x - a)^2}{b^2} + \dfrac{(y - c)^2}{d^2} = 1$ (*e*) $\left(\dfrac{y}{b}\right)^{\frac{2}{3}} - \left(\dfrac{x}{a}\right)^{\frac{2}{3}} = 1$

11. (*a*) $\frac{1}{2}$ (*c*) $+\dfrac{\sqrt{10}}{2}$ (*e*) none (*g*) $+\dfrac{\sqrt{21}}{14}$,

(*b*) $\sqrt{3}$ (*d*) $\sqrt{3}$, (*f*) $\frac{1}{4}$, (*h*) 13

§82. Page 200

1. (*a*) $6i$ (*c*) $7i$ (*e*) $4xi$ (*g*) $5x^2y\sqrt{5}i$

(*b*) $3\sqrt{3}i$ (*d*) $\sqrt{\frac{5}{12}}i$ (*f*) $\dfrac{2}{x}i$ (*h*) $i\sqrt{4ac - b^2}$

2. (*a*) $\pm 4i$; (*b*) $\pm 3xi$; (*c*) $\pm\sqrt{13}i$; (*d*) $a^2x\sqrt{7}i$
3. (*a*) i; (*b*) 1; (*c*) -1; (*d*) -1; (*e*) $-i$; (*f*) 1; (*g*) -1; (*h*) 1

§84. Pages 201, 202

1. (*a*) $x = 2,\ y = -3$; (*b*) $x = \frac{5}{3},\ y = \frac{-7}{2}$; (*c*) $x = \frac{2}{3},\ y = 4$; (*d*) $x = 3,\ y = \frac{3}{2}$; (*e*) $x = -1,\ y = 0$
2. (*a*) $7 - 2i$; (*b*) $x + yi$; (*c*) $-3i$; (*d*) 14
3. (*a*) $5 - i$ (*b*) $-4 + 8i$ (*c*) $6 - 3i$ (*d*) $3 + 4i$ (*e*) 6 (*f*) $3 + 7i$ (*g*) $2 - 2i$ (*h*) $8i$
5. (*a*) $28 + 24i$ (*b*) $20 - 48i$ (*c*) $2 + 16i$ (*d*) 65 (*e*) $5 + 2i$ (*f*) $32 - 26i$
7. (*a*) $\frac{22}{85} - \frac{71}{85}i$ (*b*) $\frac{7}{5} - \frac{1}{5}i$ (*c*) $\frac{11}{26} + \frac{3}{26}i$ (*d*) $\frac{5}{41} + \frac{4}{41}i$ (*e*) $4 - 5i$ (*f*) $-\frac{4}{25} + \frac{3}{25}i$ (*g*) $\frac{37}{65} - \frac{16}{65}i$ (*h*) $-\frac{121}{481} + \frac{578}{481}i$ (*i*) $0.02 - 0.64i$

§86. Pages 204, 205

1. $-3\sqrt{3} - 3i$, $6(\cos 210° + i \sin 210°)$
2. $\sqrt{2}(\cos 45° + i \sin 45°)$
3. (*a*) $\sqrt{3} + i$ (*b*) $\frac{3}{2} + \frac{3}{2}\sqrt{3}i$ (*c*) $-\sqrt{2} + \sqrt{2}i$ (*d*) -4 (*e*) $-\frac{11}{2}\sqrt{3} - \frac{11}{2}i$ (*f*) $-7i$ (*g*) $3 - 3\sqrt{3}i$ (*h*) $3 - 3\sqrt{3}i$
4. (*a*) $\sqrt{2}$ cis 315° (*b*) $\sqrt{13}$ cis 236°19′ (*c*) $\sqrt{13}$ cis 123°41′ (*d*) 4 cis 0° (*e*) 4 cis 90° (*f*) 5 cis 0° (*g*) 7 cis 90° (*h*) $\sqrt{1.7}$ cis 57°32′ (*i*) 1.5 cis 270° (*f*) 1 cis 270° (*k*) 8.60 cis 324°28′ (*l*) 6.28 cis 300°39′ (*m*) 7.41 cis 145°28′ (*n*) 7.38 cis 243°26′ (*o*) 8.35 cis 328°13′

§88. Pages 207, 208

1. (*a*) $9.70 + 17.5i$ (*b*) $4.86 + 27.6i$ (*c*) $2.16 - 2.08i$ (*d*) $-0.82 + 1.13i$
2. (*a*) i (*b*) $\sqrt{2}i$ (*c*) $2\sqrt{2}$ (*d*) -2 (*e*) $-0.518 + 1.93i$ (*f*) $4.39 + 16.4i$ (*g*) $0.228 - 0.0610i$ (*h*) $0.518 + 1.93i$ (*i*) $0.185 + 2.11i$ (*j*) $-0.958 + 0.804i$
3. (*a*) $-7.42 + 12.85i$ (*b*) $-0.101 + 0.175i$ (*c*) $6i$ (*d*) $-50.9 - 88.2i$
4. (*a*) 5 cis 280° (*b*) $5.54 - 5.28i$ (*c*) $17.1 - 38.4i$

§89. Pages 210, 211

1. (*a*) 16 cis 120° (*b*) 4^7 cis $\frac{8}{5}\pi$ (*c*) cis 30° (*d*) cis 180° $= -1$ (*e*) $972\sqrt{2}$ cis 135° (*f*) $\frac{1}{4}$ cis 180°

2. (*a*) 3.44 cis 344°31′, 3.44 cis 164°31′
(*b*) cis 60°, cis 132°, cis 204°, cis 276°, cis 348°
(*c*) cis 18°, cis 90°, cis 162°, cis 234°, cis 306°
(*d*) cis 60°, cis 180°, cis 300°
(*e*) 1.74 cis 76°58′ 1.74 cis 168°58′, 1.74 cis 256°58′, 1.74 cis 346°58′
(*f*) 1.341 cis 5°, 1.341 cis 45°, 1.341 cis 85°, 1.341 cis 125°, 1.341 cis 165°, 1.341 cis 205°, 1.341 cis 245°, 1.341 cis 285°, 1.341 cis 325°
(*g*) cis 20°, cis 60°, cis 100°, cis 140°, cis 180°, cis 220°, cis 260°, cis 300°, cis 340°

3. (*a*) $x = -1$, $x = 0.5 \pm 0.866i$
(*b*) $x = -2$, $x = 1.62 \pm 1.18i$, $x = -0.618 \pm 1.89i$
(*c*) $x = i$, $x = \pm 0.866 - 0.5i$
(*d*) $x = 0.855 \pm 1.48i$, $x = 1.71$, $x = 1.913$, $x = -0.956 \pm 1.66i$
(*e*) $x = 1$, $x = \pm 0.707 \pm 0.707i$, $x = -0.5 \pm 0.866i$

§90. Page 212

1. -1, i, $-0.41655 + 0.90911i$, i **2.** 3.7622, $-3.6269i$

§91. Pages 213, 214

1. 1, 0, 1.5431, 1.1752

§92. Page 214, 215

1. (*a*) $\frac{3}{2}\sqrt{2} + \frac{3}{2}\sqrt{2}i$
(*b*) $-2\sqrt{3} + 2i$
(*c*) $\frac{5}{2} - \frac{5}{2}\sqrt{3}i$
(*d*) $7i$
(*e*) $2.64960 + 4.24025i$
(*f*) $-4.47352 + 6.63232i$
(*g*) $4.85412 - 3.52674i$
(*h*) $-1.52458 - 1.29446i$

2. (*a*) $2\sqrt{2}$ cis 45°
(*b*) $3\sqrt{2}$ cis 315°
(*c*) $\sqrt{10}$ cis 161°34′
(*d*) $\sqrt{13}$ cis 303°41′
(*e*) 5 cis 233°8′
(*f*) $\sqrt{61}$ cis 309°48′
(*g*) $2\sqrt{10}$ cis 341°34′
(*h*) 4 cis 216°52′
(*i*) $\sqrt{19.6}$ cis 161°34′

3. (*a*) 14 cis 210° (*b*) 3.2966 cis 141°3′ (*d*) 10.181 cis 159°26′

4. (*a*) 32 cis 225°
(*b*) $(2.6)^3$ cis 219°
(*c*) 16 cis 120°
(*d*) 5^5 cis 274°20′

5. (*a*) 1.4142 cis (−15°)
1.4142 cis 165°
(*b*) 1.4953 cis (−9°13′)
1.4953 cis 80°47′
1.4953 cis 170°47′
1.4953 cis 260°47′
(*c*) 1.8301 cis 78°46′
1.8301 cis 198°46′
1.8301 cis 318°46′
(*d*) 1.4554 cis 12°53′
1.4554 cis 84°53′
1.4554 cis 156°53′
1.4554 cis 228°53′
1.4554 cis 300°53′
(*e*) cis (−30°)
cis 90°
cis 210°

6. (*a*) 2, $-1 \pm \sqrt{3}i$ (*b*) $\pm\frac{1}{2}\sqrt{3} + \frac{1}{2}i$, $-i$

(*c*) 1.3077 cis $-8°51'$	(*d*) 1.3446 cis $34°30'$
1.3077 cis $51°9'$	1.3446 cis $85°56'$
1.3077 cis $111°9'$	1.3446 cis $137°22'$
1.3077 cis $171°9'$	1.3446 cis $188°48'$
1.3077 cis $231°9'$	1.3446 cis $240°14'$
1.3077 cis $291°9'$	1.3446 cis $291°40'$
	1.3446 cis $343°6'$

§97. Page 223

1. 0	**5.** 2	**9.** 4	**13.** 3
2. 5	**6.** 1	**10.** 2	**14.** 4
3. 1	**7.** 8 − 10	**11.** 5 − 10	**15.** 9 − 10
4. 0	**8.** 9 − 10	**12.** 7 − 10	**16.** 6 − 10

§101. Page 226

1. 1.60733	**5.** 9.33333 − 10	**9.** 8.43198 − 10
2. 0.48391	**6.** 7.58371 − 10	**10.** 9.26133 − 10
3. 4.00864	**7.** 8.93677 − 10	
4. 2.03411	**8.** 5.88152 − 10	

§102. Page 227

1. 0.04592	**5.** 0.0093962	**9.** 12.954
2. 7903	**6.** 997.15	**10.** 0.00035304
3. 207,320	**7.** 7.4962	
4. 0.50119	**8.** 2.6448	

11. (*a*) 0.45347	(*c*) 0.00074363
(*b*) 0.0038615	(*d*) 0.68973

§103. Page 229

1. 433.90	**3.** 3.1414	**5.** 0.51514	**7.** 0.24406
2. 224.09	**4.** 1.3205	**6.** 5.2686	**8.** 0.062086

§104. Pages 229, 230

2. (*a*) 5.0187	(*c*) 0.00041391
(*b*) 147.54	(*d*) 5058.6

§106. Pages 232 to 234

1. 8.5398	**12.** 3.1414	**23.** 1.6478
2. 0.010894	**13.** 18.636	**24.** 3463.4
3. 33,451	**14.** 0.72132	**25.** 27.278
4. 1019.4	**15.** 0.26868	**26.** −22.582
5. 200,530	**16.** 0.39770	**27.** 15.353
6. 0.19835	**17.** 0.39510	**28.** 0.00021360
7. 24.682	**18.** 1.2390	**29.** 18.666
8. 17.843	**19.** 1.1605	**30.** −22.302
9. 0.65684	**20.** 0.53670	**31.** −1.2552
10. 0.0067010	**21.** 107.42	**32.** −5.2060
11. 437.88	**22.** 3630.8	

33. 0.0074500
34. 1.56026; (−)1.46098; 9.05621 − 10; 2.08309
35. 46.693
36. 8.6458
37. 0.028375
38. 266.46 lb.
39. 2283.2 lb.
40. 6.2691 ft.
41. 151,370 gal.
42. 1.01 sec.
43. 142.5 tons
44. Volume = 13,330, Surface = 2719.
45. 1051×10^7
46. 11,660.
47. 834,200.
48. 1,476,000.
49. 0.608.

§108. Pages 236, 237

1. 2.3666
2. −90.006
3. −1.7354
4. −1.9034
5. 1.5372
6. 4.9168
7. −0.15421
8. −0.76206
9. 6.0110
10. 1.7895
11. 339.86
12. 2.7183
13. 0.42767
14. 0.41639
15. 0.11699
16. −0.37979
17. $x = 3.0484$, $y = 2.0484$
18. 17.677
19. 0, ± 1.3169
20. 3.96
21. 0.00003772
22. 18,360
23. $k = 0.126$
24. 5.5 minutes
25. $x = \dfrac{e^2 - 1}{3}$
26. $x = 25$ and -4

§110. Pages 239, 240

1. 222.91
2. 0.037367
3. 72.888
4. 0.0093936
5. 24.491
6. 1.2142
7. 12.377
8. 4.4787
9. 3.0675
10. 0.00079018
11. 0.37665
12. 0.28926
13. 0.96048
14. 1.7867
15. 34.801
16. 67.535
17. 42.620
18. 2362.9
19. −4.2098
20. −0.86048
21. −0.21423
22. 0.031072
23. 4.6249
24. 3.5064
25. 1.5509
26. 0.036016
27. (*a*) 0.093180; (*b*) 168.20; (*c*) 0.44668
28. 35.239
29. 4.251
30. (*a*) 100,100; more accurate value 100,081; (*b*) 85,450; more accurate value, 85,442
31. 1547 miles
32. 146,700 sq. km.

§115. Page 245

1. 6
2. 7
3. 10
4. 9.1
5. 6.75
6. 9.62
7. 49.8
8. 340
9. 47.0
10. 0.0826
11. 3220
12. 0.836
13. 9.86
14. 3.08

§116. Page 246

1. 15
2. 15.8
3. 3530
4. 42.1
5. 0.001322
6. 1737
7. 9.98
8. 1,340

§117. Page 247

1. 2.32	**4.** 106.1	**6.** 77.5	**8.** 26.3
2. 165.2	**5.** 0.000713	**7.** 1861	**9.** 1.154
3. 0.0767			**10.** 0.0419

§118. Page 248

1. 36.7	**5.** 0.00357	**9.** 0.01311	**13.** 249
2. 8.35	**6.** 13,970	**10.** 2.36	**14.** 0.275
3. 0.0000632	**7.** 1586	**11.** 0.0414	**15.** 0.1604
4. 3400	**8.** 0.0223	**12.** 2460	**16.** 0.0977

§119. Page 250

1. $x = 5.22$
2. $x = 2.30$, $y = 31.8$
3. $x = 51.7$, $y = 3370$
4. $x = 3.97$, $y = 9.84$, $z = 0.272$
5. $x = 0.1013$, $z = 0.0769$
6. $x = 1.586$, $y = 41.4$
7. $x = 106.2$, $y = 30.4$
8. $x = 0.1170$, $y = 0.927$
9. $x = 186$, $y = 13.42$, $z = 50.3$

§120. Page 251

1. 10,570	**3.** 0.0337	**5.** 73,100	**7.** 0.002224	**9.** 1.799
2. 92,200	**4.** 1.765	**6.** 249,000	**8.** 0.314	**10.** 0.1555

§121. Page 253

1. 0.001156	**5.** 96.1	**9.** 9.76	**13.** 0.279
2. 1.512	**6.** 0.1111	**10.** 0.00288	**14.** 41.3
3. 1.015	**7.** 150,800	**11.** 144,700	**15.** 111.1
4. 17.2	**8.** 15.32	**12.** 0.0267	**16.** 3430

§122. Page 254

1. 2.83, 3.46, 4.12, 9.43, 2.98, 29.8, 0.943, 85.3, 0.252, 252, 316
2. (*a*) 231 ft., (*b*) 0.279 ft., (*c*) 5720 ft.
3. (*a*) 18.05 ft., (*b*) 0.992 ft., (*c*) 49.7 ft.

§123. Page 255

1. 64.2	**3.** 1092	**5.** 9.65	**7.** 1.525×10^5
2. 11.41	**4.** 0.428	**6.** 0.0602	**8.** 1.589

§124. Page 257

2.		(*a*) 0.5	(*c*) 0.0581	(*e*) 0.999	(*g*) 0.253	(*i*) 0.204
		(*b*) 0.616	(*d*) 1	(*f*) 0.0276	(*h*) 0.381	(*j*) 0.783
3.		(*a*) 0.866	(*c*) 0.998	(*e*) 0.0393	(*g*) 0.968	(*i*) 0.979
		(*b*) 0.788	(*d*) 0	(*f*) 1.00	(*h*) 0.924	(*j*) 0.623
4.	*A*.	(*a*) 30°	(*c*) 22°2′	(*e*) 51′34″	(*g*) 3°33′	(*i*) 66°56′
		(*b*) 61°6′	(*d*) 5°44′	(*f*) 38°19′	(*h*) 1°46′34″	(*j*) 62°15′
	B.	(*a*) 60°	(*c*) 67°58′	(*e*) 89°8′26″	(*g*) 86°27′	(*i*) 23°4′
		(*b*) 28°54′	(*d*) 84°16′	(*f*) 51°41′	(*h*) 88°13′26″	(*j*) 27°45′
5.		(*a*) 2	(*c*) 17.21	(*e*) 1.001	(*g*) 3.95	(*i*) 4.90
		(*b*) 1.623	(*d*) 1	(*f*) 36.2	(*h*) 2.63	(*j*) 1.277

6.	(*a*) 1.155	(*c*) 1.002	(*e*) 25.5	(*g*) 1.033	(*i*) 1.021
	(*b*) 1.27	(*d*) ∞	(*f*) 1	(*h*) 1.082	(*j*) 1.605
7. *A*.	(*a*) 30°	(*c*) 36°	(*d*) 9°24′	(*e*) 0°43′	(*f*) 12°14′
	(*b*) 24°38′				
B.	(*a*) 60°	(*c*) 54°	(*d*) 80°36′	(*e*) 89°17′	(*f*) 77°46′
	(*b*) 65°22′				

§125. Page 258

1. 0.1423, 0.515, 1.906, 0.01949, 3.55, 19.08, 1.09
7.03, 1.942, 0.525, 51.3, 0.282, 0.0524, 0.917

2.	(*a*) 13°30′	(*d*) 28°22′	(*g*) 23°22′	(*j*) 20°30′	(*m*) 86°38′
	(*b*) 38°8′	(*e*) 3°23′	(*h*) 2°28′	(*k*) 74°57′	(*n*) 45°51′
	(*c*) 42°37′	(*f*) 4°42′	(*i*) 51′13″	(*l*) 77°55′	(*o*) 50°56′
3.	(*a*) 76°30′	(*d*) 61°38′	(*g*) 66°38′	(*j*) 69°30′	(*m*) 3°22′
	(*b*) 51°52′	(*e*) 86°37′	(*h*) 87°32′	(*k*) 15°3′	(*n*) 44°9′
	(*c*) 47°23′	(*f*) 85°18′	(*i*) 89°8′47″	(*l*) 12°5′	(*o*) 39°4′

§126. Page 259

1. 30.5	**7.** 5.29	**13.** 2.033	**19.** 38.1
2. 0.360	**8.** 254	**14.** 0.720	**20.** 0.00319
3. 4.61	**9.** 0.0679	**15.** 4.24	**21.** 0.001091
4. 24.2	**10.** 0.267	**16.** 1.226	**22.** 5.08
5. 14.25	**11.** 1.349	**17.** 0.0771	**23.** 0.01375
6. 16.79	**12.** 16.47	**18.** 0.0961	**24.** 0.0433

§127. Pages 261, 262

1. $C = 75°$	**4.** $A = 2°47'$	**7.** $C = 55°20'$	**10.** Impossible
$b = 35.46$	$B = 87°13'$	$b = 568$	**11.** $B = 30°3'$
$c = 53.3$	$c = 4570$	$c = 664$	$C = 90°$
2. $C = 55°$	**5.** $B = 35°16'$	**8.** $b = 279$	$b = 5.01$
$b = 70.7$	$C = 84°44'$	$c = 284$	**12.** $c = 123.8$
$a = 56.1$	$c = 138$	$C = 100°50'$	$B = 3°18'35''$
3. $C = 123°12'$	**6.** $A = 17°41'$	**9.** $A = 87°41'$	$C = 116°41'25''$
$b = 2257$	$C = 53°19'$	$C = 41°12'$	**13.** 1253 ft.
$c = 2599$	$a = 0.0751$	$a = 116.9$	**14.** 1034.8 yd.

15. $B_1 = 66°10'$	**17.** $A_1 = 70°12'$	**19.** $B_1 = 45°16'$
$C_1 = 58°26'$	$B_1 = 57°24'$	$C_1 = 99°8'$
$c_1 = 18.6$	$b_1 = 28.79$	$c_1 = 300$
$B_2 = 113°50'$	$A_2 = 109°48'$	$B_2 = 134°44'$
$C_2 = 10°46'$	$B_2 = 17°48'$	$C_2 = 9°40'$
$c_2 = 4.08$	$b_2 = 10.45$	$c_2 = 51.1$
16. $B_1 = 16°43'$	**18.** $A_1 = 68°47'$	**20.** $A_1 = 51°19'$
$A_1 = 147°28'$	$C_1 = 67°10'$	$C_1 = 88°41'$
$a_1 = 35.5$	$a_1 = 6.92$	$c_1 = 21{,}850$
$B_2 = 163°17'$	$A_2 = 23°7'$	$A_2 = 128°41'$
$A_2 = 0°54'$	$C_2 = 112°50'$	$C_2 = 11°19'$
$a_2 = 1.04$	$a_2 = 2.91$	$c_2 = 4290$

21. $p = 3.13$; (*a*) none, (*b*) 2, (*c*) 1

§128. Page 263

1. $A = 31°20'$, $B = 58°40'$, $c = 23.7$
2. $A = 41°2'$, $B = 48°58'$, $c = 153.8$
3. $A = 65°$, $B = 25°$, $c = 55.2$
4. $A = 33°9'$, $B = 56°51'$, $c = 499$
5. $A = 39°30'$, $B = 50°30'$, $c = 44$
6. $A = 67°23'$, $B = 22°37'$, $c = 13$
7. $A = 45°$, $B = 45°$, $c = 18.67$
8. $A = 30°37'$, $B = 59°23'$, $c = 82.5$
9. $A = 3°42'$, $B = 86°18'$, $c = 4.8$

§129. Page 264

1. $A = 119°54'$, $B = 31°6'$, $c = 52.6$
2. $A = 49°4'$, $C = 79°7'$, $b = 104.1$
3. $A = 55°2'$, $B = 40°21'$, $c = 285$
4. $B = 39°16'$, $C = 78°44'$, $a = 3.21$
5. $A = 100°57'$, $C = 33°3'$, $b = 19.8$
6. $A = 46°26$, $C = 6°24'$, $b = 7.43$
7. $A = 121°4'$, $C = 2°26'$, $b = 0.0828$
8. $A = 77°12'$, $B = 43°30'$, $c = 15$
9. $B = 13°22'$, $C = 28°17'$, $a = 7420$
10. 10 and 4.68
11. 4.93 miles

§130. Page 265

1. $A = 106°47'$, $B = 46°53'$, $C = 26°20'$
2. $A = 27°21'$, $B = 143°8'$, $C = 9°32'$
3. $A = 52°26'$, $B = 59°23'$, $C = 68°12'$
4. $A = 49°12'$, $B = 37°36'$, $C = 93°12'$
5. $A = 44°42'$, $B = 49°37'$, $C = 85°40'$
6. $A = 83°42'$, $B = 59°22'$, $C = 36°56'$

§131. Page 266

1. (*a*) 0.785 (*b*) 1.047 (*c*) 1.571 (*d*) 3.14 (*e*) 2.09
   (*f*) 2.36 (*g*) 0.393 (*h*) 3.49 (*i*) 52.4
2. (*a*) 60° (*c*) 2.5° (*d*) 210° (*e*) 1200° (*f*) 176.4°
   (*b*) 135°
3. (*a*) 0.01745 (*c*) 0.00000485 (*e*) 3.152
   (*b*) 0.0002909 (*d*) 0.1778 (*f*) 5.24
4. (*a*) $5°44'$ (*b*) $143°15'$ (*c*) $91°40'$ (*d*) $343°46'$

§133. Page 271

3. Each side $= 5\pi$ in.
5. 3000 miles, 3638 miles, $2750\frac{1}{3}$ miles
8. (*a*) $c = 30°$, $a = 90°$, $b = 90°$

§135. Pages 275 to 277

1. (a) $c = \cos^{-1} \dfrac{\sqrt{3}}{4}$
   (b) $B = \sec^{-1} \sqrt{3}$
   (c) $c = \tan^{-1} 2$
   (d) $A = \sec^{-1} 4$
   (e) $b = \tan^{-1} \sqrt{\dfrac{3}{2}}$
   (f) Impossible

3. (a) $A = \tan^{-1} 2$
   (b) Impossible
   (c) $a = \tan^{-1} \frac{3}{2}$
   (d) $c = \pi - \sec^{-1} \sqrt{3}$
   (e) $A = \cos^{-1} \frac{3}{4}$
   (f) $B = \sec^{-1} \sqrt{3}$

8. (a) $\cos c = \cot A \cot B$

§137. Pages 280, 281

1. $b = 2°14'5''$, $c = 10°45'55''$, $A = 78°9'22''$
2. $a = 44°43'49''$, $b = 14°59'33''$, $A = 75°21'53''$
3. $b = 10°49'17''$, $c = 118°20'20''$, $A = 95°55'2''$
4. $A = 52°16'26''$, $B = 57°26'33''$, $b = 47°7'32''$
5. $a = 58°21'28''$, $A = 65°11'30''$, $B = 53°6'40''$
6. $b = 27°37'26''$, $B = 68°42'11''$, $A = 155°48'0''$
7. $a = 127°4'30''$, $b = 50°0'0''$, $A = 120°3'50''$
8. $a = 22°15'43''$, $b = 24°24'19''$, $B = 50°8'21''$
9. $a = 119°59'46''$, $b = 120°10'3''$, $c = 75°26'58''$
10. $a = 50°0'0''$, $b = 56°50'49''$, $B = 63°25'4''$
11. $b = 51°53'$, $A = 27°28'38''$, $B = 73°27'11''$
12. $c = 54°20'$, $A = 46°59'43''$, $B = 57°59'19''$
13. $b = 155°27'54''$, $c = 142°9'13''$, $A = 54°1'16''$
14. $c = 133°32'26''$, $A = 126°40'24''$, $B = 47°13'43''$
15. $c = 54°20'$, $B = 46°59'43''$, $A = 57°59'19''$
16. $a = 50°0'4''$, $b = 143°5'12''$, $c = 120°55'34''$
17. $a = 67°33'27''$, $b = 100°45'$, $c = 94°5'$
18. $a = 51°53'$, $B = 27°28'38''$, $A = 73°27'11''$
19. $b = 96°21'59''$, $c = 86°58'0''$, $A = 118°21'15''$
20. $a = 49°59'58''$, $c = 91°47'40''$, $B = 92°8'23''$
22. $D = 690.98$ miles, $L_2 = 39°31'18''$, $C = 80°19'23''$
24. $B = 53°48'27''$

§138. Page 282

1. $a_1 = 69°50'24''$, $c_1 = 73°45'15''$, $A_1 = 77°54'$
   $a_2 = 110°9'36''$, $c_2 = 106°14'45''$, $A_2 = 102°6'$
2. $a_1 = 18°54'38''$, $c_1 = 127°2'27''$, $A_1 = 23°57'19''$
   $a_2 = 161°5'22''$, $c_2 = 52°57'33''$, $A_2 = 156°2'41''$
3. $a_1 = 25°59'28''$, $c_1 = 33°20'13''$, $A_1 = 52°53'0''$
   $a_2 = 154°0'32''$, $c_2 = 146°39'47''$, $A_2 = 127°7'0''$
4. $b_1 = 28°14'31''$, $c_1 = 78°53'20''$, $B_1 = 28°49'57''$
   $b_2 = 151°45'29''$, $c_2 = 101°6'40''$, $B_2 = 151°10'3''$
5. $b_1 = 39°4'51''$, $c_1 = 136°50'23''$, $B_1 = 67°9'43''$
   $b_2 = 140°55'9''$, $c_2 = 43°9'37''$, $B_2 = 112°50'17''$

6. $a_1 = 60°36'10''$, $c_1 = 68°42'59''$, $A_1 = 69°13'47''$
$a_2 = 119°23'50''$, $c_2 = 111°17'1''$, $A_2 = 110°46'13''$

§139. Page 284

1. (a) $a' = 44°0.9'$, $b' = 79°49.9'$, $c' = 81°16.7'$, $C' = 90°$, $A' = 44°40'$; $B' = 81°28.5'$
2. (a) $\sin A' = \sin C' \sin a'$
3. (b) $a' = 133°9.7'$, $B' = 108°18.3'$, $c' = 73°35.3'$

§140. Page 285

1. $a = 68°36'13''$, $b = 59°19'4''$, $C = 103°26'36''$
2. $a = 67°46'12''$, $b = 78°21'32''$, $B = 77°24'34''$
3. $b = 117°45'28''$, $A = 96°27'1''$, $C = 93°0'51''$
4. $a = 94°22'46''$, $b = 69°48'42''$, $C = 88°23'11''$
5. $a = 106°56'53''$, $B = 8°49'46''$, $C = 28°3'4''$
6. $A = 105°21'16''$, $B = 160°13'48''$, $C = 104°25'45''$

§141. Pages 285 to 288

1. (a) $c = 66°32'6''$, $A = 41°55'45''$, $B = 70°19'15''$
(b) $a = 104°53'1''$, $b = 133°39'48''$, $C = 104°41'37''$
(c) $a = 54°41'35''$, $b = 104°21'28''$, $c = 98°14'24''$
(d) $a_1 = 20°11'16''$, $c_1 = 129°16'38''$, $A_1 = 26°28'31''$
$a_2 = 159°48'44''$, $c_2 = 50°43'22''$, $A_2 = 153°31'29''$
(e) $b = 85°17'16''$, $A = 17°35'57''$, $C = 104°31'13''$
(f) Impossible
2. (a) $a = b = 32°45'6''$, $C = 105°49'32''$
(b) $c = 46°15'12''$, $a = b = 112°32'20''$
3. $60°20'56''$
5. $C_1 = 65°22'31''$, $C_2 = 114°37'29''$, $b_1 = 130°24'35''$, $b_2 = 77°35'39''$
$B_1 = 135°20'37''$, $B_2 = 64°21'40''$
7. 247.95 miles
9. $8°56'31''$, $8°56'44''$
10. $L = 39°55'24''$ N, $\lambda = 60°53'17''$ W, $C = 98°29'7''$
11. $L = 24°8'22''$ N, $D = 3067.7$ miles
12. $L = 53°8'42''$ N, $\lambda = 176°49'56''$ W
13. 1973.9 nautical miles
14. $L = 55°17'42''$ N, $\lambda = 180°$

§142. Pages 290 to 291

3. (a) $A = 71°23'00''$
(b) $B = 53°37'47''$
4. (a) $b = 44°13'45''$
(b) $B = 131°18'$

§144. Pages 294, 295

1. (a) $a = 42°20'12''$
(b) $a = 64°10'34''$
(c) $a = 100°10'58''$
2. (a) $137°40'$
(b) $79°49'$
3. $A = 33°11'19''$
7. (a) $B = 114°35'50''$, $C = 31°39'55''$
(b) $B = 42°52'8''$, $C = 28°45'18''$
(c) $B = 21°3'6''$, $C = 26°6'0''$

8. (*a*) $A' = 137°39'48''$, $b' = 65°24'10''$, $c' = 148°20'5''$
(*b*) $A' = 115°49'26''$, $b' = 137°7'52''$, $c' = 151°14'42''$
(*c*) $A' = 79°49'2''$, $b' = 158°56'54''$, $c' = 153°54'$

§147. Pages 299, 300

2. (*a*) $A = 33°11'20''$, $B = 50°43'44''$, $C = 108°31'52''$
(*b*) $A = 34°46'44''$, $B = 81°6'4''$, $C = 81°6'4''$
(*c*) $A = 145°13'20''$, $B = 98°54'0''$, $C = 81°6'4''$
(*d*) $a = 76°9'49''$, $b = 127°33'10''$, $c = 76°9'49''$
(*e*) $a = 81°6'0''$, $b = 34°46'42''$, $c = 98°53'56''$
(*f*) $a = 146°48'40''$, $b = 71°28'8''$, $c = 129°16'16''$
3. (*a*) $A = 118°44'10''$, $B = 29°38'9''$, $C = 68°7'32''$
(*b*) $A = 123°53'48''$, $B = 57°46'56''$, $C = 46°51'50''$
(*c*) $A = 81°52'32''$, $B = 97°31'5''$, $C = 111°3'42''$
(*d*) $A = 34°59'19''$, $B = 150°13'15''$, $C = 33°11'39''$
(*e*) $a = 56°51'48''$, $b = 126°57'52''$, $c = 139°21'22''$
(*f*) $a = 51°17'31''$, $b = 64°2'47''$, $c = 51°17'31''$
(*g*) $a = 97°44'19''$, $b = 53°49'25''$, $c = 104°25'9''$
(*h*) $a = 115°10'$, $b = 84°18'28''$, $c = 31°9'14''$
4. (*a*) $a' = 146°48'40''$, $b' = 129°16'16''$, $c' = 71°28'8''$

§149. Page 304

1. (*a*) $b = 42°20'12''$, $A = 31°39'54''$, $C = 114°35'50''$
(*b*) $a = 85°26'28''$, $B = 149°53'42''$, $C = 37°54'6''$
(*c*) $A = 39°13'54''$, $B = 63°26'6''$, $c = 156°42'58''$
(*d*) $a = 165°29'53''$, $b = 154°17'43''$, $C = 93°19'34''$
(*f*) $a = 50°11'37''$, $B = 77°29'48''$, $c = 153°40'13''$
2. (*a*) $49°28'$ (*b*) $69°35'$ (*c*) $15°20'$ (*d*) $104°19'$
3. (*a*) $a = 57°56'56''$, $b = 137°20'32''$, $C = 94°48'13''$
(*b*) $b = 100°47'46''$, $A = 96°2'12''$, $C = 125°43'44''$
(*c*) $c = 104°12'55''$, $A = 63°48'26''$, $B = 51°46'38''$
(*d*) $c = 108°39'11''$, $A = 64°48'54''$, $B = 40°23'16''$
(*e*) $c = 156°18'49''$, $A = 29°42'0''$, $B = 41°2'38''$
(*f*) $a = 23°57'11''$, $b = 118°2'13''$, $C = 102°5'46''$
4. (*a*) $c = 9°5'14''$, $A = 56°30'0''$, $B = 115°33'56''$
(*b*) $c = 73°41'2''$, $A = 130°25'0''$, $B = 128°26'27''$

§150. Pages 306, 307

1. $c_1 = 104°19'10''$, $A_1 = 52°19'33''$, $C_1 = 124°42'2''$
$c_2 = 18°10'14''$, $A_2 = 127°40'27''$, $C_2 = 15°20'32''$
2. $b = 15°18'34''$, $c = 38°59'34''$, $C = 98°40'56''$
3. $b_1 = 55°25'2''$, $c_1 = 81°27'26''$, $C_1 = 119°22'28''$
$b_2 = 124°34'58''$, $c_2 = 162°34'27''$, $C_2 = 164°41'55''$
4. $b_1 = 81°15'15''$, $c_1 = 110°10'50''$, $C_1 = 119°43'48''$
$b_2 = 98°44'45''$, $c_2 = 138°45'26''$, $C_2 = 142°24'59''$
5. Impossible
6. $c = 88°57'44''$, $A = 51°44'11''$, $B = 139°29'35''$

§151. Pages 307, 308

1. $A = 126°18'42''$, $B = 119°42'8''$, $C = 111°51'42''$
2. $c = 89°37'43''$, $A = 29°42'0''$, $B = 138°57'22''$
3. $a = 123°34'46''$, $b = 75°56'32''$, $c = 105°0'18''$
4. $b = 88°12'19''$, $C = 78°15'46''$, $a = 152°43'49''$
5. $a = 114°26'50''$, $c = 82°33'31''$, $C = 79°10'30''$
6. $c = 153°38'40''$, $A = 29°42'34''$, $B = 42°37'18''$
7. $a_1 = 42°37'18''$, $c_1 = 129°41'5''$, $C_1 = 89°54'19''$
   $a_2 = 137°22'42''$, $c_2 = 19°58'36''$, $C_2 = 26°21'18''$
8. $A = 59°29'42''$, $B = 62°49'42''$, $C = 65°50'48''$
9. $a = 110°30'23''$, $b = 36°47'37''$, $C = 135°12'15''$
10. $a = 51°17'31''$, $b = 64°2'47''$, $c = 51°17'31''$

§154. Page 312

1. $c = 135°49'19''$, $b = 146°37'15''$, $A = 105°8'17''$
2. $a = 40°1'5''$, $b = 38°31'5''$, $C = 130°3'48''$
3. $c = 120°10'52''$, $A = 65°13'4''$, $B = 49°27'53''$
4. $a = 69°34'44''$, $B = 135°5'14''$, $C = 50°29'54''$
5. $c = 104°12'52''$, $B = 51°46'38''$, $A = 63°48'24''$
6. $b = 100°47'46''$, $A = 96°2'12''$, $C = 125°43'46''$
7. $c = 108°39'11''$, $B = 40°23'17''$, $A = 64°48'55''$
8. $a = 65°28'34''$, $B = 148°14'43''$, $C = 44°9'3''$
9. $a = 145°24'53''$, $b = 139°45'58''$, $C = 49°46'16''$
10. $a = 23°57'9''$, $c = 118°2'15''$, $B = 102°5'52''$

§155. Pages 314, 315

1. $c = 120°10'52''$, $A = 65°13'4''$, $B = 49°27'53''$
2. $a = 69°34'44''$, $B = 135°5'14''$, $C = 50°29'54''$
3. $c = 104°12'52''$, $B = 51°46'38''$, $A = 63°48'24''$
4. $b = 100°47'46''$, $A = 96°2'12''$, $C = 125°43'46''$
5. $c = 108°39'11''$, $B = 40°23'17''$, $A = 64°48'55''$
6. $a = 65°28'34''$, $B = 148°14'43''$, $C = 44°9'3''$
7. $a = 145°24'53''$, $b = 139°45'58''$, $C = 49°56'16''$
8. $a = 23°57'9''$, $c = 118°2'15''$, $B = 102°5'52''$
10. $c = 135°49'19''$, $b = 146°37'15''$, $A = 105°8'17''$
11. $a = 40°1'5''$, $b = 38°31'5''$, $C = 130°3'48''$

§156. Page 316

1. $a = 112°10'4''$
2. $c = 73°41'0''$
3. $c = 88°57'41''$
4. $c = 37°3'52''$
5. $A = 51°44'7''$, $B = 139°29'36''$

§158. Page 319

1. $B_1 = 42°37'30''$, $C_1 = 160°1'43''$, $c_1 = 153°39'4''$
   $B_2 = 137°22'30''$, $C_2 = 50°19'3''$, $c_2 = 90°5'18''$
2. $B = 131°25'11''$, $C = 108°18'55''$, $c = 78°21'6''$
3. $B_1 = 120°47'28''$, $C_1 = 97°42'38''$, $c_1 = 55°41'57''$
   $B_2 = 59°12'18''$, $C_2 = 29°9'0''$, $c_2 = 23°57'27''$

4. $C_1 = 59°24'20''$, $B_1 = 115°40'1''$, $b_1 = 97°33'11''$
$C_2 = 120°35'40''$, $B_2 = 26°59'51''$, $b_2 = 29°57'19''$
5(*a*). $b = 76°47'13''$, $a = 96°46'12''$, $A = 99°24'13''$
5(*b*). $b_1 = 109°49'57''$, $c_1 = 98°21'33''$, $C_1 = 109°55'11''$
$b_2 = 70°10'3''$, $c_2 = 168°48'53''$, $C_2 = 169°22'45''$
6(*a*). $c_1 = 120°56'49''$, $b_1 = 48°18'43''$, $B_1 = 58°55'29''$
$c_2 = 59°3'11''$, $b_2 = 120°8'55''$, $B_2 = 97°21'31''$
6(*b*). $b_1 = 59°0'17''$, $c_1 = 118°21'34''$, $C_1 = 95°12'4''$
$b_2 = 120°59'43''$, $c_2 = 43°52'14''$, $C_2 = 51°39'22''$

§159. Page 320

1. $A = 68°33'42''$, $B = 130°48'18''$, $C = 94°0'48''$
3. Impossible.
4. $a = 165°2'6''$, $b = 163°49'24''$, $c = 11°25'6''$
5. $A = 65°49'48''$, $B = 56°32'48''$, $C = 116°56'48''$
6. No solution. Examine the polar triangle.

§160. Pages 320, 321

1. $A = 63°48'35''$, $B = 51°46'12''$, $c = 104°13'27''$
2. $B = 95°38'4''$, $C = 97°26'29''$, $a = 64°23'15''$
3. $a = 40°1'5''$, $b = 38°31'3''$, $C = 130°3'50''$
4. $B_1 = 42°37'17''$, $C_1 = 160°1'24''$, $c_1 = 153°38'42''$
$B_2 = 137°22'42''$, $C_2 = 50°18'55''$, $c_2 = 90°5'41''$
5. $B = 65°33'10''$, $C = 97°26'29''$, $c = 100°49'30''$
6. $b = 41°52'35''$, $c = 41°35'4''$, $C = 60°42'46''$
7. $A = 21°1'2''$, $B = 8°38'46''$, $C = 155°31'36''$
8. $a = 87°20'28''$, $b = 76°44'2''$, $c = 93°55'31''$
9. $44°23'16''$ N
10. $L = 22°44'22''$ S, $\gamma = 166°3'$ E
11. $L = 42°54'52''$ N, $\gamma = 99°3'30''$ E
12. $L = 41°3'50''$ N, $\gamma = 168°19'20''$ W
13. $C = 224°8'45''$, $D = 5832$ mile
14. $A = 110°51'5''$, $B = 48°56'16''$, $C = 38°26'56''$

§163. Pages 326 to 328

5. $C_n = 311°3'38''$, $D = 6386.7$ miles
6. $C_n = 217°1'18''$
7. $D = 6779.9$ miles
8. $C_n = 241°29'52''$
9. $C_n = 86°18'15''$, $D = 5213.7$ miles
$L_v = 34°32'27''$ N, $\lambda_v = 168°1'41''$ W
10. $C_n = 224°8'48''$, $D = 5832$ miles
11. $L = 44°55'16''$
12. (*a*) $43°9'$ W (*d*) $20°31'28''$ N
(*b*) $35°53'$ N (*e*) $C_n = 31°56'17''$ or $211°56'17''$, 6988.9 miles
(*c*) $32°34'36''$ W (*f*) 2870.4 miles
13. $C_n = 297°42'24''$, $C_n = 225°44'48''$, $D = 5992.0$ miles

§166. Pages 332, 333

3. $Z_n = 208°12'00''$, $h = 59°10'22''$
4. $Z_n = 203°46'46''$, $h = 21°42'43''$
5. $Z_n = 44°40'43''$, $h = 51°39'30''$
6. $Z_n = 73°11'42''$, $h = 64°13'50''$
7. $Z_n = 312°14'54''$, $h = 31°13'24''$
8. $Z_n = 145°3'31''$, $h = 35°33'10''$
9. $Z_n = 125°18'40''$, $h = 45°53'20''$
10. $Z_n = 85°59'36''$, $h = 36°40'18''$
11. $h = 22°42'25''$
12. $h = 64°13'52''$
13. $h = 31°13'25''$
14. $h = 55°36'22''$
15. $h = 51°39'30''$
16. $h = 59°10'15''$
18. $h = 2°11'50''$

§167. Page 335

1. A = E 29°28′6″ S
2. $4^h\ 37^m\ 48^s$ A.M.
3. Summer: sunrise at $4^h\ 37^m\ 48^s$ A.M., sunset at $7^h\ 22^m\ 12^s$ P.M.
   Winter: sunrise at $7^h\ 22^m\ 12^s$ A.M., sunset at $4^h\ 37^m\ 48^s$ P.M.
4. (*a*) March 21: sunrise at $6^h\ 0^m\ 0^s$ A.M., sunset at $6^h\ 0^m\ 0^s$ P.M.
   December 21: sunrise at $10^h\ 19^m\ 7^s$ A.M., sunset at $1^h\ 40^m\ 53^s$ P.M.
   June 21: sunrise at $1^h\ 40^m\ 53^s$ A.M., sunset at $10^h\ 19^m\ 7^s$ P.M.
   (*b*) March 21: $A = 0°0'0''$ at sunrise; $A = 0°0'0''$ at sunset
   December 21: A = E 66°59′30″ S at sunrise; A = W 66°59′30″ S at sunset
   June 21: A = E 66°59′30″ N at sunrise; A = W 66°59′30″ N at sunset
   (*c*) Length of longest day: $20^h\ 38^m\ 14^s$
   Length of shortest day: $3^h\ 21^m\ 46^s$
6. (*a*) 10°N
   (*b*) 10°S
   (*c*) $h = 13°27$, $h = 33°27'$
   (*d*) 10°S
   (*e*) 30.25 ft.

§168. Page 337

2. (*a*) $t = 7^h\ 8^m\ 2^s$ A.M., $Z_n = 79°26'13''$
   (*b*) $t = 7^h\ 10^m\ 41^s$ A.M., $Z_n = 84°58'52''$
   (*c*) $t = 6^h\ 50^m\ 25^s$ A.M., $Z_n = 81°31'5''$
3. $t = 8^h\ 23^m\ 50^s$ A.M., $Z_n = 100°44'48''$
4. $t = 9^h\ 10^m\ 46^s$ A.M., $Z_n = 125°46'0''$
5. $t = 4^h\ 37^m\ 46^s$ P.M., $Z_n = 272°43'40''$
6. $t = 3^h\ 5^m\ 18^s$ P.M., $Z_n = 261°6'0''$

§169. Pages 339, 340

1. 60° E
2. $15^h\ 42^m\ 30^s$
3. (*a*) $16^h\ 22^m$; (*b*) $3^h\ 38^m$
4. $9^h\ 48^m\ 40^s$
5. $\lambda_2 = ST_1 - ST_2 + \lambda_1$
6. $18^h\ 19^m\ 40^s$
7. $23^h\ 45^m\ 22^s$

§170. Page 341

1. $\lambda = 176°23'15''$ W
2. $\lambda = 12°9'15''$ E
3. $\lambda = 124°23'45''$ W
4. $\lambda = 60°29'0''$ W
5. $\lambda = 111°7'30''$ W
6. $\lambda = 116°0'15''$ W

§171. Page 343

1. $L = 0°$
2. $L = 30°$ N
3. $L = 50°$ N
4. $L = 4°6'$ N
5. $L = 72°40'$ S
6. $L = 46°58'$ N
7. $L = 33°50'$ N
8. $L = 12°24'$ S
9. $L = 8°41'$ S
10. $L = 0°$
11. $L = 7°11'$ N
12. $L = 37°33'$ N
13. $L = 74°22'$ N
14. $L = 37°24'$ S
15. $L = 45°32'$ N
16. Impossible

§172. Page 344

1. (a) $L_1 = 13°26'28''$ S
$L_2 = 61°21'31''$ N
(b) $L_1 = 58°21'19''$ S
$L_2 = 42°22'21''$ N
2. (a) $L_1 = 25°41'32''$ N
$Z_1 = 255°0'0''$
$L_2 = 8°41'32''$ N
$Z_2 = 285°0'0''$
(b) $L_1 = 13°07'20''$ S
$L_2 = 72°55'50''$ N
$Z_1 = 321°33'20''$
$Z_2 = 218°26'40''$
(c) $L_1 = 10°15'58''$ N
$L_2 = 24°58'58''$ N
$Z_1 = 77°29'28''$
$Z_2 = 102°30'32''$
(d) $L = 44°22'51''$ N
$Z = 170°4'0''$

§173. Pages 344 to 349

2. $Z_n = 237°53'17''$
3. $h = 13°48'1''$, $Z_n = 125°26'9''$
4. $L_1 = 26°53'48''$ N, $L_2 = 71°19'0''$ N, $Z_1 =$ N $45°0'0''$ W, $Z_2 =$ N $135°0'0''$ W
5. $L_1 = 25°42'1''$ S, $L_2 = 8°41'1''$ S, $Z_1 =$ S $105°0'0''$ E, $Z_2 =$ S $75°0'0''$ E
6. (a) $L_1 = 3°14'46''$ S, $L_2 = 43°23'16''$ S, $Z_1 =$ S $25°15'29''$ E, $Z_2 =$ S $154°44'31''$ E
(b) $L_1 = 11°29'32''$ S, $L_2 = 62°39'40''$ N, $Z_1 =$ N $41°1'54''$ E, $Z_2 =$ N $138°58'5''$ E
7. (a) $t = 4^h\ 27^m\ 46^s$ P.M., $Z_n = 272°43'40''$
(b) $t = 10^h\ 7^m\ 44^s$ A.M., $Z_n = 34°56'36''$
8. Comes within 7.6 nautical miles of the Chicago position
9. $D = 3355.2$ miles, $C_n = 86°48'48''$
10. $D = 6748.6$ miles, $C_n = 82°4'28''$, $L_r = 28°29'44''$ S, $\lambda_r = 136°13'45''$ E
11. $D = 4461.7$ miles, $C_n = 302°13'45''$
12. $D = 6430.6$ miles, $C_n = 300°40'2''$
13. $L = 43°25'37''$ N, 1329.5 miles north of Honolulu
14. $169°7'4''$ W
15. $L = 66°10'2''$ N, $\lambda = 167°34'16''$ E
16. (a) $L = 57°21'21''$ N, $\lambda = 17°33'33''$ W
(b) $L = 44°37'18''$ N, $\lambda = 68°20'35''$ W
17. $152°23'$
18. $99°57'30''$
19. $d = 32°40'36''$ S
20. $3^h\ 26^m\ 0^s$ E
21. $55°45'$ N

22. (*a*) $4^h\ 50^m\ 59^s$ A.M., $7^h\ 9^m 1^s$ P.M

(*b*) $5^h\ 47^m\ 56^s$ A.M., $6^h\ 12^m\ 4^s$ P.M.

(*c*) $5^h\ 50^m$ A.M., $6^h\ 10^m$ P.M.

(*d*) $6^h\ 12^m$ A.M., $5^h\ 48^m$ P.M.

23. (*a*) $18^h\ 28^m\ 24^s$; (*b*) $5^h\ 31^m\ 36^s$

24. $t = 4^h\ 29^m\ 19^s$ E, A = E 33°35′3″ N

25. (*a*) $2^h\ 4^m\ 28^s$, $5^h\ 6^m\ 40^s$, $14^h\ 44^m\ 25^s$, $2^h\ 4^m\ 28^s$

(*b*) $1^h\ 41^m\ 5^s$, $11^h\ 22^m\ 15^s$, $9^h\ 15^m\ 35^s$, $1^h\ 41^m\ 5^s$

(*c*) $1^h\ 33^m\ 42^s$, $8^h\ 52^m\ 37^s$, $12^h\ 0^m\ 0^s$, $1^h\ 33^m\ 42^s$

26. (*a*) 46°58′ N (*b*) 41°42′ N (*c*) 19°40′ S (*d*) 72°40′ S (*e*) 4°6′ N (*f*) 9°30′ S

27. For visible lower culmination, L, d, and bearing must all be of the same name, with $L + d > 90°$ and at a lower culmination $h < d$.

28. (*a*) 38°30′ N

(*b*) 75°53′ S

(*c*) 74°22′ N

(*d*) 37°24′ S

29. (*a*) $7^h\ 43^m\ 15^s$

(*b*) 6.91

(*c*) S 57°14′39″ E

30. $3^h\ 59^m\ 23^s$ P.M.

31. $2^h\ 58^m\ 44^s$ P M

32. (*a*) 93°19′15″ E

(*b*) 9°2′27″ E

33. The shadow stretches from foot of pole S 71°22′ W

34. $Z_n = 75°11′$

35. 13.8 ft.

36. 120°

37. $6^h\ 58^m$ A M., $5^h\ 2^m$ P.M.

38. 89.7 miles, 341 36 miles

39. 17°14′40″

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PREFACE

A table of logarithms should be accurate, it should be easy to understand, and it should be as easy to use as possible. The authors, in the tables offered here, have attempted to make improvements along these three lines.

The tables used in trigonometry and its applications have been checked many times and have been carefully read against other tables. If, in spite of this thoroughness in compilation, errors are discovered, the authors would appreciate having them pointed out.

Frequently students fail to understand the process of linear interpolation. It is explained in this book by means of a simple diagram which gives the idea almost at a glance.

The table of logarithms of trigonometric functions (Table II), the most important one for trigonometry, has a number of new features. The proportional parts are tabulated for each second from 0″ to 60″, and bold-faced numbers have been so used as to avoid ambiguity. Whenever there is a choice of two numbers one of which is written in bold face, the bold-faced number is always chosen. The simplicity of operation introduced by this plan gives a gain both in speed and in accuracy. In the table proper all six functions are tabulated, and bold-faced numbers are used in such a way as to enable the user to locate approximate position by using them only. It is believed that the gains due to these innovations are decidedly worth while.

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ANNAPOLIS, MD.,
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CONTENTS

FIVE-PLACE LOGARITHMIC AND TRIGONOMETRIC TABLES

TABLE I

COMMON LOGARITHMS OF NUMBERS

1. Introduction.* *The power L to which a given number b must be raised to produce a number N is called the logarithm of N to the base b.* This relation expressed in symbols is

$$b^L = N.$$

It appears at once that b must not be unity and it must not be negative. In the following set of tables, 10 is used as base. This system is called the *common system* or the *Briggs system.* Another important system, called the *natural system,* has e as base, where $e = 2.71828$ accurate to six figures.

2. Characteristic and mantissa. The common logarithm of any real, positive number may be written as an integer, positive or negative, plus a positive decimal fraction. The integral part is called the *characteristic* and the decimal part the *mantissa.* The characteristic may be written by using the following rules:

Rule 1. *The characteristic of the common logarithm of a number greater than* 1 *is obtained by subtracting* 1 *from the number of digits to the left of the decimal point.*

Rule 2. *The characteristic of the common logarithm of a positive number less than* 1 *is negative and its magnitude is obtained by adding* 1 *to the number of zeros immediately following the decimal point.*

If the characteristic of a number is $-n$ (n positive), it should be written in the form $(10 - n) - 10$. *To obtain directly the logarithm of a number less than* 1, *subtract from* 9 *the number of zeros immediately following the decimal point, and write the result before the mantissa and* -10 *after it.*

The method of finding the mantissa of the logarithm of a number will be explained in the succeeding articles.

* Since the theory of logarithms is treated completely in algebra and in trigonometry, only the actual manipulation of the tables is explained here.

EXERCISES

Verify the characteristic of the logarithm of each of the numbers N written below.

	N	$\log N$		N	$\log N$
1.	6.830	0.83442.	**8.**	58.73	1.76886.
2.	68.30	1.83442.	**9.**	0.6740	9.82866 − 10.
3.	6830	3.83442.	**10.**	0.007500	7.87506 − 10.
4.	683,000	5.83442.	**11.**	6.870×10^5	5.83696.
5.	0.7860	9.89542 − 10.	**12.**	5.860×10^{-4}	6.76790 − 10.
6.	0.007860	7.89542 − 10.	**13.**	3.990×10^{-6}	4.60097 − 10.
7.	0.0007860	6.89542 − 10.	**14.**	7.330×10^2	2.86510.

3. To find the mantissa. Special case. The mantissa, or decimal part of the logarithm of a number, depends only on the sequence of the digits and not on the position of the decimal point. Table I lists the mantissas, accurate to five decimal places, of the logarithms of all integers from 1 to 10,000.

The change in the mantissas of the logarithms is so slow that the first two figures do not change for several lines of the table. Consequently the appropriate first two figures are printed in the first column before the first full row to which they apply. Also the appropriate first two figures appear at the left of the first line of mantissas on each page An asterisk in any row indicates that the first two figures are to be found at the left of the next row.

To find the mantissa of the logarithm of a number locate the first three digits of this number in the left-hand column headed N and the fourth digit in the row at the top of the page. Then the mantissa of the given number containing four significant figures is in the row whose first three figures are the first three significant figures of the given number, and in the column headed by the fourth. Thus to find the logarithm of 76.64 find 766 in the column headed N, follow the corresponding row to the entry in the column headed by 4. This entry 88446 represents the mantissa required. Hence we have

$$\log 76.64 = 1.88446. \quad \textit{Ans.}$$

EXERCISES

Verify the logarithms in the exercise of §2.

4. Interpolation. When a number contains a fifth significant figure, we find the logarithm corresponding to the first four figures as in §3 and then add an increment obtained by a process called interpolation. This process is based on the assumption that *for relatively small changes in the number N the changes in $\log N$ are proportional to the changes in N.* The following example will serve to illustrate the process of interpolation.

The expression *tabular difference* will be used frequently in what follows. The tabular difference, when used in connection with a table,

means the result of subtracting the lesser of two successive entries from the greater.

Example. Find log 235.47.

Solution. We first find the logarithms in the following form and then compute the difference indicated:

$$\left.\begin{array}{l}\left.\begin{array}{l}\log 235.40\\ \log 235.47\end{array}\right\}7\\ \log 235.50\end{array}\right\}10 \quad \left.\begin{array}{l}\left.\begin{array}{l}= 2.37181\\ = \ ?\end{array}\right\}d\\ = 2.37199\end{array}\right\}18 \text{ (tabular difference*)}$$

By the principle of proportional parts, we have

$$\frac{7}{10} = \frac{d}{18}, \qquad \text{or} \qquad d = \frac{7}{10}(18) = 12.6 = 13 \text{ (nearly)}.$$

Adding 0.00013 to 2.37181, we obtain

$$\log 235.47 = 2.37194. \quad \textit{Ans.}$$

The increment 12.6 was rounded off to 13 because we are not justified in writing more than five decimal places in the mantissa.

The essence of this procedure is embodied in the following statement. To find the logarithm of a number composed of five significant figures, first find the logarithm corresponding to the first four figures and to it add one-tenth of the tabular difference multiplied by the fifth digit.

To shorten the process of interpolation, 10^5 times each tabular difference occurring in the table has been multiplied by 0.1, 0.2, . . . 0.9, and the results have been tabulated on the right-hand sides of the pages on which these differences occur. The abbreviation Prop. Parts written at the top of the page over these small tables abbreviates the words *proportional parts.* To interpolate in the example just solved, locate the Prop. Parts table headed 18 and find opposite 7 in its left-hand column the entry 12.6 (=13 nearly). In general, this difference should not be computed but should be obtained from the number opposite the fifth digit in the appropriate table of proportional parts.

EXERCISES

Verify the following logarithms:

1. log 7012.6 = 3.84588
2. log 54.725 = 1.73819.
3. log 0.87364 = 9.94133 − 10.
4. log 3.7245 = 0.57107.
5. log 0.00065931 = 6.81909. — 10
6. log 25.819 = 1.41194.
7. log 2.3454 = 0.37022.
8. log 0.056321 = 8.75067 − 10.
9. log 4,574,000 = 6.66030.
10. log 568.91 = 2.75504.
11. log 4.3965×10^5 = 5.64311.
12. log 10.905 = 1.03763.
13. log 0.0025725 = 7.41036. — 10
14. log 0.000032026 = 5.50550 − 10.

5. To find the number corresponding to a given logarithm. If log $N = L$, the number N is called the *antilogarithm* of L. The sequence of

* For convenience the decimal point has been omitted.

digits of a number N corresponding to a given logarithm L is found from its mantissa, and the decimal point is then placed in accordance with the rules of §2.

Example. Given $\log N = 1.60334$, find N.

Solution. The mantissa .60334 lies between the entries .60325 and .60336 of Table I. Using the table and computing the differences indicated, we write the following form:

$$\left.\begin{array}{l}\left.\begin{array}{l}1.60325\\1.60334\end{array}\right\}9\\1.60336\end{array}\right\}11\quad\begin{array}{l}=\log 40.110\\=\log N\\=\log 40.120\end{array}\quad\left.\begin{array}{l}\left.\begin{array}{l}\\ \end{array}\right\}x\\ \end{array}\right\}10$$

Assuming that changes in the logarithm are proportional to the corresponding changes in the number, we write

$$\frac{9}{11} = \frac{x}{10}, \qquad \text{or} \qquad x = 10\left(\frac{9}{11}\right) = 8 \text{ (nearly)}.$$

Hence

$$N = 40.118. \quad \textit{Ans.}$$

The essence of the process of interpolation is indicated in the foregoing procedure. However, in practice, the student should always interpolate by using the table of proportional parts. The fifth figure 8 should have been obtained from the table of proportional parts. In the small Prop. Parts table corresponding to the tabular difference 11, we read the fifth figure 8 in the left-hand column opposite the entry 8.8, the entry nearest to 9.

EXERCISES

Verify the following antilogarithms:

1. $3.57351 = \log 3745.5$.

2. $2.82315 = \log 665.50$.

3. $0.12112 = \log 1.3217$.

4. $1.92594 = \log 84.321$.

5. $9.47954 - 10 = \log 0.30167$.

6. $8.65636 - 10 = \log 0.045327$.

7. $0.37976 = \log 2.3975$.

8. $4.76224 = \log 57842$.

9. $6.51738 - 10 = \log 0.00032914$.

10. $1.49715 = \log 31.416$.

11. $4.21691 - 10 = \log 16478$.

12. $5.09873 = \log 125520$.

13. $9.27951 - 10 = \log 0.19033$.

14. $7.88000 - 10 = \log 0.0075858$.

TABLE II

LOGARITHMS OF TRIGONOMETRIC FUNCTIONS

6. Table of logarithms of trigonometric functions. Table II gives the logarithms of the sines, cosines, tangents, cotangents, secants, and cosecants of angles at intervals of 1′ from 0° to 90°. The names of the functions written at the top of any page apply to angles having the number of degrees written at the top of the page, and the function names written at the bottom apply to angles having the number of degrees written at the bottom. The left-hand or the right-hand minute column applies according as the number of degrees in the angle is written on the left side or on the right side of the block of numbers under consideration.

For example, to find log sin 32° 46′, we find the page at the top of which 32° appears, find the row containing 46 in the left-hand minute column, and read 73337 in this row and in the column headed l sin. Hence log sin 32° 46′ = 9.73337 − 10. The number 9 was found at the head of the l sin column and the number −10 is to be applied to every logarithm in the table. Again, to find log tan 142° 36′, find the page at the top of which 142° appears, find the row containing 36 in the right-hand minute column, and read 88341 in this row and in the column headed l tan. Hence log tan 142° 36′ = (−) 9.88341 − 10. The minus sign in parentheses before the log indicates that a negative number is under consideration. The characteristic was obtained as in the first example.

EXERCISES

Verify the following:

1. log sin 37° 27′ = 9.78395 − 10.
2. log tan 36° 41′ = 9.87211 − 10.
3. log cot 28° 16′ = 0.26946.
4. log cos 62° 20′ = 9.66682 − 10.
5. log csc 69° 54′ = 0.02729.
6. log sin 131° 10′ = 9.87668 − 10.
7. log tan 142° 27′ = (−) 9.88577 − 10.
8. log sec 134° 47′ = (−) 0.15216.
9. log cos 45° 47′ = 9.84347 − 10.
10. log csc 135° 13′ = (−) 0.15216.
11. log cot 132° 0′ = (−) 9.95444 − 10.

7. Given the angle, to find the logarithm of a trigonometric function. The principles involved here are the same as those involved in finding

logarithms and antilogarithms of numbers. Interpolation for seconds is accomplished by direct interpolation or by using the columns headed *d* 1′ and the columns headed proportional parts. The following example will illustrate the procedure.

Example. Find log tan 65° 42′ 17″.

Solution. Using the table to find logarithms and computing differences, we write the following form:

$$\left.\begin{array}{l}\left.\begin{array}{l}\log\tan 65^\circ\ 42'\ 00''\\ \log\tan 65^\circ\ 42'\ 17''\end{array}\right\}17''\\ \log\tan 65^\circ\ 43'\ 00''\end{array}\right\}60'' \qquad \left.\begin{array}{l}\left.\begin{array}{l}= 0.34533\\ \quad ?\end{array}\right\}x\\ = 0.34566\end{array}\right\}33$$

Hence assuming that, for small changes, change of logarithm is proportional to change of angle, we have

$$\frac{x}{33} = \frac{17}{60}, \quad \text{or} \quad x = 33\left(\frac{17}{60}\right) = 9.35 = 9 \text{ (nearly)}.$$

Therefore

$$\log\tan 65^\circ\ 42'\ 17'' = 0.34533 + 0.00009 = 0.34542. \quad \textit{Ans.}$$

The essence of the process of interpolation is indicated in the foregoing procedure. However, in practice, the student should always interpolate by using the columns headed *d* 1′ and the proportional parts column.

Each entry in the column headed *d* 1′ gives the difference of the logarithms between which it is spaced in each of the adjacent columns. In each column headed by *proportional parts* appears $\frac{1}{60}$, $\frac{2}{60}$, $\frac{3}{60}$ . . . of the number heading the column. Hence the difference 9 to be applied in the case of the foregoing example is found in the proportional parts column headed by 33 (the tabular difference for 1′ written between 0.34533 and 0.34566) and in the row with the 17 of the seconds column. Again, to find log cot 10° 28′ 36″, we find the entry 73345 for log cot 10° 28′, note the appropriate number 71 in the adjacent column headed *d* 1′, enter the proportional parts column headed by 71, read in this column 43 opposite the 36 of the seconds column; subtract 43 from 73345, and write log cot 10° 28′ 36″ = 0.73302.

It is worthy of note that *the changes of logarithms due to the seconds of an angle* must be added or subtracted according as the value of the function for angles near the one under consideration is increasing or decreasing with increasing angle.

EXERCISES

Verify the following:

1. log sin 35° 17′ 8″ = 9.76166 − 10.

2. log cos 48° 24′ 21″ = 9.82207 − 10.

3. log sec 142° 37′ 15″ = (−) 0.09984.

4. log csc 56° 21′ 57″ = 0.07956.
5. log cot 23° 16′ 50″ = 0.36626.
6. log csc 128° 47′ 52″ = 0.10826.
7. log tan − 69° 38′ 54″ = (−) 0.43070.
8. log sin 197° 36′ 57″ = 9.48092 − 10.
9. log sin 137° 45′ 22″ = 9.82756 − 10.
10. log cos 137° 45′ 22″ = (−) 9.86940 − 10.
11. log sin 209° 32′ 50″ = 9.69297 − 10.
12. log cos 330° 27′ 10″ = 9.93949 − 10.

8. Given the logarithm of a trigonometric function, to find the angle. The following example will indicate the procedure necessary to find the angle when the logarithm of a trigonometric function of the angle is given:

Example. Find θ if log cos θ is 9.85391 − 10.

Solution. Using the table to find logarithms and computing differences, we write the following form:

$$\left.\begin{array}{l}\left.\begin{array}{l}\log\cos 44^\circ\,24'\,00''\\ \log\cos 44^\circ\,24'\;\;?''\end{array}\right\}x\\ \log\cos 44^\circ\,25'\,00''\end{array}\right\}60'' \quad \left.\begin{array}{l}\left.\begin{array}{l}= 9.85399\\ = 9.85391\end{array}\right\}8\\ = 9.85386\end{array}\right\}13$$

Hence

$$\frac{x}{60} = \frac{8}{13}, \qquad \text{or} \qquad x = \frac{8}{13}(60) = 37''\ \text{(nearly)},$$

and

$$\theta = 44^\circ\ 24'\ 37''. \quad \textit{Ans.}$$

The essence of the process of interpolation is indicated in the foregoing procedure. In practice, however, the columns headed d 1′ and the proportional parts columns should be used in interpolation. Thus, to find θ in the example just considered, we first find 44° 24′ and difference 8 as above, then read 13 in the column headed d 1′ adjacent to and slightly below the entry 85399, enter the corresponding proportional parts column, opposite the bold-faced one of the five 8's tabulated read 37″ in the seconds column, and then write θ = 44° 24′ 37″.

When finding the number of seconds in an angle corresponding to a given logarithm of a trigonometric function, the student may find several identical entries in the proportional parts column involved. In this case, and *in any case where there is a choice between two or more entries one of which is printed in* **bold face,** *always give preference to the* **bold-faced** *entry.*

EXERCISES

Find the value of θ less than 360° in the following:

1. log sin θ = 9.96162 − 10. *Ans.* 66° 16′ 0″ and 113° 44′ 0″.
2. log cos θ = 9.99537 − 10. *Ans.* 8° 21′ 0″ and 351° 39′ 0″.
3. log cot θ = 0.52368. *Ans.* 16° 40′ 13″ and 196° 40′ 13″.

4. $\log \tan \theta = 9.50368 - 10.$ *Ans.* $17° 41' 18''$ and $197° 41' 18''$.
5. $\log \cos \theta = 9.96301 - 10.$ *Ans.* $23° 18' 48''$ and $336° 41' 12''$.
6. $\log \sin \theta = 9.84963 - 10.$ *Ans.* $45° 1' 9''$ and $134° 58' 51''$.
7. $\log \cot \theta = 9.50064 - 10.$ *Ans.* $72° 25' 38''$ and $252° 25' 38''$.
8. $\log \tan \theta = 0.96236.$ *Ans.* $83° 46' 34''$ and $263° 46' 34''$.
9. $\log \sec \theta = 0.12358.$ *Ans.* $41° 12' 22''$ and $318° 47' 38''$.
10. $\log \csc \theta = 0.71238.$ *Ans.* $11° 10' 53''$ and $168° 49' 7''$.

9. Angles near 0° and 90°. When angles are near 0° or near 90°, interpolation based on the assumption of proportional change in angle and logarithm may give results considerably in error. For this reason it is convenient to introduce the functions S and T defined by the equations $S = \alpha/\sin \alpha$ and $T = \alpha/\tan \alpha$. The relative change of the functions S and T with respect to α is very small when α is less than 3° and, as a consequence, the required accuracy of the results is obtained by using them. On the first three pages of Table II the columns headed log S* and log T give the common logarithms of S and T, respectively.

The following formulas apply when the angle involved is less than 3°:

1. For angles less in magnitude than 3°.

(*a*) $\log \sin \alpha = \log \alpha''$† $- \log S$.
(*b*) $\log \tan \alpha = \log \alpha'' - \log T$.
(*c*) $\log \cot \alpha = \operatorname{colog} \alpha'' + \log T$,
$\qquad = \operatorname{colog} \tan \alpha$.
(*d*) $\log \csc \alpha = \operatorname{colog} \alpha'' + \log S$.
(*e*) $\log \alpha'' = \log \sin \alpha + \log S$.
(*f*) $\log \alpha'' = \log \tan \alpha + \log T$.
(*g*) $\log \alpha'' = \operatorname{colog} \cot \alpha + \log T$.
(*h*) $\log \alpha'' = \operatorname{colog} \csc \alpha + \log S$.

2. For angles α such that $90° - \alpha$‡ is less in magnitude than 3°.

(*i*) $\log \cos \alpha = \log (90° - \alpha)'' - \log S$.
(*j*) $\log \cot \alpha = \log (90° - \alpha)'' - \log T$.
(*k*) $\log \tan \alpha = \operatorname{colog} (90° - \alpha)'' + \log T$,
$\qquad = \operatorname{colog} \cot \alpha$.
(*l*) $\log \sec \alpha = \operatorname{colog} (90° - \alpha)'' + \log S$.
(*m*) $\log (90° - \alpha)'' = \log \cos \alpha + \log S$.
(*n*) $\log (90° - \alpha)'' = \log \cot \alpha + \log T$.
(*o*) $\log (90° - \alpha)'' = \operatorname{colog} \tan \alpha + \log T$.
(*p*) $\log (90° - \alpha)'' = \operatorname{colog} \sec \alpha + \log S$.

To find θ when $\log \sin \theta = 8.46932 - 10$, we first find in the column headed l sin the entry nearest to 8.46932, namely, 8.46799. On one side of 8.46799 we read $\log S = 5.31449$, and on the other $1° 41' = 6060''$. Hence, using formula (*e*), we write $\log \alpha = 8.46932 - 10 + 5.31449 =$

* The function log S is often written cpl S, and the function log T, is written cpl T.

† The symbol $\log \alpha''$ means in this connection the logarithm of the number of seconds in the angle.

‡ Since $\cos \alpha = \sin (90° - \alpha)$, in this case $S = \dfrac{(90° - \alpha)''}{\sin (90° - \alpha)}$.

3.78381. Therefore $\alpha = 6078.7''$. Since $1° \ 41' = 6060''$, $6078.7'' = 1° \ 41' \ 19''$.

EXERCISES

Verify the following:

1. $\log \sin 0° \ 44' \ 13'' = 8.10930 - 10$.

2. $\log \cos 89° \ 21' \ 31'' = 8.04899 - 10$.

3. $\log \tan 0° \ 32' \ 23'' = 7.97406 - 10$.

4. $\log \cot 0° \ 25' \ 56'' = 2.12241$.

5. $\log \tan 1° \ 10' \ 9'' = 8.30981 - 10$.

6. $\log \cot 89° \ 3' \ 11'' = 8.21824 - 10$.

7. $\log \cos 88° \ 41' \ 20'' = 8.35948 - 10$.

8. $\log \sin 0° \ 59' \ 8'' = 8.23554 - 10$.

9. $\log \tan 1° \ 29' \ 10'' = 8.41403 - 10$.

10. $\log \sec 88° \ 16' \ 10'' = 1.52000$.

Verify the following:

11. $\log \cos \theta = 8.32967 - 10$; $\theta = 88° \ 46' \ 33''$ and $271° \ 13' \ 27''$.

12. $\log \tan \theta = 8.11584 - 10$; $\theta = 0° \ 44' \ 53''$ and $180° \ 44' \ 53''$.

13. $\log \sin \theta = 8.23468 - 10$; $\theta = 0° \ 59' \ 1''$ and $179° \ 0' \ 59''$.

TABLE III

NATURAL TRIGONOMETRIC FUNCTIONS

10. Table of natural values of trigonometric functions. Table III contains the numerical values of the sines, cosines, tangents, and cotangents of angles from 0° to 90° at intervals of 1′. In the case of an angle in the range from 0° to 45°, the number of degrees in the angle and the names of the functions are found at the top of the page and the left-hand minute column applies; in the case of angles in the range from 45° to 90°, the number of degrees in the angle and the names of the functions are found at the bottom of the page and the right-hand minute column applies. Interpolation must be carried out without the aid of difference columns or tables of proportional parts.

The following examples illustrate the method of using the tables.

Example 1. Find sin 68° 28′.

Solution. We first find the page at the bottom of which 68° appears and then find the row of the 68° block containing 28′ in the right-hand minute column. In this row and in the column having sin at its foot we find 020 to which we must prefix 0.93 to obtain sin 68° 28′ = 0.93020.

Example 2. Find sin 38° 38′ 27″.

Solution. Using the tables and computing differences, we find the values exhibited in the following form:

$$\left.\begin{array}{l}\left.\begin{array}{l}\sin 38^\circ\ 38'\ 00''\\ \sin 38^\circ\ 38'\ 27''\end{array}\right\}27''\\ \sin 38^\circ\ 39'\ 00''\end{array}\right\}60''\quad \left.\begin{array}{l}\left.\begin{array}{l}= 0.62433\\ = \ ?\end{array}\right\}x\\ = 0.62456\end{array}\right\}23$$

Hence

$$\frac{x}{23} = \frac{27}{60}, \qquad \text{or} \qquad x = \left(\frac{27}{60}\right)23 = 10 \text{ (nearly).}$$

Therefore

$$\sin 38^\circ\ 38'\ 27'' = 0.62433 + 0.00010 = 0.62443. \quad \textit{Ans.}$$

Example 3. If cot θ = 0.37806, find θ.

Solution. Using the tables and computing differences, we find the values exhibited in the following form:

$$\left.\begin{array}{l}\left.\begin{array}{l}\cot 69^\circ\ 17'\ 00''\\ \cot \qquad ?\end{array}\right\}x\\ \cot 69^\circ\ 18'\ 00''\end{array}\right\}60 \quad \left.\begin{array}{l}\left.\begin{array}{l}= 0.37820\\ = 0.37806\end{array}\right\}14\\ = 0.37787\end{array}\right\}33$$

Hence

$$\frac{x}{60} = \frac{14}{33}, \quad \text{or} \quad x = \frac{14}{33}(60) = 25'' \text{ (nearly), and } \theta = 69^\circ\ 17'\ 25''. \quad \textit{Ans.}$$

Since cot θ is positive in the third quadrant, we may also write an answer $180^\circ + 69^\circ\ 17'\ 25'' = 249^\circ\ 17'\ 25''$. *Ans.*

EXERCISES

Verify the following:

1. $\sin 53^\circ\ 42'\ 0'' = 0.80593$

2. $\cos 31^\circ\ 53'\ 9'' = 0.84911$.

3. $\tan 156^\circ\ 42'\ 13'' = -0.43059$.

4. $\cot 27^\circ\ 51'\ 17'' = 1.8923$

5. $\cos 83^\circ\ 17'\ 38'' = 0.11678$.

6. $\sin 87^\circ\ 37'\ 25'' = 0.99914$

7. $\cot 13^\circ\ 14'\ 52'' = 4.2475$.

8. $\tan 83^\circ\ 40'\ 30'' = 9.0218$.

Find the values of θ less than 360° in the following:

9. $\sin \theta = 0.89742$ *Ans.* $63^\circ\ 49'\ 12''$ and $116^\circ\ 10'\ 48''$

10. $\cos \theta = 0.43750$. *Ans.* $64^\circ\ 3'\ 20''$ and $295^\circ\ 56'\ 40''$.

11. $\tan \theta = -0.92834$ *Ans.* $137^\circ\ 7'\ 41''$ and $317^\circ\ 7'\ 41''$.

12. $\cot \theta = 1.8923$. *Ans.* $27^\circ\ 51'\ 17''$ and $207^\circ\ 51'\ 17''$

13. $\cos \theta = 0.95140$. *Ans.* $17^\circ\ 56'\ 14''$ and $342^\circ\ 3'\ 46''$

14. $\sin \theta = 0.13552$. *Ans.* $7^\circ\ 47'\ 19''$ and $172^\circ\ 12'\ 41''$.

TABLE I

FIVE-PLACE TABLE OF COMMON LOGARITHMS OF NUMBERS

From 1 to 10,000

TABLE I

FIVE-PLACE TABLE OF COMMON LOGARITHMS OF NUMBERS

From 1 to 10,000

N.	Log.	N.	Log.	N.	Log.	N.	Log.	N.	Log.
0	—	**20**	1.30 103	**40**	1.60 206	**60**	1.77 815	**80**	1.90 309
1	0.00 000	21	1.32 222	41	1.61 278	61	1.78 533	81	1.90 849
2	0.30 103	22	1.34 242	42	1.62 325	62	1.79 239	82	1.91 381
3	0.47 712	23	1.36 173	43	1.63 347	63	1.79 934	83	1.91 908
4	0.60 206	24	1.38 021	44	1.64 345	64	1.80 618	84	1.92 428
5	0.69 897	25	1.39 794	45	1.65 321	65	1.81 291	85	1.92 942
6	0.77 815	26	1.41 497	46	1.66 276	66	1.81 954	86	1.93 450
7	0.84 510	27	1.43 136	47	1.67 210	67	1.82 607	87	1.93 952
8	0.90 309	28	1.44 716	48	1.68 124	68	1.83 251	88	1.94 448
9	0.95 424	29	1.46 240	49	1.69 020	69	1.83 885	89	1.94 939
10	1.00 000	**30**	1.47 712	**50**	1.69 897	**70**	1.84 510	**90**	1.95 424
11	1.04 139	31	1.49 136	51	1.70 757	71	1.85 126	91	1.95 904
12	1.07 918	32	1.50 515	52	1.71 600	72	1.85 733	92	1.96 379
13	1.11 394	33	1.51 851	53	1.72 428	73	1.86 332	93	1.96 848
14	1.14 613	34	1.53 148	54	1.73 239	74	1.86 923	94	1.97 313
15	1.17 609	35	1.54 407	55	1.74 036	75	1.87 506	95	1.97 772
16	1.20 412	36	1.55 630	56	1.74 819	76	1.88 081	96	1.98 227
17	1.23 045	37	1.56 820	57	1.75 587	77	1.88 649	97	1.98 677
18	1.25 527	38	1.57 978	58	1.76 343	78	1.89 209	98	1.99 123
19	1.27 875	39	1.59 106	59	1.77 085	79	1.89 763	99	1.99 564
20	1.30 103	**40**	1.60 206	**60**	1.77 815	**80**	1.90 309	**100**	2.00 000

N.	L. 0	1	2	3	4	5	6	7	8	9
0		00 000	30 103	47 712	60 206	69 897	77 815̇	84 510	90 309	95 424
1	00 000	04 139	07 918	11 394	14 613	17 609	20 412	23 045̄	25 527	27 875̇
2	30 103	32 222	34 242	36 173	38 021	39 794	41 497	43 136	44 716	46 240
3	47 712	49 136	50 515̄	51 851	53 148	54 407	55 630	56 820	57 978	59 106
4	60 206	61 278	62 325̄	63 347	64 345̇	65 321	66 276	67 210	68 124	69 020
5	69 897	70 757	71 600	72 428	73 239	74 036	74 819	75 587	76 343	77 085̄
6	77 815̇	78 533	79 239	79 934	80 618	81 291	81 954	82 607	83 251	83 885̄
7	84 510	85 126	85 733	86 332	86 923	87 506	88 081	88 649	89 209	89 763
8	90 309	90 849	91 381	91 908	92 428	92 942	93 450	93 952	94 448	94 939
9	95 424	95 904	96 379	96 848	97 313	97 772	98 227	98 677	99 123	99 564
10	00 000	00 432	00 860	01 284	01 703	02 119	02 531	02 938	03 342	03 743
11	04 139	04 532	04 922	05 308	05 690	06 070	06 446	06 819	07 188	07 555̄
12	07 918	08 279	08 636	08 991	09 342	09 691	10 037	10 380	10 721	11 059
13	11 394	11 727	12 057	12 385̇	12 710	13 033	13 354	13 672	13 988	14 301
14	14 613	14 922	15 229	15 534	15 836	16 137	16 435̇	16 732	17 026	17 319
15	17 609	17 898	18 184	18 469	18 752	19 033	19 312	19 590	19 866	20 140
16	20 412	20 683	20 952	21 219	21 484	21 748	22 011	22 272	22 531	22 789
17	23 045̄	23 300	23 553	23 805̄	24 055̄	24 304	24 551	24 797	25 042	25 285̇
18	25 527	25 768	26 007	26 245̇	26 482	26 717	26 951	27 184	27 416	27 646
19	27 875̇	28 103	28 330	28 556	28 780	29 003	29 226	29 447	29 667	29 885̄
20	30 103	30 320	30 535̇	30 750	30 963	31 175̇	31 387	31 597	31 806	32 015̄
21	32 222	32 428	32 634	32 838	33 041	33 244	33 445̇	33 646	33 846	34 044
22	34 242	34 439	34 635̇	34 830	35 025̄	35 218	35 411	35 603	35 793	35 984
23	36 173	36 361	36 549	36 736	36 922	37 107	37 291	37 475̄	37 658	37 840
24	38 021	38 202	38 382	38 561	38 739	38 917	39 094	39 270	39 445̇	39 620
25	39 794	39 967	40 140	40 312	40 483	40 654	40 824	40 993	41 162	41 330
26	41 497	41 664	41 830	41 996	42 160	42 325̄	42 488	42 651	42 813	42 975̇
27	43 136	43 297	43 457	43 616	43 775̇	43 933	44 091	44 248	44 404	44 560
28	44 716	44 871	45 025̄	45 179	45 332	45 484	45 637	45 788	45 939	46 090
29	46 240	46 389	46 538	46 687	46 835	46 982	47 129	47 276	47 422	47 567
30	47 712	47 857	48 001	48 144	48 287	48 430	48 572	48 714	48 855̇	48 996
31	49 136	49 276	49 415̇	49 554	49 693	49 831	49 969	50 106	50 243	50 379
32	50 515̄	50 651	50 786	50 920	51 055̄	51 188	51 322	51 455̄	51 587	51 720
33	51 851	51 983	52 114	52 244	52 375̄	52 504	52 634	52 763	52 892	53 020
34	53 148	53 275̇	53 403	53 529	53 656	53 782	53 908	54 033	54 158	54 283
35	54 407	54 531	54 654	54 777	54 900	55 023	55 145̄	55 267	55 388	55 509
36	55 630	55 751	55 871	55 991	56 110	56 229	56 348	56 467	56 585̄	56 703
37	56 820	56 937	57 054	57 171	57 287	57 403	57 519	57 634	57 749	57 864
38	57 978	58 092	58 206	58 320	58 433	58 546	58 659	58 771	58 883	58 995̄
39	59 106	59 218	59 329	59 439	59 550	59 660	59 770	59 879	59 988	60 097
40	60 206	60 314	60 423	60 531	60 638	60 746	60 853	60 959	61 066	61 172
41	61 278	61 384	61 490	61 595̇	61 700	61 805̄	61 909	62 014	62 118	62 221
42	62 325̄	62 428	62 531	62 634	62 737	62 839	62 941	63 043	63 144	63 246
43	63 347	63 448	63 548	63 649	63 749	63 849	63 949	64 048	64 147	64 246
44	64 345̇	64 444	64 542	64 640	64 738	64 836	64 933	65 031	65 128	65 225̄
45	65 321	65 418	65 514	65 610	65 706	65 801	65 896	65 992	66 087	66 181
46	66 276	66 370	66 464	66 558	66 652	66 745̇	66 839	66 932	67 025̄	67 117
47	67 210	67 302	67 394	67 486	67 578	67 669	67 761	67 852	67 943	68 034
48	68 124	68 215̄	68 305̄	68 395̄	68 485̄	68 574	68 664	68 753	68 842	68 931
49	69 020	69 108	69 197	69 285̄	69 373	69 461	69 548	69 636	69 723	69 810
50	69 897	69 984	70 070	70 157	70 243	70 329	70 415̇	70 501	70 586	70 672
N.	L. 0	1	2	3	4	5	6	7	8	9

50-100

N.	L. 0	1	2	3	4	5	6	7	8	9
50	69 897	69 984	70 070	70 157	70 243	70 329	70 415	70 501	70 586	70 672
51	70 757	70 842	70 927	71 012	71 096	71 181	71 265	71 349	71 433	71 517
52	71 600	71 684	71 767	71 850	71 933	72 016	72 099	72 181	72 263	72 346
53	72 428	72 509	72 591	72 673	72 754	72 835	72 916	72 997	73 078	73 159
54	73 239	73 320	73 400	73 480	73 560	73 640	73 719	73 799	73 878	73 957
55	74 036	74 115	74 194	74 273	74 351	74 429	74 507	74 586	74 663	74 741
56	74 819	74 896	74 974	75 051	75 128	75 205	75 282	75 358	75 435	75 511
57	75 587	75 664	75 740	75 815	75 891	75 967	76 042	76 118	76 193	76 268
58	76 343	76 418	76 492	76 567	76 641	76 716	76 790	76 864	76 938	77 012
59	77 085	77 159	77 232	77 305	77 379	77 452	77 525	77 597	77 670	77 743
60	77 815	77 887	77 960	78 032	78 104	78 176	78 247	78 319	78 390	78 462
61	78 533	78 604	78 675	78 746	78 817	78 888	78 958	79 029	79 099	79 169
62	79 239	79 309	79 379	79 449	79 518	79 588	79 657	79 727	79 796	79 865
63	79 934	80 003	80 072	80 140	80 209	80 277	80 346	80 414	80 482	80 550
64	80 618	80 686	80 754	80 821	80 889	80 956	81 023	81 090	81 158	81 224
65	81 291	81 358	81 425	81 491	81 558	81 624	81 690	81 757	81 823	81 889
66	81 954	82 020	82 086	82 151	82 217	82 282	82 347	82 413	82 478	82 543
67	82 607	82 672	82 737	82 802	82 866	82 930	82 995	83 059	83 123	83 187
68	83 251	83 315	83 378	83 442	83 506	83 569	83 632	83 696	83 759	83 822
69	83 885	83 948	84 011	84 073	84 136	84 198	84 261	84 323	84 386	84 448
70	84 510	84 572	84 634	84 696	84 757	84 819	84 880	84 942	85 003	85 065
71	85 126	85 187	85 248	85 309	85 370	85 431	85 491	85 552	85 612	85 673
72	85 733	85 794	85 854	85 914	85 974	86 034	86 094	86 153	86 213	86 273
73	86 332	86 392	86 451	86 510	86 570	86 629	86 688	86 747	86 806	86 864
74	86 923	86 982	87 040	87 099	87 157	87 216	87 274	87 332	87 390	87 448
75	87 506	87 564	87 622	87 679	87 737	87 795	87 852	87 910	87 967	88 024
76	88 081	88 138	88 195	88 252	88 309	88 366	88 423	88 480	88 536	88 593
77	88 649	88 705	88 762	88 818	88 874	88 930	88 986	89 042	89 098	89 154
78	89 209	89 265	89 321	89 376	89 432	89 487	89 542	89 597	89 653	89 708
79	89 763	89 818	89 873	89 927	89 982	90 037	90 091	90 146	90 200	90 255
80	90 309	90 363	90 417	90 472	90 526	90 580	90 634	90 687	90 741	90 795
81	90 849	90 902	90 956	91 009	91 062	91 116	91 169	91 222	91 275	91 328
82	91 381	91 434	91 487	91 540	91 593	91 645	91 698	91 751	91 803	91 855
83	91 908	91 960	92 012	92 065	92 117	92 169	92 221	92 273	92 324	92 376
84	92 428	92 480	92 531	92 583	92 634	92 686	92 737	92 788	92 840	92 891
85	92 942	92 993	93 044	93 095	93 146	93 197	93 247	93 298	93 349	93 399
86	93 450	93 500	93 551	93 601	93 651	93 702	93 752	93 802	93 852	93 902
87	93 952	94 002	94 052	94 101	94 151	94 201	94 250	94 300	94 349	94 399
88	94 448	94 498	94 547	94 596	94 645	94 694	94 743	94 792	94 841	94 890
89	94 939	94 988	95 036	95 085	95 134	95 182	95 231	95 279	95 328	95 376
90	95 424	95 472	95 521	95 569	95 617	95 665	95 713	95 761	95 809	95 856
91	95 904	95 952	95 999	96 047	96 095	96 142	96 190	96 237	96 284	96 332
92	96 379	96 426	96 473	96 520	96 567	96 614	96 661	96 708	96 755	96 802
93	96 848	96 895	96 942	96 988	97 035	97 081	97 128	97 174	97 220	97 267
94	97 313	97 359	97 405	97 451	97 497	97 543	97 589	97 635	97 681	97 727
95	97 772	97 818	97 864	97 909	97 955	98 000	98 046	98 091	98 137	98 182
96	98 227	98 272	98 318	98 363	98 408	98 453	98 498	98 543	98 588	98 632
97	98 677	98 722	98 767	98 811	98 856	98 900	98 945	98 989	99 034	99 078
98	99 123	99 167	99 211	99 255	99 300	99 344	99 388	99 432	99 476	99 520
99	99 564	99 607	99 651	99 695	99 739	99 782	99 826	99 870	99 913	99 957
100	00 000	00 043	00 087	00 130	00 173	00 217	00 260	00 303	00 346	00 389
N.	L. 0	1	2	3	4	5	6	7	8	9

N.	L.	0	1	2	3	4	5	6	7	8	9
100	00	000	043	087	130	173	217	260	303	346	389
101		432	475	518	561	604	647	689	732	775	817
102		860	903	945	988	*030	*072	*115	*157	*199	*242
103	01	284	326	368	410	452	494	536	578	620	662
104		703	745	787	828	870	912	953	995	*036	*078
105	02	119	160	202	243	284	325	366	407	449	490
106		531	572	612	653	694	735	776	816	857	898
107		938	979	*019	*060	*100	*141	*181	*222	*262	*302
108	03	342	383	423	463	503	543	583	623	663	703
109		743	782	822	862	902	941	981	*021	*060	*100
110	04	139	179	218	258	297	336	376	415	454	493
111		532	571	610	650	689	727	766	805	844	883
112		922	961	999	*038	*077	*115	*154	*192	*231	*269
113	05	308	346	385	423	461	500	538	576	614	652
114		690	729	767	805	843	881	918	956	994	*032
115	06	070	108	145	183	221	258	296	333	371	408
116		446	483	521	558	595	633	670	707	744	781
117		819	856	893	930	967	*004	*041	*078	*115	*151
118	07	188	225	262	298	335	372	408	445	482	518
119		555	591	628	664	700	737	773	809	846	882
120		918	954	990	*027	*063	*099	*135	*171	*207	*243
121	08	279	314	350	386	422	458	493	529	565	600
122		636	672	707	743	778	814	849	884	920	955
123		991	*026	*061	*096	*132	*167	*202	*237	*272	*307
124	09	342	377	412	447	482	517	552	587	621	656
125		691	726	760	795	830	864	899	934	968	*003
126	10	037	072	106	140	175	209	243	278	312	346
127		380	415	449	483	517	551	585	619	653	687
128		721	755	789	823	857	890	924	958	992	*025
129	11	059	093	126	160	193	227	261	294	327	361
130		394	428	461	494	528	561	594	628	661	694
131		727	760	793	826	860	893	926	959	992	*024
132	12	057	090	123	156	189	222	254	287	320	352
133		385	418	450	483	516	548	581	613	646	678
134		710	743	775	808	840	872	905	937	969	*001
135	13	033	066	098	130	162	194	226	258	290	322
136		354	386	418	450	481	513	545	577	609	640
137		672	704	735	767	799	830	862	893	925	956
138		988	*019	*051	*082	*114	*145	*176	*208	*239	*270
139	14	301	333	364	395	426	457	489	520	551	582
140		613	644	675	706	737	768	799	829	860	891
141		922	953	983	*014	*045	*076	*106	*137	*168	*198
142	15	229	259	290	320	351	381	412	442	473	503
143		534	564	594	625	655	685	715	746	776	806
144		836	866	897	927	957	987	*017	*047	*077	*107
145	16	137	167	197	227	256	286	316	346	376	406
146		435	465	495	524	554	584	613	643	673	702
147		732	761	791	820	850	879	909	938	967	997
148	17	026	056	085	114	143	173	202	231	260	289
149		319	348	377	406	435	464	493	522	551	580
150		609	638	667	696	725	754	782	811	840	869

Prop. Parts

	44	43	42
1	4.4	4.3	4.2
2	8.8	8.6	8.4
3	13.2	12.9	12.6
4	17.6	17.2	16.8
5	22.0	21.5	21.0
6	26.4	25.8	25.2
7	30.8	30.1	29.4
8	35.2	34.4	33.6
9	39.6	38.7	37.8

	41	40	39
1	4.1	4.0	3.9
2	8.2	8.0	7.8
3	12.3	12.0	11.7
4	16.4	16.0	15.6
5	20.5	20.0	19.5
6	24.6	24.0	23.4
7	28.7	28.0	27.3
8	32.8	32.0	31.2
9	36.9	36.0	35.1

	38	37	36
1	3.8	3.7	3.6
2	7.6	7.4	7.2
3	11.4	11.1	10.8
4	15.2	14.8	14.4
5	19.0	18.5	18.0
6	22.8	22.2	21.6
7	26.6	25.9	25 2
8	30.4	29.6	28 8
9	34.2	33.3	32.4

	35	34	33
1	3.5	3.4	3.3
2	7.0	6.8	6.6
3	10.5	10.2	9.9
4	14.0	13.6	13.2
5	17.5	17.0	16 5
6	21.0	20.4	19.8
7	24.5	23.8	23.1
8	28.0	27.2	26.4
9	31.5	30.6	29.7

	32	31	30
1	3.2	3.1	3.0
2	6.4	6.2	6.0
3	9.6	9.3	9.0
4	12.8	12.4	12.0
5	16.0	15.5	15 0
6	19.2	18.6	18.0
7	22.4	21.7	21 0
8	25.6	24.8	24 0
9	28.8	27.9	27.0

N.	L. 0	1	2	3	4	5	6	7	8	9
150	17 609	638	667	696	725	754	782	811	840	869
151	898	926	955	984	*013	*041	*070	*099	*127	*156
152	18 184	213	241	270	298	327	355	384	412	441
153	469	498	526	554	583	611	639	667	696	724
154	752	780	808	837	865	893	921	949	977	*005
155	19 033	061	089	117	145	173	201	229	257	285
156	312	340	368	396	424	451	479	507	535	562
157	590	618	645	673	700	728	756	783	811	838
158	866	893	921	948	976	*003	*030	*058	*085	*112
159	20 140	167	194	222	249	276	303	330	358	385
160	412	439	466	493	520	548	575	602	629	656
161	683	710	737	763	790	817	844	871	898	925
162	952	978	*005	*032	*059	*085	*112	*139	*165	*192
163	21 219	245	272	299	325	352	378	405	431	458
164	484	511	537	564	590	617	643	669	696	722
165	748	775	801	827	854	880	906	932	958	985
166	22 011	037	063	089	115	141	167	194	220	246
167	272	298	324	350	376	401	427	453	479	505
168	531	557	583	608	634	660	686	712	737	763
169	789	814	840	866	891	917	943	968	994	*019
170	23 045	070	096	121	147	172	198	223	249	274
171	300	325	350	376	401	426	452	477	502	528
172	553	578	603	629	654	679	704	729	754	779
173	805	830	855	880	905	930	955	980	*005	*030
174	24 055	080	105	130	155	180	204	229	254	279
175	304	329	353	378	403	428	452	477	502	527
176	551	576	601	625	650	674	699	724	748	773
177	797	822	846	871	895	920	944	969	993	*018
178	25 042	066	091	115	139	164	188	212	237	261
179	285	310	334	358	382	406	431	455	479	503
180	527	551	575	600	624	648	672	696	720	744
181	768	792	816	840	864	888	912	935	959	983
182	26 007	031	055	079	102	126	150	174	198	221
183	245	269	293	316	340	364	387	411	435	458
184	482	505	529	553	576	600	623	647	670	694
185	717	741	764	788	811	834	858	881	905	928
186	951	975	998	*021	*045	*068	*091	*114	*138	*161
187	27 184	207	231	254	277	300	323	346	370	393
188	416	439	462	485	508	531	554	577	600	623
189	646	669	692	715	738	761	784	807	830	852
190	875	898	921	944	967	989	*012	*035	*058	*081
191	28 103	126	149	171	194	217	240	262	285	307
192	330	353	375	398	421	443	466	488	511	533
193	556	578	601	623	646	668	691	713	735	758
194	780	803	825	847	870	892	914	937	959	981
195	29 003	026	048	070	092	115	137	159	181	203
196	226	248	270	292	314	336	358	380	403	425
197	447	469	491	513	535	557	579	601	623	645
198	667	688	710	732	754	776	798	820	842	863
199	885	907	929	951	973	994	*016	*038	*060	*081
200	30 103	125	146	168	190	211	233	255	276	298
N.	L. 0	1	2	3	4	5	6	7	8	9

Prop. Parts

	29	28
1	2.9	2.8
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	:14.5	14.0
6	17.4	16 8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	27	26
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	25
1	2.5
2	5.0
3	7.5
4	10 0
5	12.5
6	15 0
7	17 5
8	20 0
9	22.5

	24	23
1	2.4	2.3
2	4.8	4.6
3	7.2	6.9
4	9.6	9.2
5	12.0	11.5
6	14.4	13.8
7	16.8	16.1
8	19.2	18.4
9	21.6	20.7

	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6 3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

N.	L. 0	1	2	3	4	5	6	7	8	9
200	30 103	125	146	168	190	211	233	255	276	298
201	320	341	363	384	406	428	449	471	492	514
202	535	557	578	600	621	643	664	685	707	728
203	750	771	792	814	835	856	878	899	920	942
204	963	984	*006	*027	*048	*069	*091	*112	*133	*154
205	31 175	197	218	239	260	281	302	323	345	366
206	387	408	429	450	471	492	513	534	555	576
207	597	618	639	660	681	702	723	744	765	785
208	806	827	848	869	890	911	931	952	973	994
209	32 015	035	056	077	098	118	139	160	181	201
210	222	243	263	284	305	325	346	366	387	408
211	428	449	469	490	510	531	552	572	593	613
212	634	654	675	695	715	736	756	777	797	818
213	838	858	879	899	919	940	960	980	*001	*021
214	33 041	062	082	102	122	143	163	183	203	224
215	244	264	284	304	325	345	365	385	405	425
216	445	465	486	506	526	546	566	586	606	626
217	646	666	686	706	726	746	766	786	806	826
218	846	866	885	905	925	945	965	985	*005	*025
219	34 044	064	084	104	124	143	163	183	203	223
220	242	262	282	301	321	341	361	380	400	420
221	439	459	479	498	518	537	557	577	596	616
222	635	655	674	694	713	733	753	772	792	811
223	830	850	869	889	908	928	947	967	986	*005
224	35 025	044	064	083	102	122	141	160	180	199
225	218	238	257	276	295	315	334	353	372	392
226	411	430	449	468	488	507	526	545	564	583
227	603	622	641	660	679	698	717	736	755	774
228	793	813	832	851	870	889	908	927	946	965
229	984	*003	*021	*040	*059	*078	*097	*116	*135	*154
230	36 173	192	211	229	248	267	286	305	324	342
231	361	380	399	418	436	455	474	493	511	530
232	549	568	586	605	624	642	661	680	698	717
233	736	754	773	791	810	829	847	866	884	903
234	922	940	959	977	996	*014	*033	*051	*070	*088
235	37 107	125	144	162	181	199	218	236	254	273
236	291	310	328	346	365	383	401	420	438	457
237	475	493	511	530	548	566	585	603	621	639
238	658	676	694	712	731	749	767	785	803	822
239	840	858	876	894	912	931	949	967	985	*003
240	38 021	039	057	075	093	112	130	148	166	184
241	202	220	238	256	274	292	310	328	346	364
242	382	399	417	435	453	471	489	507	525	543
243	561	578	596	614	632	650	668	686	703	721
244	739	757	775	792	810	828	846	863	881	899
245	917	934	952	970	987	*005	*023	*041	*058	*076
246	39 094	111	129	146	164	182	199	217	235	252
247	270	287	305	322	340	358	375	393	410	428
248	445	463	480	498	515	533	550	568	585	602
249	620	637	655	672	690	707	724	742	759	777
250	794	811	829	846	863	881	898	915	933	950
N.	L. 0	1	2	3	4	5	6	7	8	9

Prop. Parts

	22	21	20	19	18	17
1	2.2	2.1	2.0	1.9	1.8	1.7
2	4.4	4.2	4.0	3.8	3.6	3.4
3	6.6	6.3	6.0	5.7	5.4	5.1
4	8.8	8.4	8.0	7.6	7.2	6.8
5	11.0	10.5	10.0	9.5	9.0	8.5
6	13.2	12.6	12.0	11.4	10 8	10.2
7	15.4	14.7	14.0	13.3	12.6	11.9
8	17.6	16.8	16.0	15.2	14.4	13.6
9	19.8	18.9	18.0	17.1	16.2	15.3

N.	L. 0	1	2	3	4	5	6	7	8	9	Prop. Parts
250	39 794	811	829	846	863	881	898	915	933	950	**18**
251	967	985	*002	*019	*037	*054	*071	*088	*106	*123	1 1.8
252	40 140	157	175	192	209	226	243	261	278	295	2 3.6
253	312	329	346	364	381	398	415	432	449	466	3 5.4
254	483	500	518	535	552	569	586	603	620	637	4 7.2
255	654	671	688	705	722	739	756	773	790	807	5 9.0
256	824	841	858	875	892	909	926	943	960	976	6 10.8
257	993	*010	*027	*044	*061	*078	*095	*111	*128	*145	7 12.6
258	41 162	179	196	212	229	246	263	280	296	313	8 14.4
259	330	347	363	380	397	414	430	447	464	481	9 16.2
260	497	514	531	547	564	581	597	614	631	647	**17**
261	664	681	697	714	731	747	764	780	797	814	1 1.7
262	830	847	863	880	896	913	929	946	963	979	2 3 4
263	996	*012	*029	*045	*062	*078	*095	*111	*127	*144	3 5.1
264	42 160	177	193	210	226	243	259	275	292	308	4 6.8
265	325	341	357	374	390	406	423	439	455	472	5 8.5
266	488	504	521	537	553	570	586	602	619	635	6 10.2
267	651	667	684	700	716	732	749	765	781	797	7 11.9
268	813	830	846	862	878	894	911	927	943	959	8 13.6
269	975	991	*008	*024	*040	*056	*072	*088	*104	*120	9 15.3
270	43 136	152	169	185	201	217	233	249	265	281	log e = 0.43429
271	297	313	329	345	361	377	393	409	425	441	**16**
272	457	473	489	505	521	537	553	569	584	600	1 1.6
273	616	632	648	664	680	696	712	727	743	759	2 3.2
274	775	791	807	823	838	854	870	886	902	917	3 4.8
275	933	949	965	981	996	*012	*028	*044	*059	*075	4 6.4
276	44 091	107	122	138	154	170	185	201	217	232	5 8.0
277	248	264	279	295	311	326	342	358	373	389	6 9.6
278	404	420	436	451	467	483	498	514	529	545	7 11.2
279	560	576	592	607	623	638	654	669	685	700	8 12 8
280	716	731	747	762	778	793	809	824	840	855	9 14.4
281	871	886	902	917	932	948	963	979	994	*010	**15**
282	45 025	040	056	071	086	102	117	133	148	163	1 1.5
283	179	194	209	225	240	255	271	286	301	317	2 3.0
284	332	347	362	378	393	408	423	439	454	469	3 4.5
285	484	500	515	530	545	561	576	591	606	621	4 6.0
286	637	652	667	682	697	712	728	743	758	773	5 7.5
287	788	803	818	834	849	864	879	894	909	924	6 9.0
288	939	954	969	984	*000	*015	*030	*045	*060	*075	7 10.5
289	46 090	105	120	135	150	165	180	195	210	225	8 12.0
290	240	255	270	285	300	315	330	345	359	374	9 13.5
291	389	404	419	434	449	464	479	494	509	523	**14**
292	538	553	568	583	598	613	627	642	657	672	1 1.4
293	687	702	716	731	746	761	776	790	805	820	2 2.8
294	835	850	864	879	894	909	923	938	953	967	3 4.2
295	982	997	*012	*026	*041	*056	*070	*085	*100	*114	4 5.6
296	47 129	144	159	173	188	202	217	232	246	261	5 7.0
297	276	290	305	319	334	349	363	378	392	407	6 8.4
298	422	436	451	465	480	494	509	524	538	553	7 9.8
299	567	582	596	611	625	640	654	669	683	698	8 11.2
300	712	727	741	756	770	784	799	813	828	842	9 12.6

N. | L. 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Parts

300-350

N.	L. 0	1	2	3	4	5	6	7	8	9
300	47 712	727	741	756	770	784	799	813	828	842
301	857	871	885	900	914	929	943	958	972	986
302	48 001	015	029	044	058	073	087	101	116	130
303	144	159	173	187	202	216	230	244	259	273
304	287	302	316	330	344	359	373	387	401	416
305	430	444	458	473	487	501	515	530	544	558
306	572	586	601	615	629	643	657	671	686	700
307	714	728	742	756	770	785	799	813	827	841
308	855	869	883	897	911	926	940	954	968	982
309	996	*010	*024	*038	*052	*066	*080	*094	*108	*122
310	49 136	150	164	178	192	206	220	234	248	262
311	276	290	304	318	332	346	360	374	388	402
312	415	429	443	457	471	485	499	513	527	541
313	554	568	582	596	610	624	638	651	665	679
314	693	707	721	734	748	762	776	790	803	817
315	831	845	859	872	886	900	914	927	941	955
316	969	982	996	*010	*024	*037	*051	*065	*079	*092
317	50 106	120	133	147	161	174	188	202	215	229
318	243	256	270	284	297	311	325	338	352	365
319	379	393	406	420	433	447	461	474	488	501
320	515	529	542	556	569	583	596	610	623	637
321	651	664	678	691	705	718	732	745	759	772
322	786	799	813	826	840	853	866	880	893	907
323	920	934	947	961	974	987	*001	*014	*028	*041
324	51 055	068	081	095	108	121	135	148	162	175
325	188	202	215	228	242	255	268	282	295	308
326	322	335	348	362	375	388	402	415	428	441
327	455	468	481	495	508	521	534	548	561	574
328	587	601	614	627	640	654	667	680	693	706
329	720	733	746	759	772	786	799	812	825	838
330	851	865	878	891	904	917	930	943	957	970
331	983	996	*009	*022	*035	*048	*061	*075	*088	*101
332	52 114	127	140	153	166	179	192	205	218	231
333	244	257	270	284	297	310	323	336	349	362
334	375	388	401	414	427	440	453	466	479	492
335	504	517	530	543	556	569	582	595	608	621
336	634	647	660	673	686	699	711	724	737	750
337	763	776	789	802	815	827	840	853	866	879
338	892	905	917	930	943	956	969	982	994	*007
339	53 020	033	046	058	071	084	097	110	122	135
340	148	161	173	186	199	212	224	237	250	263
341	275	288	301	314	326	339	352	364	377	390
342	403	415	428	441	453	466	479	491	504	517
343	529	542	555	567	580	593	605	618	631	643
344	656	668	681	694	706	719	732	744	757	769
345	782	794	807	820	832	845	857	870	882	895
346	908	920	933	945	958	970	983	995	*008	*020
347	54 033	045	058	070	083	095	108	120	133	145
348	158	170	183	195	208	220	233	245	258	270
349	283	295	307	320	332	345	357	370	382	394
350	407	419	432	444	456	469	481	494	506	518
N.	L. 0	1	2	3	4	5	6	7	8	9

Prop. Parts

	15
1	1.5
2	3.0
3	4.5
4	6.0
5	7.5
6	9.0
7	10.5
8	12.0
9	13.5

$\log \pi = 0.49715$

	14
1	1.4
2	2.8
3	4.2
4	5.6
5	7.0
6	8.4
7	9.8
8	11.2
9	12.6

	13
1	1.3
2	2.6
3	3.9
4	5.2
5	6.5
6	7.8
7	9.1
8	10.4
9	11.7

	12
1	1.2
2	2.4
3	3.6
4	4.8
5	6.0
6	7.2
7	8.4
8	9.6
9	10.8

Prop. Parts

N.	L. 0	1	2	3	4	5	6	7	8	9
350	54 407	419	432	444	456	469	481	494	506	518
351	531	543	555	568	580	593	605	617	630	642
352	654	667	679	691	704	716	728	741	753	765
353	777	790	802	814	827	839	851	864	876	888
354	900	913	925	937	949	962	974	986	998	*011
355	55 023	035	047	060	072	084	096	108	121	133
356	145	157	169	182	194	206	218	230	242	255
357	267	279	291	303	315	328	340	352	364	376
358	388	400	413	425	437	449	461	473	485	497
359	509	522	534	546	558	570	582	594	606	618
360	630	642	654	666	678	691	703	715	727	739
361	751	763	775	787	799	811	823	835	847	859
362	871	883	895	907	919	931	943	955	967	979
363	991	*003	*015	*027	*038	*050	*062	*074	*086	*098
364	56 110	122	134	146	158	170	182	194	205	217
365	229	241	253	265	277	289	301	312	324	336
366	348	360	372	384	396	407	419	431	443	455
367	467	478	490	502	514	526	538	549	561	573
368	585	597	608	620	632	644	656	667	679	691
369	703	714	726	738	750	761	773	785	797	808
370	820	832	844	855	867	879	891	902	914	926
371	937	949	961	972	984	996	*008	*019	*031	*043
372	57 054	066	078	089	101	113	124	136	148	159
373	171	183	194	206	217	229	241	252	264	276
374	287	299	310	322	334	345	357	368	380	392
375	403	415	426	438	449	461	473	484	496	507
376	519	530	542	553	565	576	588	600	611	623
377	634	646	657	669	680	692	703	715	726	738
378	749	761	772	784	795	807	818	830	841	852
379	864	875	887	898	910	921	933	944	955	967
380	978	990	*001	*013	*024	*035	*047	*058	*070	*081
381	58 092	104	115	127	138	149	161	172	184	195
382	206	218	229	240	252	263	274	286	297	309
383	320	331	343	354	365	377	388	399	410	422
384	433	444	456	467	478	490	501	512	524	535
385	546	557	569	580	591	602	614	625	636	647
386	659	670	681	692	704	715	726	737	749	760
387	771	782	794	805	816	827	838	850	861	872
388	883	894	906	917	928	939	950	961	973	984
389	995	*006	*017	*028	*040	*051	*062	*073	*084	*095
390	59 106	118	129	140	151	162	173	184	195	207
391	218	229	240	251	262	273	284	295	306	318
392	329	340	351	362	373	384	395	406	417	428
393	439	450	461	472	483	494	506	517	528	539
394	550	561	572	583	594	605	616	627	638	649
395	660	671	682	693	704	715	726	737	748	759
396	770	780	791	802	813	824	835	846	857	868
397	879	890	901	912	923	934	945	956	966	977
398	988	999	*010	*021	*032	*043	*054	*065	*076	*086
399	60 097	108	119	130	141	152	163	173	184	195
400	206	217	228	239	249	260	271	282	293	304
N.	L. 0	1	2	3	4	5	6	7	8	9

Prop. Parts

	13
1	1.3
2	2.6
3	3.9
4	5.2
5	6.5
6	7.8
7	9.1
8	10.4
9	11.7

	12
1	1.2
2	2.4
3	3.6
4	4.8
5	6.0
6	7.2
7	8.4
8	9.6
9	10.8

	11
1	1.1
2	2.2
3	3.3
4	4.4
5	5.5
6	6.6
7	7.7
8	8.8
9	9.9

	10
1	1.0
2	2.0
3	3.0
4	4.0
5	5.0
6	6.0
7	7.0
8	8.0
9	9.0

N.	L.	0	1	2	3	4	5	6	7	8	9
400	60	206	217	228	239	249	260	271	282	293	304
401		314	325	336	347	358	369	379	390	401	412
402		423	433	444	455	466	477	487	498	509	520
403		531	541	552	563	574	584	595	606	617	627
404		638	649	660	670	681	692	703	713	724	73$\bar{5}$
405		746	756	767	778	788	799	810	821	831	842
406		853	863	874	88$\bar{5}$	895	906	917	927	938	949
407		959	970	981	991	*002	*013	*023	*034	*04$\bar{5}$	*055
408	61	066	077	087	098	109	119	130	140	151	162
409		172	183	194	204	21$\bar{5}$	225	236	247	257	268
410		278	289	300	310	321	331	342	352	363	374
411		384	39$\bar{5}$	405	416	426	437	448	458	469	479
412		490	500	511	521	532	542	553	563	574	584
413		595	606	616	627	637	648	658	669	679	690
414		700	711	721	731	742	752	763	773	784	794
415		80$\bar{5}$	815	826	836	847	857	868	878	888	899
416		909	920	930	941	951	962	972	982	993	*003
417	62	014	024	034	04$\bar{5}$	055	066	076	086	097	107
418		118	128	138	149	159	170	180	190	201	211
419		221	232	242	252	263	273	284	294	304	31$\bar{5}$
420		32$\bar{5}$	335	346	356	366	377	387	397	408	418
421		428	439	449	459	469	480	490	500	511	521
422		531	542	552	562	572	583	593	603	613	624
423		634	644	65$\bar{5}$	66$\bar{5}$	675	685	696	706	716	726
424		737	747	757	767	778	788	798	808	818	829
425		839	849	859	870	880	890	900	910	921	931
426		941	951	961	972	982	992	*002	*012	*022	*033
427	63	043	053	063	073	083	094	104	114	124	134
428		144	15$\bar{5}$	16$\bar{5}$	17$\bar{5}$	18$\bar{5}$	195	205	215	225	236
429		246	256	266	276	286	296	306	317	327	337
430		347	357	367	377	387	397	407	417	428	438
431		448	458	468	478	488	498	508	518	528	538
432		548	558	568	579	589	599	609	619	629	639
433		649	659	669	679	689	699	709	719	729	739
434		749	759	769	779	789	799	809	819	829	839
435		849	859	869	879	889	899	909	919	929	939
436		949	959	969	979	988	998	*008	*018	*028	*038
437	64	048	058	068	078	088	098	108	118	128	137
438		147	157	167	177	187	197	207	217	227	237
439		246	256	266	276	286	296	306	316	326	335
440		345	355	365	37$\bar{5}$	38$\bar{5}$	39$\bar{5}$	404	414	424	434
441		444	454	464	473	483	493	503	513	523	532
442		542	552	562	572	582	591	601	611	621	631
443		640	650	660	670	680	689	699	709	719	729
444		738	748	758	768	777	787	797	807	816	826
445		836	846	856	865	875	88$\bar{5}$	89$\bar{5}$	904	914	924
446		933	943	953	963	972	982	992	*002	*011	*021
447	65	031	040	050	060	070	079	089	099	108	118
448		128	137	147	157	167	176	186	196	205	21$\bar{5}$
449		22$\bar{5}$	234	244	254	263	273	283	292	302	312
450		321	331	341	350	360	369	379	389	398	408
N.	L.	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	11	10	9
1	1.1	1.0	0.9
2	2.2	2.0	1.8
3	3.3	3.0	2.7
4	4.4	4.0	3.6
5	5.5	5 0	4.5
6	6 6	6 0	5.4
7	7 7	7.0	6.3
8	8 8	8.0	7.2
9	9.9	9.0	8.1

N.	L. 0	1	2	3	4	5	6	7	8	9
450	65 321	331	341	350	360	369	379	389	398	408
451	418	427	437	447	456	466	475	485	495	504
452	514	523	533	543	552	562	571	581	591	600
453	610	619	629	639	648	658	667	677	686	696
454	706	715	725	734	744	753	763	772	782	792
455	801	811	820	830	839	849	858	868	877	887
456	896	906	916	925	935	944	954	963	973	982
457	992	*001	*011	*020	*030	*039	*049	*058	*068	*077
458	66 087	096	106	115	124	134	143	153	162	172
459	181	191	200	210	219	229	238	247	257	266
460	276	285	295	304	314	323	332	342	351	361
461	370	380	389	398	408	417	427	436	445	455
462	464	474	483	492	502	511	521	530	539	549
463	558	567	577	586	596	605	614	624	633	642
464	652	661	671	680	689	699	708	717	727	736
465	745	755	764	773	783	792	801	811	820	829
466	839	848	857	867	876	885	894	904	913	922
467	932	941	950	960	969	978	987	997	*006	*015
468	67 025	034	043	052	062	071	080	089	099	108
469	117	127	136	145	154	164	173	182	191	201
470	210	219	228	237	247	256	265	274	284	293
471	302	311	321	330	339	348	357	367	376	385
472	394	403	413	422	431	440	449	459	468	477
473	486	495	504	514	523	532	541	550	560	569
474	578	587	596	605	614	624	633	642	651	660
475	669	679	688	697	706	715	724	733	742	752
476	761	770	779	788	797	806	815	825	834	843
477	852	861	870	879	888	897	906	916	925	934
478	943	952	961	970	979	988	997	*006	*015	*024
479	68 034	043	052	061	070	079	088	097	106	115
480	124	133	142	151	160	169	178	187	196	205
481	215	224	233	242	251	260	269	278	287	296
482	305	314	323	332	341	350	359	368	377	386
483	395	404	413	422	431	440	449	458	467	476
484	485	494	502	511	520	529	538	547	556	565
485	574	583	592	601	610	619	628	637	646	655
486	664	673	681	690	699	708	717	726	735	744
487	753	762	771	780	789	797	806	815	824	833
488	842	851	860	869	878	886	895	904	913	922
489	931	940	949	958	966	975	984	993	*002	*011
490	69 020	028	037	046	055	064	073	082	090	099
491	108	117	126	135	144	152	161	170	179	188
492	197	205	214	223	232	241	249	258	267	276
493	285	294	302	311	320	329	338	346	355	364
494	373	381	390	399	408	417	425	434	443	452
495	461	469	478	487	496	504	513	522	531	539
496	548	557	566	574	583	592	601	609	618	627
497	636	644	653	662	671	679	688	697	705	714
498	723	732	740	749	758	767	775	784	793	801
499	810	819	827	836	845	854	862	871	880	888
500	897	906	914	923	932	940	949	958	966	975

| N. | L. 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Parts |

Prop. Parts

	10
1	1.0
2	2 0
3	3.0
4	4 0
5	5 0
6	6.0
7	7.0
8	8.0
9	9.0

	9
1	0.9
2	1.8
3	2.7
4	3.6
5	4.5
6	5.4
7	6.3
8	7.2
9	8.1

	8
1	0.8
2	1.6
3	2.4
4	3 2
5	4.0
6	4 8
7	5 6
8	6.4
9	7.2

N.	L. 0	1	2	3	4	5	6	7	8	9	Prop. Parts
500	69 897	906	914	923	932	940	949	958	966	975	
501	984	992	*001	*010	*018	*027	*036	*044	*053	*062	
502	70 070	079	088	096	105	114	122	131	140	148	
503	157	165	174	183	191	200	209	217	226	234	
504	243	252	260	269	278	286	295	303	312	321	
505	329	338	346	355	364	372	381	389	398	406	9
506	415	424	432	441	449	458	467	475	484	492	1 | 0.9
507	501	509	518	526	535	544	552	561	569	578	2 | 1.8
508	586	595	603	612	621	629	638	646	655	663	3 | 2.7
509	672	680	689	697	706	714	723	731	740	749	4 | 3.6
510	757	766	774	783	791	800	808	817	825	834	5 | 4 5
511	842	851	859	868	876	885	893	902	910	919	6 | 5.4
512	927	935	944	952	961	969	978	986	995	*003	7 | 6 3
513	71 012	020	029	037	046	054	063	071	079	088	8 | 7 2
514	096	105	113	122	130	139	147	155	164	172	9 | 8.1
515	181	189	198	206	214	223	231	240	248	257	
516	265	273	282	290	299	307	315	324	332	341	
517	349	357	366	374	383	391	399	408	416	425	
518	433	441	450	458	466	475	483	492	500	508	
519	517	525	533	542	550	559	567	575	584	592	
520	600	609	617	625	634	642	650	659	667	675	
521	684	692	700	709	717	725	734	742	750	759	8
522	767	775	784	792	800	809	817	825	834	842	1 | 0.8
523	850	858	867	875	883	892	900	908	917	925	2 | 1.6
524	933	941	950	958	966	975	983	991	999	*008	3 | 2.4
525	72 016	024	032	041	049	057	066	074	082	090	4 | 3.2
526	099	107	115	123	132	140	148	156	165	173	5 | 4.0
527	181	189	198	206	214	222	230	239	247	255	6 | 4 8
528	263	272	280	288	296	304	313	321	329	337	7 | 5 6
529	346	354	362	370	378	387	395	403	411	419	8 | 6.4
530	428	436	444	452	460	469	477	485	493	501	9 | 7.2
531	509	518	526	534	542	550	558	567	575	583	
532	591	599	607	616	624	632	640	648	656	665	
533	673	681	689	697	705	713	722	730	738	746	
534	754	762	770	779	787	795	803	811	819	827	
535	835	843	852	860	868	876	884	892	900	908	
536	916	925	933	941	949	957	965	973	981	989	7
537	997	*006	*014	*022	*030	*038	*046	*054	*062	*070	1 | 0.7
538	73 078	086	094	102	111	119	127	135	143	151	2 | 1.4
539	159	167	175	183	191	199	207	215	223	231	3 | 2 1
540	239	247	255	263	272	280	288	296	304	312	4 | 2.8
541	320	328	336	344	352	360	368	376	384	392	5 | 3 5
542	400	408	416	424	432	440	448	456	464	472	6 | 4.2
543	480	488	496	504	512	520	528	536	544	552	7 | 4.9
544	560	568	576	584	592	600	608	616	624	632	8 | 5.6
545	640	648	656	664	672	679	687	695	703	711	9 | 6.3
546	719	727	735	743	751	759	767	775	783	791	
547	799	807	815	823	830	838	846	854	862	870	
548	878	886	894	902	910	918	926	933	941	949	
549	957	965	973	981	989	997	*005	*013	*020	*028	
550	74 036	044	052	060	068	076	084	092	099	107	
N.	L. 0	1	2	3	4	5	6	7	8	9	Prop. Parts

N.	L. 0	1	2	3	4	5	6	7	8	9
550	74 036	044	052	060	068	076	084	092	099	107
551	115	123	131	139	147	155	162	170	178	186
552	194	202	210	218	225	233	241	249	257	265
553	273	280	288	296	304	312	320	327	335	343
554	351	359	367	374	382	390	398	406	414	421
555	429	437	445	453	461	468	476	484	492	500
556	507	515	523	531	539	547	554	562	570	578
557	586	593	601	609	617	624	632	640	648	656
558	663	671	679	687	695	702	710	718	726	733
559	741	749	757	764	772	780	788	796	803	811
560	819	827	834	842	850	858	865	873	881	889
561	896	904	912	920	927	935	943	950	958	966
562	974	981	989	997	*005	*012	*020	*028	*035	*043
563	75 051	059	066	074	082	089	097	105	113	120
564	128	136	143	151	159	166	174	182	189	197
565	205	213	220	228	236	243	251	259	266	274
566	282	289	297	305	312	320	328	335	343	351
567	358	366	374	381	389	397	404	412	420	427
568	435	442	450	458	465	473	481	488	496	504
569	511	519	526	534	542	549	557	565	572	580
570	587	595	603	610	618	626	633	641	648	656
571	664	671	679	686	694	702	709	717	724	732
572	740	747	755	762	770	778	785	793	800	808
573	815	823	831	838	846	853	861	868	876	884
574	891	899	906	914	921	929	937	944	952	959
575	967	974	982	989	997	*005	*012	*020	*027	*035
576	76 042	050	057	065	072	080	087	095	103	110
577	118	125	133	140	148	155	163	170	178	185
578	193	200	208	215	223	230	238	245	253	260
579	268	275	283	290	298	305	313	320	328	335
580	343	350	358	365	373	380	388	395	403	410
581	418	425	433	440	448	455	462	470	477	485
582	492	500	507	515	522	530	537	545	552	559
583	567	574	582	589	597	604	612	619	626	634
584	641	649	656	664	671	678	686	693	701	708
585	716	723	730	738	745	753	760	768	775	782
586	790	797	805	812	819	827	834	842	849	856
587	864	871	879	886	893	901	908	916	923	930
588	938	945	953	960	967	975	982	989	997	*004
589	77 012	019	026	034	041	048	056	063	070	078
590	085	093	100	107	115	122	129	137	144	151
591	159	166	173	181	188	195	203	210	217	225
592	232	240	247	254	262	269	276	283	291	298
593	305	313	320	327	335	342	349	357	364	371
594	379	386	393	401	408	415	422	430	437	444
595	452	459	466	474	481	488	495	503	510	517
596	525	532	539	546	554	561	568	576	583	590
597	597	605	612	619	627	634	641	648	656	663
598	670	677	685	692	699	706	714	721	728	735
599	743	750	757	764	772	779	786	793	801	808
600	815	822	830	837	844	851	859	866	873	880
N.	L. 0	1	2	3	4	5	6	7	8	9

Prop. Parts

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

N.	L. 0	1	2	3	4	5	6	7	8	9
600	77 815	822	830	837	844	851	859	866	873	880
601	887	895	902	909	916	924	931	938	945	952
602	960	967	974	981	988	996	*003	*010	*017	*025
603	78 032	039	046	053	061	068	075	082	089	097
604	104	111	118	125	132	140	147	154	161	168
605	176	183	190	197	204	211	219	226	233	240
606	247	254	262	269	276	283	290	297	305	312
607	319	326	333	340	347	355	362	369	376	383
608	390	398	405	412	419	426	433	440	447	455
609	462	469	476	483	490	497	504	512	519	526
610	533	540	547	554	561	569	576	583	590	597
611	604	611	618	625	633	640	647	654	661	668
612	675	682	689	696	704	711	718	725	732	739
613	746	753	760	767	774	781	789	796	803	810
614	817	824	831	838	845	852	859	866	873	880
615	888	895	902	909	916	923	930	937	944	951
616	958	965	972	979	986	993	*000	*007	*014	*021
617	79 029	036	043	050	057	064	071	078	085	092
618	099	106	113	120	127	134	141	148	155	162
619	169	176	183	190	197	204	211	218	225	232
620	239	246	253	260	267	274	281	288	295	302
621	309	316	323	330	337	344	351	358	365	372
622	379	386	393	400	407	414	421	428	435	442
623	449	456	463	470	477	484	491	498	505	511
624	518	525	532	539	546	553	560	567	574	581
625	588	595	602	609	616	623	630	637	644	650
626	657	664	671	678	685	692	699	706	713	720
627	727	734	741	748	754	761	768	775	782	789
628	796	803	810	817	824	831	837	844	851	858
629	865	872	879	886	893	900	906	913	920	927
630	934	941	948	955	962	969	975	982	989	996
631	80 003	010	017	024	030	037	044	051	058	065
632	072	079	085	092	099	106	113	120	127	134
633	140	147	154	161	168	175	182	188	195	202
634	209	216	223	229	236	243	250	257	264	271
635	277	284	291	298	305	312	318	325	332	339
636	346	353	359	366	373	380	387	393	400	407
637	414	421	428	434	441	448	455	462	468	475
638	482	489	496	502	509	516	523	530	536	543
639	550	557	564	570	577	584	591	598	604	611
640	618	625	632	638	645	652	659	665	672	679
641	686	693	699	706	713	720	726	733	740	747
642	754	760	767	774	781	787	794	801	808	814
643	821	828	835	841	848	855	862	868	875	882
644	889	895	902	909	916	922	929	936	943	949
645	956	963	969	976	983	990	996	*003	*010	*017
646	81 023	030	037	043	050	057	064	070	077	084
647	090	097	104	111	117	124	131	137	144	151
648	158	164	171	178	184	191	198	204	211	218
649	224	231	238	245	251	258	265	271	278	285
650	291	298	305	311	318	325	331	338	345	351
N.	L. 0	1	2	3	4	5	6	7	8	9

Prop. Parts

	8	7	6
1	0.8	0 7	0.6
2	1.6	1.4	1.2
3	2.4	2.1	1.8
4	3.2	2.8	2.4
5	4.0	3.5	3.0
6	4.8	4.2	3.6
7	5.6	4.9	4.2
8	6.4	5.6	4.8
9	7.2	6.3	5.4

650-700

N.	L. 0	1	2	3	4	5	6	7	8	9
650	81 291	298	30$\bar{5}$	311	318	32$\bar{5}$	331	338	34$\bar{5}$	351
651	358	36$\bar{5}$	371	378	38$\bar{5}$	391	398	40$\bar{5}$	411	418
652	42$\bar{5}$	431	438	44$\bar{5}$	451	458	46$\bar{5}$	471	478	48$\bar{5}$
653	491	498	50$\bar{5}$	511	518	52$\bar{5}$	531	538	544	551
654	558	564	571	578	584	591	598	604	611	617
655	624	631	637	644	651	657	664	671	677	684
656	690	697	704	710	717	723	730	737	743	7$\bar{5}$0
657	757	763	770	776	783	790	796	803	809	816
658	823	829	836	842	849	856	862	869	875	882
659	889	895	902	908	91$\bar{5}$	921	928	93$\bar{5}$	941	948
660	954	961	968	974	981	987	994	*000	*007	*014
661	82 020	027	033	040	046	053	060	066	073	079
662	086	092	099	105	112	119	125	132	138	14$\bar{5}$
663	151	158	164	171	178	184	191	197	204	210
664	217	223	230	236	243	249	256	263	269	276
665	282	289	295	302	308	31$\bar{5}$	321	328	334	341
666	347	354	360	367	373	380	387	393	400	406
667	413	419	426	432	439	445	452	458	46$\bar{5}$	471
668	478	484	491	497	504	510	517	523	530	536
669	543	549	556	562	569	575	582	588	59$\bar{5}$	601
670	607	614	620	627	633	640	646	653	659	666
671	672	679	685	692	698	70$\bar{5}$	711	718	724	730
672	737	743	7$\bar{5}$0	756	763	769	776	782	789	795
673	802	808	814	821	827	834	840	847	853	860
674	866	872	879	885	892	898	90$\bar{5}$	911	918	924
675	930	937	943	9$\bar{5}$0	956	963	969	975	982	988
676	99$\bar{5}$	*001	*008	*014	*020	*027	*033	*040	*046	*052
677	83 059	065	072	078	08$\bar{5}$	091	097	104	110	117
678	123	129	136	142	149	15$\bar{5}$	161	168	174	181
679	187	193	200	206	213	219	225	232	238	24$\bar{5}$
680	251	257	264	270	276	283	289	296	302	308
681	31$\bar{5}$	321	327	334	340	347	353	359	366	372
682	378	38$\bar{5}$	391	398	404	410	417	423	429	436
683	442	448	45$\bar{5}$	461	467	474	480	487	493	499
684	506	512	518	52$\bar{5}$	531	537	544	550	556	563
685	569	575	582	588	594	601	607	613	620	626
686	632	639	645	651	658	664	670	677	683	689
687	696	702	708	71$\bar{5}$	721	727	734	740	746	753
688	759	765	771	778	784	790	797	803	809	816
689	822	828	83$\bar{5}$	841	847	853	860	866	872	879
690	88$\bar{5}$	891	897	904	910	916	923	929	935	942
691	948	954	960	967	973	979	985	992	998	*004
692	84 011	017	023	029	036	042	048	05$\bar{5}$	061	067
693	073	080	086	092	098	10$\bar{5}$	111	117	123	130
694	136	142	148	15$\bar{5}$	161	167	173	180	186	192
695	198	20$\bar{5}$	211	217	223	230	236	242	248	25$\bar{5}$
696	261	267	273	280	286	292	298	30$\bar{5}$	311	317
697	323	330	336	342	348	354	361	367	373	379
698	386	392	398	404	410	417	423	429	435	442
699	448	454	460	466	473	479	48$\bar{5}$	491	497	504
700	510	516	522	528	53$\bar{5}$	541	547	553	559	566

Prop. Parts

	7
1	0.7
2	1 4
3	2 1
4	2 8
5	3 5
6	4 2
7	4.9
8	5.6
9	6.3

	6
1	0 6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4 2
8	4.8
9	5.4

N. | L. 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Parts

N.	L. 0	1	2	3	4	5	6	7	8	9
700	84 510	516	522	528	535	541	547	553	559	566
701	572	578	584	590	597	603	609	615	621	628
702	634	640	646	652	658	665	671	677	683	689
703	696	702	708	714	720	726	733	739	745	751
704	757	763	770	776	782	788	794	800	807	813
705	819	825	831	837	844	850	856	862	868	874
706	880	887	893	899	905	911	917	924	930	936
707	942	948	954	960	967	973	979	985	991	997
708	85 003	009	016	022	028	034	040	046	052	058
709	065	071	077	083	089	095	101	107	114	120
710	126	132	138	144	150	156	163	169	175	181
711	187	193	199	205	211	217	224	230	236	242
712	248	254	260	266	272	278	285	291	297	303
713	309	315	321	327	333	339	345	352	358	364
714	370	376	382	388	394	400	406	412	418	425
715	431	437	443	449	455	461	467	473	479	485
716	491	497	503	509	516	522	528	534	540	546
717	552	558	564	570	576	582	588	594	600	606
718	612	618	625	631	637	643	649	655	661	667
719	673	679	685	691	697	703	709	715	721	727
720	733	739	745	751	757	763	769	775	781	788
721	794	800	806	812	818	824	830	836	842	848
722	854	860	866	872	878	884	890	896	902	908
723	914	920	926	932	938	944	950	956	962	968
724	974	980	986	992	998	*004	*010	*016	*022	*028
725	86 034	040	046	052	058	064	070	076	082	088
726	094	100	106	112	118	124	130	136	141	147
727	153	159	165	171	177	183	189	195	201	207
728	213	219	225	231	237	243	249	255	261	267
729	273	279	285	291	297	303	308	314	320	326
730	332	338	344	350	356	362	368	374	380	386
731	392	398	404	410	415	421	427	433	439	445
732	451	457	463	469	475	481	487	493	499	504
733	510	516	522	528	534	540	546	552	558	564
734	570	576	581	587	593	599	605	611	617	623
735	629	635	641	646	652	658	664	670	676	682
736	688	694	700	705	711	717	723	729	735	741
737	747	753	759	764	770	776	782	788	794	800
738	806	812	817	823	829	835	841	847	853	859
739	864	870	876	882	888	894	900	906	911	917
740	923	929	935	941	947	953	958	964	970	976
741	982	988	994	999	*005	*011	*017	*023	*029	*035
742	87 040	046	052	058	064	070	075	081	087	093
743	099	105	111	116	122	128	134	140	146	151
744	157	163	169	175	181	186	192	198	204	210
745	216	221	227	233	239	245	251	256	262	268
746	274	280	286	291	297	303	309	315	320	326
747	332	338	344	349	355	361	367	373	379	384
748	390	396	402	408	413	419	425	431	437	442
749	448	454	460	466	471	477	483	489	495	500
750	506	512	518	523	529	535	541	547	552	558
N.	L. 0	1	2	3	4	5	6	7	8	9

Prop. Parts

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

N.	L. 0	1	2	3	4	5	6	7	8	9	Prop. Parts
750	87 506	512	518	523	529	535	541	547	552	558	
751	564	570	576	581	587	593	599	604	610	616	
752	622	628	633	639	645	651	656	662	668	674	
753	679	685	691	697	703	708	714	720	726	731	
754	737	743	749	754	760	766	772	777	783	789	
755	795	800	806	812	818	823	829	835	841	846	
756	852	858	864	869	875	881	887	892	898	904	
757	910	915	921	927	933	938	944	950	955	961	
758	967	973	978	984	990	996	*001	*007	*013	*018	
759	88 024	030	036	041	047	053	058	064	070	076	
760	081	087	093	098	104	110	116	121	127	133	6
761	138	144	150	156	161	167	173	178	184	190	1 0.6
762	195	201	207	213	218	224	230	235	241	247	2 1.2
763	252	258	264	270	275	281	287	292	298	304	3 1.8
764	309	315	321	326	332	338	343	349	355	360	4 2.4
765	366	372	377	383	389	395	400	406	412	417	5 3.0
766	423	429	434	440	446	451	457	463	468	474	6 3.6
767	480	485	491	497	502	508	513	519	525	530	7 4.2
768	536	542	547	553	559	564	570	576	581	587	8 4.8
769	593	598	604	610	615	621	627	632	638	643	9 5.4
770	649	655	660	666	672	677	683	689	694	700	
771	705	711	717	722	728	734	739	745	750	756	
772	762	767	773	779	784	790	795	801	807	812	
773	818	824	829	835	840	846	852	857	863	868	
774	874	880	885	891	897	902	908	913	919	925	
775	930	936	941	947	953	958	964	969	975	981	
776	986	992	997	*003	*009	*014	*020	*025	*031	*037	
777	89 042	048	053	059	064	070	076	081	087	092	
778	098	104	109	115	120	126	131	137	143	148	
779	154	159	165	170	176	182	187	193	198	204	
780	209	215	221	226	232	237	243	248	254	260	
781	265	271	276	282	287	293	298	304	310	315	5
782	321	326	332	337	343	348	354	360	365	371	1 0.5
783	376	382	387	393	398	404	409	415	421	426	2 1.0
784	432	437	443	448	454	459	465	470	476	481	3 1.5
785	487	492	498	504	509	515	520	526	531	537	4 2.0
786	542	548	553	559	564	570	575	581	586	592	5 2.5
787	597	603	609	614	620	625	631	636	642	647	6 3.0
788	653	658	664	669	675	680	686	691	697	702	7 3.5
789	708	713	719	724	730	735	741	746	752	757	8 4.0
790	763	768	774	779	785	790	796	801	807	812	9 4.5
791	818	823	829	834	840	845	851	856	862	867	
792	873	878	883	889	894	900	905	911	916	922	
793	927	933	938	944	949	955	960	966	971	977	
794	982	988	993	998	*004	*009	*015	*020	*026	*031	
795	90 037	042	048	053	059	064	069	075	080	086	
796	091	097	102	108	113	119	124	129	135	140	
797	146	151	157	162	168	173	179	184	189	195	
798	200	206	211	217	222	227	233	238	244	249	
799	255	260	266	271	276	282	287	293	298	304	
800	309	314	320	325	331	336	342	347	352	358	
N.	L. 0	1	2	3	4	5	6	7	8	9	Prop. Parts

TABLE I

800-850

N.	L. 0	1	2	3	4	5	6	7	8	9	Prop. Parts
800	90 309	314	320	325	331	336	342	347	352	358	
801	363	369	374	380	38$\bar{5}$	390	396	401	407	412	
802	417	423	428	434	439	44$\bar{5}$	4$\bar{5}$0	455	461	466	
803	472	477	482	488	493	499	504	509	51$\bar{5}$	520	
804	526	531	536	542	547	553	558	563	569	574	
805	580	58$\bar{5}$	590	596	601	607	612	617	623	628	
806	634	639	644	6$\bar{5}$0	655	660	666	671	677	682	
807	687	693	698	703	709	714	720	725	730	736	
808	741	747	752	757	763	768	773	779	784	789	
809	79$\bar{5}$	800	806	811	816	822	827	832	838	843	
810	849	854	859	86$\bar{5}$	870	875	881	886	891	897	
811	902	907	913	918	924	929	934	940	94$\bar{5}$	950	**6**
812	956	961	966	972	977	982	988	993	998	*004	1 | 0.6
813	91 009	014	020	025	030	036	041	046	052	057	2 | 1.2
814	062	068	073	078	084	089	094	100	105	110	3 | 1 8
815	116	121	126	132	137	142	148	153	158	164	4 | 2 4
816	169	174	180	18$\bar{5}$	190	196	201	206	212	217	5 | 3 0
817	222	228	233	238	243	249	254	259	26$\bar{5}$	270	6 | 3 6
818	275	281	286	291	297	302	307	312	318	323	7 | 4 2
819	328	334	339	344	3$\bar{5}$0	35$\bar{5}$	360	365	371	376	8 | 4.8
820	381	387	392	397	403	408	413	418	424	429	9 | 5.4
821	434	440	44$\bar{5}$	450	455	461	466	471	477	482	
822	487	492	498	503	508	514	519	524	529	53$\bar{5}$	
823	540	545	551	556	561	566	572	577	582	587	
824	593	598	603	609	614	619	624	630	63$\bar{5}$	640	
825	645	651	656	661	666	672	677	682	687	693	
826	698	703	709	714	719	724	730	73$\bar{5}$	740	745	
827	751	756	761	766	772	777	782	787	793	798	
828	803	808	814	819	824	829	834	840	84$\bar{5}$	850	
829	855	861	866	871	876	882	887	892	897	903	
830	908	913	918	924	929	934	939	944	9$\bar{5}$0	95$\bar{5}$	
831	960	965	971	976	981	986	991	997	*002	*007	**5**
832	92 012	018	023	028	033	038	044	049	054	059	1 | 0 5
833	06$\bar{5}$	070	07$\bar{5}$	080	085	091	096	101	106	111	2 | 1 0
834	117	122	127	132	137	143	148	153	158	163	3 | 1 5
835	169	174	179	184	189	19$\bar{5}$	200	205	210	215	4 | 2 0
836	221	226	231	236	241	247	252	257	262	267	5 | 2 5
837	273	278	283	288	293	298	304	309	314	319	6 | 3.0
838	324	330	33$\bar{5}$	340	345	350	355	361	366	371	7 | 3.5
839	376	381	387	392	397	402	407	412	418	423	8 | 4.0
840	428	433	438	443	449	454	459	464	469	474	9 | 4.5
841	480	48$\bar{5}$	490	495	500	505	511	516	521	526	
842	531	536	542	547	552	557	562	567	572	578	
843	583	588	593	598	603	609	614	619	624	629	
844	634	639	64$\bar{5}$	6$\bar{5}$0	65$\bar{5}$	660	665	670	675	681	
845	686	691	696	701	706	711	716	722	727	732	
846	737	742	747	752	758	763	768	773	778	783	
847	788	793	799	804	809	814	819	824	829	834	
848	840	84$\bar{5}$	8$\bar{5}$0	85$\bar{5}$	860	865	870	875	881	886	
849	891	896	901	906	911	916	921	927	932	937	
850	942	947	952	957	962	967	973	978	983	988	
N.	L. 0	1	2	3	4	5	6	7	8	9	Prop. Parts

N.	L. 0	1	2	3	4	5	6	7	8	9	Prop. Parts
850	92 942	947	952	957	962	967	973	978	983	988	
851	993	998	*003	*008	*013	*018	*024	*029	*034	*039	
852	93 044	049	054	059	064	069	075	080	085	090	
853	095	100	105	110	115	120	125	131	136	141	
854	146	151	156	161	166	171	176	181	186	192	
855	197	202	207	212	217	222	227	232	237	242	6
856	247	252	258	263	268	273	278	283	288	293	1 | 0.6
857	298	303	308	313	318	323	328	334	339	344	2 | 1.2
858	349	354	359	364	369	374	379	384	389	394	3 | 1.8
859	399	404	409	414	420	425	430	435	440	445	4 | 2.4
860	450	455	460	465	470	475	480	485	490	495	5 | 3.0
861	500	505	510	515	520	526	531	536	541	546	6 | 3.6
862	551	556	561	566	571	576	581	586	591	596	7 | 4.2
863	601	606	611	616	621	626	631	636	641	646	8 | 4.8
864	651	656	661	666	671	676	682	687	692	697	9 | 5.4
865	702	707	712	717	722	727	732	737	742	747	
866	752	757	762	767	772	777	782	787	792	797	
867	802	807	812	817	822	827	832	837	842	847	
868	852	857	862	867	872	877	882	887	892	897	
869	902	907	912	917	922	927	932	937	942	947	
870	952	957	962	967	972	977	982	987	992	997	
871	94 002	007	012	017	022	027	032	037	042	047	5
872	052	057	062	067	072	077	082	086	091	096	1 | 0.5
873	101	106	111	116	121	126	131	136	141	146	2 | 1.0
874	151	156	161	166	171	176	181	186	191	196	3 | 1.5
875	201	206	211	216	221	226	231	236	240	245	4 | 2.0
876	250	255	260	265	270	275	280	285	290	295	5 | 2.5
877	300	305	310	315	320	325	330	335	340	345	6 | 3.0
878	349	354	359	364	369	374	379	384	389	394	7 | 3.5
879	399	404	409	414	419	424	429	433	438	443	8 | 4.0
880	448	453	458	463	468	473	478	483	488	493	9 | 4.5
881	498	503	507	512	517	522	527	532	537	542	
882	547	552	557	562	567	571	576	581	586	591	
883	596	601	606	611	616	621	626	630	635	640	
884	645	650	655	660	665	670	675	680	685	689	
885	694	699	704	709	714	719	724	729	734	738	
886	743	748	753	758	763	768	773	778	783	787	4
887	792	797	802	807	812	817	822	827	832	836	1 | 0.4
888	841	846	851	856	861	866	871	876	880	885	2 | 0.8
889	890	895	900	905	910	915	919	924	929	934	3 | 1.2
890	939	944	949	954	959	963	968	973	978	983	4 | 1.6
891	988	993	998	*002	*007	*012	*017	*022	*027	*032	5 | 2.0
892	95 036	041	046	051	056	061	066	071	075	080	6 | 2.4
893	085	090	095	100	105	109	114	119	124	129	7 | 2.8
894	134	139	143	148	153	158	163	168	173	177	8 | 3.2
895	182	187	192	197	202	207	211	216	221	226	9 | 3.6
896	231	236	240	245	250	255	260	265	270	274	
897	279	284	289	294	299	303	308	313	318	323	
898	328	332	337	342	347	352	357	361	366	371	
899	376	381	386	390	395	400	405	410	415	419	
900	424	429	434	439	444	448	453	458	463	468	
N.	L. 0	1	2	3	4	5	6	7	8	9	Prop. Parts

N.	L. 0	1	2	3	4	5	6	7	8	9	Prop. Parts
900	95 424	429	434	439	444	448	453	458	463	468	
901	472	477	482	487	492	497	501	506	511	516	
902	521	525	530	535	540	545̄	550̄	554	559	564	
903	569	574	578	583	588	593	598	602	607	612	
904	617	622	626	631	636	641	646	650	655	660	
905	665̄	670	674	679	684	689	694	698	703	708	
906	713	718	722	727	732	737	742	746	751	756	
907	761	766	770	775	780	785̄	789	794	799	804	
908	809	813	818	823	828	832	837	842	847	852	
909	856	861	866	871	875	880	885	890	895̄	899	
910	904	909	914	918	923	928	933	938	942	947	5
911	952	957	961	966	971	976	980	985	990	995̄	1 | 0.5
912	999	*004	*009	*014	*019	*023	*028	*033	*038	*042	2 | 1 0
913	96 047	052	057	061	066	071	076	080	085	090	3 | 1.5
914	095̄	099	104	109	114	118	123	128	133	137	4 | 2.0
915	142	147	152	156	161	166	171	175	180	185̄	5 | 2.5
916	190	194	199	204	209	213	218	223	227	232	6 | 3 0
917	237	242	246	251	256	261	265	270	275̄	280	7 | 3 5
918	284	289	294	298	303	308	313	317	322	327	8 | 4.0
919	332	336	341	346	350	355	360	365̄	369	374	9 | 4 5
920	379	384	388	393	398	402	407	412	417	421	
921	426	431	435	440	445̄	450̄	454	459	464	468	
922	473	478	483	487	492	497	501	506	511	515	
923	520	525̄	530	534	539	544	548	553	558	562	
924	567	572	577	581	586	591	595	600	605̄	609	
925	614	619	624	628	633	638	642	647	652	656	
926	661	666	670	675	680	685̄	689	694	699	703	
927	708	713	717	722	727	731	736	741	745	750	
928	755̄	759	764	769	774	778	783	788	792	797	
929	802	806	811	816	820	825̄	830	834	839	844	
930	848	853	858	862	867	872	876	881	886	890	4
931	895̄	900	904	909	914	918	923	928	932	937	1 | 0.4
932	942	946	951	956	960	965̄	970	974	979	984	2 | 0.8
933	988	993	997	*002	*007	*011	*016	*021	*025	*030	3 | 1.2
934	97 035̄	039	044	049	053	058	063	067	072	077	4 | 1.6
935	081	086	090	095	100	104	109	114	118	123	5 | 2.0
936	128	132	137	142	146	151	155	160	165̄	169	6 | 2.4
937	174	179	183	188	192	197	202	206	211	216	7 | 2.8
938	220	225̄	230	234	239	243	248	253	257	262	8 | 3.2
939	267	271	276	280	285	290	294	299	304	308	9 | 3.6
940	313	317	322	327	331	336	340	345	350̄	354	
941	359	364	368	373	377	382	387	391	396	400	
942	405	410	414	419	424	428	433	437	442	447	
943	451	456	460	465̄	470	474	479	483	488	493	
944	497	502	506	511	516	520	525̄	529	534	539	
945	543	548	552	557	562	566	571	575	580	585̄	
946	589	594	598	603	607	612	617	621	626	630	
947	635̄	640	644	649	653	658	663	667	672	676	
948	681	685	690	695̄	699	704	708	713	717	722	
949	727	731	736	740	745̄	749	754	759	763	768	
950	772	777	782	786	791	795	800	804	809	813	
N.	L. 0	1	2	3	4	5	6	7	8	9	Prop. Parts

N.	L. 0	1	2	3	4	5	6	7	8	9	Prop. Parts
950	97 772	777	782	786	791	795	800	804	809	813	
951	818	823	827	832	836	841	845	850	855	859	
952	864	868	873	877	882	886	891	896	900	905	
953	909	914	918	923	928	932	937	941	946	950	
954	955	959	964	968	973	978	982	987	991	996	
955	98 000	005	009	014	019	023	028	032	037	041	
956	046	050	055	059	064	068	073	078	082	087	
957	091	096	100	105	109	114	118	123	127	132	
958	137	141	146	150	155	159	164	168	173	177	
959	182	186	191	195	200	204	209	214	218	223	
960	227	232	236	241	245	250	254	259	263	268	5
961	272	277	281	286	290	295	299	304	308	313	1 0.5
962	318	322	327	331	336	340	345	349	354	358	2 1.0
963	363	367	372	376	381	385	390	394	399	403	3 1.5
964	408	412	417	421	426	430	435	439	444	448	4 2.0
965	453	457	462	466	471	475	480	484	489	493	5 2.5
966	498	502	507	511	516	520	525	529	534	538	6 3.0
967	543	547	552	556	561	565	570	574	579	583	7 3.5
968	588	592	597	601	605	610	614	619	623	628	8 4.0
969	632	637	641	646	650	655	659	664	668	673	9 4.5
970	677	682	686	691	695	700	704	709	713	717	
971	722	726	731	735	740	744	749	753	758	762	
972	767	771	776	780	784	789	793	798	802	807	
973	811	816	820	825	829	834	838	843	847	851	
974	856	860	865	869	874	878	883	887	892	896	
975	900	905	909	914	918	923	927	932	936	941	
976	945	949	954	958	963	967	972	976	981	985	
977	989	994	998	*003	*007	*012	*016	*021	*025	*029	
978	99 034	038	043	047	052	056	061	065	069	074	
979	078	083	087	092	096	100	105	109	114	118	
980	123	127	131	136	140	145	149	154	158	162	4
981	167	171	176	180	185	189	193	198	202	207	1 0.4
982	211	216	220	224	229	233	238	242	247	251	2 0.8
983	255	260	264	269	273	277	282	286	291	295	3 1.2
984	300	304	308	313	317	322	326	330	335	339	4 1.6
985	344	348	352	357	361	366	370	374	379	383	5 2.0
986	388	392	396	401	405	410	414	419	423	427	6 2.4
987	432	436	441	445	449	454	458	463	467	471	7 2.8
988	476	480	484	489	493	498	502	506	511	515	8 3.2
989	520	524	528	533	537	542	546	550	555	559	9 3.6
990	564	568	572	577	581	585	590	594	599	603	
991	607	612	616	621	625	629	634	638	642	647	
992	651	656	660	664	669	673	677	682	686	691	
993	695	699	704	708	712	717	721	726	730	734	
994	739	743	747	752	756	760	765	769	774	778	
995	782	787	791	795	800	804	808	813	817	822	
996	826	830	835	839	843	848	852	856	861	865	
997	870	874	878	883	887	891	896	900	904	909	
998	913	917	922	926	930	935	939	944	948	952	
999	957	961	965	970	974	978	983	987	991	996	
1000	00 000	004	009	013	017	022	026	030	035	039	
N.	L. 0	1	2	3	4	5	6	7	8	9	Prop. Parts

TABLE II

LOGARITHMS OF TRIGONOMETRIC FUNCTIONS

"	′	*l* sin	log *S*	*l* csc	*l* tan	log *T*	*l* cot	*l* sec	*l* cos	′
0	**0**	Inf. neg.	—	Infinite.	Inf. neg.	—	Infinite.	10 00000	10 00000	**60**
60	1	6.46373	5.31 443	13.53627	6 46373	5 31 443	13.53627	00000	00000	59
120	2	76476	5.31 443	23524	76476	5.31 443	23524	00000	00000	58
180	3	94085	5.31 443	05915	94085	5.31 443	05915	00000	00000	57
240	4	7.06579	5.31 443	12.93421	7 06579	5 31 442	12.93421	00000	00000	56
300	**5**	7.16270	5.31 443	12 83730	7.16270	5.31 442	12.83730	10.00000	10.00000	**55**
360	6	24188	5.31 443	75812	24188	5.31 442	75812	00000	00000	54
420	7	30882	5.31 443	69118	30882	5.31 442	69118	00000	00000	53
480	8	36682	5 31 443	63318	36682	5.31 442	63318	00000	00000	52
540	9	41797	5.31 443	58203	41797	5.31 442	58203	00000	00000	51
600	**10**	7.46373	5.31 443	12.53627	7.46373	5.31 442	12.53627	10.00000	10.00000	**50**
660	11	50512	5.31 443	49488	50512	5 31 442	49488	00000	00000	49
720	12	54291	5.31 443	45709	54291	5 31 442	45709	00000	00000	48
780	13	57767	5.31 443	42233	57767	5.31 442	42233	00000	00000	47
840	14	60985	5.31 443	39015	60986	5.31 442	39014	00000	00000	46
900	**15**	7.63982	5.31 443	12.36018	7 63982	5 31 442	12 36018	10 00000	10 00000	**45**
960	16	66784	5 31 443	33216	66785	5.31 442	33215	00000	00000	44
1020	17	69417	5.31 443	30583	69418	5 31 442	30582	00001	9 99999	43
1080	18	71900	5.31 443	28100	71900	5 31 442	28100	00001	99999	42
1140	19	74248	5.31 443	25752	74248	5 31 442	25752	00001	99999	41
1200	**20**	7 76475	5 31 443	12 23525	7.76476	5 31 442	12 23524	10 00001	9 99999	**40**
1260	21	78594	5 31 443	21406	78595	5.31 442	21405	00001	99999	39
1320	22	80615	5 31 443	19385	80615	5 31 442	19385	00001	99999	38
1380	23	82545	5 31 443	17455	82546	5 31 442	17454	00001	99999	37
1440	24	84393	5.31 443	15607	84394	5 31 442	15606	00001	99999	36
1500	**25**	7.86166	5 31 443	12.13834	7 86167	5 31 442	12 13833	10 00001	9 99999	**35**
1560	26	87870	5 31 443	12130	87871	5.31 442	12129	00001	99999	34
1620	27	89509	5 31 443	10491	89510	5 31 442	10490	00001	99999	33
1680	28	91088	5 31 443	08912	91089	5 31 442	08911	00001	99999	32
1740	29	92612	5 31 443	07388	92613	5 31 441	07387	00002	99998	31
1800	**30**	7 94084	5 31 443	12 05916	7 94086	5 31 441	12.05914	10 00002	9.99998	**30**
1860	31	95508	5 31 443	04492	95510	5 31 441	04490	00002	99998	29
1920	32	96887	5.31 443	03113	96889	5.31 441	03111	00002	99998	28
1980	33	98223	5.31 443	01777	98225	5 31 441	01775	00002	99998	27
2040	34	99520	5 31 443	00480	99522	5 31 441	00478	00002	99998	26
2100	**35**	8 00779	5.31 443	11 99221	8.00781	5 31 441	11 99219	10 00002	9 99998	**25**
2160	36	02002	5 31 443	97998	02004	5 31 441	97996	00002	99998	24
2220	37	03192	5 31 443	96808	03194	5 31 441	96806	00003	99997	23
2280	38	04350	5 31 443	95650	04353	5 31 441	95647	00003	99997	22
2340	39	05478	5 31 443	94522	05481	5 31 441	94519	00003	99997	21
2400	**40**	8 06578	5 31 443	11 93422	8 06581	5 31 441	11 93419	10 00003	9 99997	**20**
2460	41	07650	5 31 444	92350	07653	5.31 440	92347	00003	99997	19
2520	42	08696	5.31 444	91304	08700	5 31 440	91300	00003	99997	18
2580	43	09718	5 31 444	90282	09722	5.31 440	90278	00003	99997	17
2640	44	10717	5 31 444	89283	10720	5.31 440	89280	00004	99996	16
2700	**45**	8 11693	5.31 444	11.88307	8 11696	5 31 440	11 88304	10.00004	9.99996	**15**
2760	46	12647	5 31 444	87353	12651	5 31 440	87349	00004	99996	14
2820	47	13581	5.31 444	86419	13585	5 31 440	86415	00004	99996	13
2880	48	14495	5 31 444	85505	14500	5.31 440	85500	00004	99996	12
2940	49	15391	5.31 444	84609	15395	5 31 440	84605	00004	99996	11
3000	**50**	8.16268	5.31 444	11.83732	8.16273	5 31 439	11 83727	10.00005	9 99995	**10**
3060	51	17128	5.31 444	82872	17133	5 31 439	82867	00005	99995	9
3120	52	17971	5.31 444	82029	17976	5.31 439	82024	00005	99995	8
3180	53	18798	5.31 444	81202	18804	5.31 439	81196	00005	99995	7
3240	54	19610	5 31 444	80390	19616	5 31 439	80384	00005	99995	6
3300	**55**	8.20407	5.31 444	11.79593	8 20413	5.31 439	11 79587	10.00006	9 99994	**5**
3360	56	21189	5.31 444	78811	21195	5.31 439	78805	00006	99994	4
3420	57	21958	5 31 445	78042	21964	5.31 439	78036	00006	99994	3
3480	58	22713	5 31 445	77287	22720	5.31 438	77280	00006	99994	2
3540	59	23456	5 31 445	76544	23462	5.31 438	76538	00006	99994	1
3600	**60**	24186	5.31 445	75814	24192	5 31 438	75808	00007	99993	**0**
	′	*l* cos		*l* sec	*l* cot		*l* tan	*l* csc	*l* sin	′

″	′	l sin	log S	l csc	l tan	log T	l cot	l sec	l cos	′
3600	**0**	8.24186	5.31 445	11.75814	8.24192	5 31 438	11.75808	10.00007	9.99993	**60**
3660	1	24903	5.31 445	75097	24910	5.31 438	75090	00007	99993	59
3720	2	25609	5.31 445	74391	25616	5.31 438	74384	00007	99993	58
3780	3	26304	5 31 445	73696	26312	5 31 438	73688	00007	99993	57
3840	4	26988	5.31 445	73012	26996	5.31 437	73004	00008	99992	56
3900	**5**	8.27661	5.31 445	11.72339	8.27669	5.31 437	11.72331	10.00008	9.99992	**55**
3960	6	28324	5.31 445	71676	28332	5.31 437	71668	00008	99992	54
4020	7	28977	5.31 445	71023	28986	5.31 437	71014	00008	99992	53
4080	8	29621	5.31 445	70379	29629	5.31 437	70371	00008	99992	52
4140	9	30255	5 31 445	69745	30263	5.31 437	69737	00009	99991	51
4200	**10**	8.30879	5 31 446	11.69121	8.30888	5.31 437	11 69112	10 00009	9 99991	**50**
4260	11	31495	5 31 446	68505	31505	5.31 436	68495	00009	99991	49
4320	12	32103	5 31 446	67897	32112	5.31 436	67888	00010	99990	48
4380	13	32702	5.31 446	67298	32711	5 31 436	67289	00010	99990	47
4440	14	33292	5 31 446	66708	33302	5.31 436	66698	00010	99990	46
4500	**15**	8.33875	5 31 446	11.66125	8.33886	5 31 436	11.66114	10.00010	9 99990	**45**
4560	16	34450	5 31 446	65550	34461	5.31 435	65539	00011	99989	44
4620	17	35018	5.31 446	64982	35029	5.31 435	64971	00011	99989	43
4680	18	35578	5 31 446	64422	35590	5 31 435	64410	00011	99989	42
4740	19	36131	5.31 446	63869	36143	5.31 435	63857	00011	99989	41
4800	**20**	8.36678	5 31 446	11.63322	8 36689	5 31 435	11.63311	10.00012	9 99988	**40**
4860	21	37217	5 31 447	62783	37229	5.31 434	62771	00012	99988	39
4920	22	37750	5 31 447	62250	37762	5 31 434	62238	00012	99988	38
4980	23	38276	5 31 447	61724	38289	5 31 434	61711	00013	99987	37
5040	24	38796	5 31 447	61204	38809	5 31 434	61191	00013	99987	36
5100	**25**	8 39310	5 31 447	11.60690	8.39323	5 31 434	11.60677	10 00013	9.99987	**35**
5160	26	39818	5 31 447	60182	39832	5 31 433	60168	00014	99986	34
5220	27	40320	5.31 447	59680	40334	5 31 433	59666	00014	99986	33
5280	28	40816	5.31 447	59184	40830	5.31 433	59170	00014	99986	32
5340	29	41307	5 31 447	58693	41321	5 31 433	58679	00015	99985	31
5400	**30**	8 41792	5.31 447	11.58208	8 41807	5.31 433	11 58193	10 00015	9.99985	**30**
5460	31	42272	5.31 448	57728	42287	5 31 432	57713	00015	99985	29
5520	32	42746	5 31 448	57254	42762	5 31 432	57238	00016	99984	28
5580	33	43216	5 31 448	56784	43232	5 31 432	56768	00016	99984	27
5640	34	43680	5.31 448	56320	43696	5 31 432	56304	00016	99984	26
5700	**35**	8 44139	5 31 448	11 55861	8 44156	5 31 431	11.55844	10.00017	9.99983	**25**
5760	36	44594	5 31 448	55406	44611	5 31 431	55389	00017	99983	24
5820	37	45044	5 31 448	54956	45061	5 31 431	54939	00017	99983	23
5880	38	45489	5 31 448	54511	45507	5 31 431	54493	00018	99982	22
5940	39	45930	5 31 449	54070	45948	5 31 431	54052	00018	99982	21
6000	**40**	8 46366	5.31 449	11.53634	8 46385	5 31 430	11 53615	10 00018	9.99982	**20**
6060	41	46799	5 31 449	53201	46817	5 31 430	53183	00019	99981	19
6120	42	47226	5.31 449	52774	47245	5 31 430	52755	00019	99981	18
6180	43	47650	5.31 449	52350	47669	5.31 430	52331	00019	99981	17
6240	44	48069	5 31 449	51931	48089	5 31 429	51911	00020	99980	16
6300	**45**	8 48485	5.31 449	11.51515	8 48505	5.31 429	11 51495	10 00020	9 99980	**15**
6360	46	48896	5.31 449	51104	48917	5.31 429	51083	00021	99979	14
6420	47	49304	5.31 450	50696	49325	5 31 428	50675	00021	99979	13
6480	48	49708	5 31 450	50292	49729	5 31 428	50271	00021	99979	12
6540	49	50108	5.31 450	49892	50130	5.31 428	49870	00022	99978	11
6600	**50**	8.50504	5 31 450	11.49496	8 50527	5.31 428	11.49473	10 00022	9.99978	**10**
6660	51	50897	5 31 450	49103	50920	5.31 427	49080	00023	99977	9
6720	52	51287	5.31 450	48713	51310	5 31 427	48690	00023	99977	8
6780	53	51673	5.31 450	48327	51696	5 31 427	48304	00023	99977	7
6840	54	52055	5 31 450	47945	52079	5.31 427	47921	00024	99976	6
6900	**55**	8.52434	5.31 451	11.47566	8.52459	5.31 426	11.47541	10.00024	9.99976	**5**
6960	56	52810	5 31 451	47190	52835	5.31 426	47165	00025	99975	4
7020	57	53183	5.31 451	46817	53208	5.31 426	46792	00025	99975	3
7080	58	53552	5 31 451	46448	53578	5 31 425	46422	00026	99974	2
7140	59	53919	5.31 451	46081	53945	5.31 425	46055	00026	99974	1
7200	**60**	54282	5.31 451	45718	54308	5.31 425	45692	00026	99974	**0**
	′	l cos		l sec	l cot		l tan	l csc	l sin	′

91°　　88°

"	'	*l* sin	log *S*	*l* csc	*l* tan	log *T*	*l* cot	*l* sec	d 1'	*l* cos	'
7200	**0**	8.54282	5.31 451	11 45718	8 54308	5.31 425	11.45692	10 00026	1	9.99974	**60**
7260	1	54642	5 31 451	45358	54669	5 31 425	45331	00027	0	99973	59
7320	2	54999	5 31 452	45001	55027	5.31 424	44973	00027	1	99973	58
7380	3	55354	5 31 452	44646	55382	5 31 424	44618	00028	0	99972	57
7440	4	55705	5 31 452	44295	55734	5.31 424	44266	00028	1	99972	56
7500	**5**	8.56054	5.31 452	11 43946	8 56083	5 31 423	11 43917	10 00029	0	9 99971	**55**
7560	6	56400	5 31 452	43600	56429	5.31 423	43571	00029	1	99971	54
7620	7	56743	5 31 452	43257	56773	5 31 423	43227	00030	0	99970	53
7680	8	57084	5 31 453	42916	57114	5 31 422	42886	00030	1	99970	52
7740	9	57421	5 31 453	42579	57452	5 31 422	42548	00031	0	99969	51
7800	**10**	8 57757	5 31 453	11 42243	8 57788	5 31 422	11 42212	10 00031	1	9 99969	**50**
7860	11	58089	5 31 453	41911	58121	5.31 421	41879	00032	0	99968	49
7920	12	58419	5 31 453	41581	58451	5.31 421	41549	00032	1	99968	48
7980	13	58747	5 31 453	41253	58779	5.31 421	41221	00033	0	99967	47
8040	14	59072	5 31 454	40928	59105	5 31 421	40895	00033	0	99967	46
8100	**15**	8.59395	5 31 454	11 40605	8 59428	5.31 420	11.40572	10 00033	1	9 99967	**45**
8160	16	59715	5 31 454	40285	59749	5 31 420	40251	00034	0	99966	44
8220	17	60033	5 31 454	39967	60068	5 31 420	39932	00034	1	99966	43
8280	18	60349	5 31 454	39651	60384	5 31 419	39616	00035	1	99965	42
8340	19	60662	5 31 454	39338	60698	5 31 419	39302	00036	0	99964	41
8400	**20**	8 60973	5 31 455	11 39027	8 61009	5 31 418	11 38991	10 00036	1	9 99964	**40**
8460	21	61282	5 31 455	38718	61319	5 31 418	38681	00037	0	99963	39
8520	22	61589	5 31 455	38411	61626	5 31 418	38374	00037	1	99963	38
8580	23	61894	5 31 455	38106	61931	5 31 417	38069	00038	0	99962	37
8640	24	62196	5 31 455	37804	62234	5 31 417	37766	00038	1	99962	36
8700	**25**	8.62497	5 31 455	11 37503	8 62535	5 31 417	11 37465	10 00039	0	9 99961	**35**
8760	26	62795	5 31 456	37205	62834	5 31 416	37166	00039	1	99961	34
8820	27	63091	5 31 456	36909	63131	5 31 416	36869	00040	0	99960	33
8880	28	63385	5 31 456	36615	63426	5 31 416	36574	00040	1	99960	32
8940	29	63678	5 31 456	36322	63718	5 31 415	36282	00041	0	99959	31
9000	**30**	8 63968	5 31 456	11 36032	8 64009	5 31 415	11 35991	10 00041	1	9 99959	**30**
9060	31	64256	5 31 456	35744	64298	5 31 415	35702	00042	0	99958	29
9120	32	64543	5 31 457	35457	64585	5 31 414	35415	00042	1	99958	28
9180	33	64827	5 31 457	35173	64870	5.31 414	35130	00043	1	99957	27
9240	34	65110	5 31 457	34890	65154	5 31 413	34846	00044	0	99956	26
9300	**35**	8 65391	5 31 457	11 34609	8 65435	5 31 413	11 34565	10 00044	1	9 99956	**25**
9360	36	65670	5 31 457	34330	65715	5 31 413	34285	00045	0	99955	24
9420	37	65947	5 31 458	34053	65993	5 31 412	34007	00045	1	99955	23
9480	38	66223	5 31 458	33777	66269	5 31 412	33731	00046	0	99954	22
9540	39	66497	5 31 458	33503	66543	5 31 412	33457	00046	1	99954	21
9600	**40**	8 66769	5 31 458	11 33231	8 66816	5.31 411	11 33184	10 00047	1	9 99953	**20**
9660	41	67039	5 31 458	32961	67087	5 31 411	32913	00048	0	99952	19
9720	42	67308	5 31 459	32692	67356	5.31 410	32644	00048	1	99952	18
9780	43	67575	5 31 459	32425	67624	5 31 410	32376	00049	0	99951	17
9840	44	67841	5 31 459	32159	67890	5 31 410	32110	00049	1	99951	16
9900	**45**	8 68104	5 31 459	11 31896	8 68154	5 31 409	11 31846	10 00050	1	9 99950	**15**
9960	46	68367	5 31 459	31633	68417	5 31 409	31583	00051	0	99949	14
10020	47	68627	5.31 460	31373	68678	5 31 408	31322	00051	1	99949	13
10080	48	68886	5 31 460	31114	68938	5 31 408	31062	00052	0	99948	12
10140	49	69144	5.31 460	30856	69196	5 31 408	30804	00052	1	99948	11
10200	**50**	8 69400	5.31 460	11 30600	8 69453	5 31 407	11 30547	10 00053	1	9 99947	**10**
10260	51	69654	5.31 460	30346	69708	5 31 407	30292	00054	0	99946	9
10320	52	69907	5.31 461	30093	69962	5 31 406	30038	00054	1	99946	8
10380	53	70159	5.31 461	29841	70214	5 31 406	29786	00055	1	99945	7
10440	54	70409	5 31 461	29591	70465	5.31 405	29535	00056	0	99944	6
10500	**55**	8 70658	5.31 461	11 29342	8 70714	5 31 405	11 29286	10 00056	1	9 99944	**5**
10560	56	70905	5 31 461	29095	70962	5 31 405	29038	00057	1	99943	4
10620	57	71151	5 31 462	28849	71208	5 31 404	28792	00058	0	99942	3
10680	58	71395	5 31 462	28605	71453	5 31 404	28547	00058	1	99942	2
10740	59	71638	5 31 462	28362	71697	5 31 403	28303	00059	1	99941	1
10800	**60**	71880	5.31 462	28120	71940	5 31 403	28060	00060		99940	**0**
	'	*l* cos		*l* sec	*l* cot		*l* tan	*l* csc	d 1'	*l* sin	'

92° 87°

′	*l* sin 8.	d 1′	*l* csc 11.	*l* tan 8.	d 1′	*l* cot 11.	*l* sec 10.	d 1′	*l* cos 9.	′
0	**71880**	240	**28120**	**71940**	241	**28060**	**00060**	0	**99940**	**60**
1	**72120**	239	**27880**	**72181**	239	**27819**	060	1	940	59
2	359	238	641	420	239	580	061	1	939	58
3	597	237	403	659	237	341	062	0	938	57
4	834	235	166	896	236	104	062	1	938	56
5	**73069**	234	**26931**	**73132**	234	**26868**	063	1	937	**55**
6	303	232	697	366	234	634	064	0	936	54
7	535	232	465	600	232	400	064	1	936	53
8	767	230	233	832	231	168	065	1	935	52
9	997	229	003	**74063**	229	**25937**	066	0	934	51
10	**74226**	228	**25774**	292	229	708	066	1	934	**50**
11	454	226	546	521	227	479	067	1	933	49
12	680	226	320	748	226	252	068	0	932	48
13	906	224	094	974	225	026	068	1	932	47
14	**75130**	223	**24870**	**75199**	224	**24801**	069	1	931	46
15	353	222	647	423	222	577	070	1	930	**45**
16	575	220	425	645	222	355	071	0	929	44
17	795	220	205	867	220	133	071	1	929	43
18	**76015**	219	**23985**	**76087**	219	**23913**	072	1	928	42
19	234	217	766	306	219	694	073	1	927	41
20	451	216	549	525	217	475	074	0	926	**40**
21	667	216	333	742	216	258	074	1	926	39
22	883	214	117	958	215	042	075	1	925	38
23	**77097**	213	**22903**	**77173**	214	**22827**	076	1	924	37
24	310	212	690	387	213	613	077	0	923	36
25	522	211	478	600	211	400	077	1	923	**35**
26	733	210	267	811	211	189	078	1	922	34
27	943	209	057	**78022**	210	**21978**	079	1	921	33
28	**78152**	208	**21848**	232	209	768	080	0	920	32
29	360	208	640	441	208	559	080	1	920	31
30	**78568**	206	**21432**	**78649**	206	**21351**	**00081**	1	**99919**	**30**
31	774	205	226	855	206	145	082	1	918	29
32	979	204	021	**79061**	205	**20939**	083	0	917	28
33	**79183**	203	**20817**	266	204	734	083	1	917	27
34	386	202	614	470	203	530	084	1	916	26
35	588	201	412	673	202	327	085	1	915	**25**
36	789	201	211	875	201	125	086	1	914	24
37	990	199	010	**80076**	201	**19924**	087	0	913	23
38	**80189**	199	**19811**	277	199	723	087	1	913	22
39	388	197	612	476	198	524	088	1	912	21
40	585	197	415	674	198	326	089	1	911	**20**
41	782	196	218	872	196	128	090	1	910	19
42	978	195	022	**81068**	196	**18932**	091	0	909	18
43	**81173**	194	**18827**	264	195	736	091	1	909	17
44	367	193	633	459	194	541	092	1	908	16
45	560	192	440	653	193	347	093	1	907	**15**
46	752	192	248	846	192	154	094	1	906	14
47	944	190	056	**82038**	192	**17962**	095	1	905	13
48	**82134**	190	**17866**	230	190	770	096	0	904	12
49	324	189	676	420	190	580	096	1	904	11
50	513	188	487	610	189	390	097	1	903	**10**
51	701	187	299	799	188	201	098	1	902	9
52	888	187	112	987	188	013	099	1	901	8
53	**83075**	186	**16925**	**83175**	186	**16825**	100	1	900	7
54	261	185	739	361	186	639	101	1	899	6
55	446	184	554	547	185	453	102	0	898	**5**
56	630	183	370	732	184	268	102	1	898	4
57	813	183	187	916	184	084	103	1	897	3
58	996	181	004	**84100**	182	**15900**	104	1	896	2
59	**84177**	181	**15823**	282	182	718	105	1	895	1
60	**84358**		**15642**	**84464**		**15536**	**00106**		**99894**	**0**
′	8. *l* cos	d 1′	11. *l* sec	8. *l* cot	d 1′	11. *l* tan	10. *l* csc	d 1′	9. *l* sin	′

″	Proportional Parts 241	239	237	235	234	232	229
0	0	0	0	0	0	0	0
1	4	4	4	4	4	4	4
2	8	8	8	8	8	8	8
3	12	12	12	12	12	12	11
4	16	16	16	16	16	15	15
5	20	20	20	20	19	19	19
6	24	24	24	24	23	23	23
7	28	28	28	27	27	27	27
8	32	32	32	31	31	31	31
9	36	36	36	35	35	35	34
10	40	40	40	39	39	39	38
11	44	44	43	43	43	43	42
12	48	48	47	47	47	46	46
13	52	52	51	51	51	50	50
14	56	56	55	55	55	54	53
15	60	60	59	59	59	58	57
16	64	64	63	63	62	62	61
17	68	68	67	67	66	66	65
18	72	72	71	70	70	70	69
19	76	76	75	74	74	73	73
20	80	80	79	78	78	77	76
21	84	84	83	82	82	81	80
22	88	88	87	86	86	85	84
23	92	92	91	90	90	89	88
24	96	96	95	94	94	93	92
25	100	100	99	98	97	97	95
26	104	104	103	102	101	101	99
27	108	108	107	106	105	104	103
28	112	112	111	110	109	108	107
29	116	116	115	114	113	112	111
30	120	120	118	118	117	116	114
31	125	123	122	121	121	120	118
32	129	127	126	125	125	124	122
33	133	131	130	129	129	128	126
34	137	135	134	133	133	131	130
35	141	139	138	137	137	135	134
36	145	143	142	141	140	139	137
37	149	147	146	145	144	143	141
38	153	151	150	149	148	147	145
39	157	155	154	153	152	151	149
40	161	159	158	157	156	155	153
41	165	163	162	161	160	159	156
42	169	167	166	164	164	162	160
43	173	171	170	168	168	166	164
44	177	175	174	172	172	170	168
45	181	179	178	176	175	174	172
46	185	183	182	180	179	178	176
47	189	187	186	184	183	182	179
48	193	191	190	188	187	186	183
49	197	195	194	192	191	189	187
50	201	199	198	196	195	193	191
51	205	203	201	200	199	197	195
52	209	207	205	204	203	201	198
53	213	211	209	208	207	205	202
54	217	215	213	212	211	209	206
55	221	219	217	215	215	213	210
56	225	223	221	219	218	217	214
57	229	227	225	223	222	220	218
58	233	231	229	227	226	224	221
59	237	235	233	231	230	228	225
60	241	239	237	235	234	232	229
″	241	239	237	235	234	232	229

Proportional Parts

TABLE II

Proportional Parts

″	227	225	223	220	217	215	213	211	208	206	203	201	199	197	195	193	192	189	187	185	183	181
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	4	4	4	4	4	4	4	4	3	3	3	3	3	3	3	3	3	3	3	3	3	3
2	8	8	7	7	7	7	7	7	7	7	7	7	7	7	6	6	6	6	6	6	6	6
3	11	11	11	11	11	11	11	11	10	10	10	10	10	10	10	10	10	9	9	9	9	9
4	15	15	15	15	14	14	14	14	14	14	14	13	13	13	13	13	13	13	12	12	12	12
5	19	19	19	18	18	18	18	18	17	17	17	17	17	16	16	16	16	16	16	15	15	15
6	23	22	22	22	22	22	21	21	21	21	20	20	20	20	20	19	19	19	19	18	18	18
7	26	26	26	26	25	25	25	25	24	24	24	23	23	23	23	23	22	22	22	22	21	21
8	30	30	30	29	29	29	28	28	28	27	27	27	27	26	26	26	26	25	25	25	24	24
9	34	34	33	33	33	32	32	32	31	31	30	30	30	30	29	29	29	28	28	28	27	27
10	38	38	37	37	36	36	36	35	35	34	34	34	33	33	32	32	32	32	31	31	30	30
11	42	41	41	40	40	39	39	39	38	38	37	37	36	36	36	35	35	35	34	34	34	33
12	45	45	45	44	43	43	43	42	42	41	41	40	40	39	39	39	38	38	37	37	37	36
13	49	49	48	48	47	47	46	46	45	45	44	44	43	43	42	42	42	41	41	40	40	39
14	53	52	52	51	51	50	50	49	49	48	47	47	46	46	46	45	45	44	44	43	43	42
15	57	56	56	55	54	54	53	53	52	51	51	50	50	49	49	48	48	47	47	46	46	45
16	61	60	59	59	58	57	57	56	55	55	54	54	53	53	52	51	51	50	50	49	49	48
17	64	64	63	62	61	61	60	60	59	58	58	57	56	56	55	55	54	54	53	52	52	51
18	68	68	67	66	65	64	64	63	62	62	61	60	60	59	58	58	58	57	56	56	55	54
19	72	71	71	70	69	68	67	67	66	65	64	64	63	62	62	61	61	60	59	59	58	57
20	76	75	74	73	72	72	71	70	69	69	68	67	66	66	65	64	64	63	62	62	61	60
21	79	79	78	77	76	75	75	74	73	72	71	70	70	69	68	68	67	66	65	65	64	63
22	83	82	82	81	80	79	78	77	76	76	74	74	73	72	72	71	70	69	69	68	67	66
23	87	86	85	84	83	82	82	81	80	79	78	77	76	76	75	74	74	72	72	71	70	69
24	91	90	89	88	87	86	85	84	83	82	81	80	80	79	78	77	77	76	75	74	73	72
25	95	94	93	92	90	90	89	88	87	86	85	84	83	82	81	80	80	79	78	77	76	75
26	98	98	97	95	94	93	92	91	90	89	88	87	86	85	84	84	83	82	81	80	79	78
27	102	101	100	99	98	97	96	95	94	93	91	90	90	89	88	87	86	85	84	83	82	81
28	106	105	104	103	101	100	99	98	97	96	95	94	93	92	91	90	90	88	87	86	85	84
29	110	109	108	106	105	104	103	102	101	100	98	97	96	95	94	93	93	91	90	89	88	87
30	114	112	112	110	108	108	106	106	104	103	102	100	100	98	98	96	96	94	94	92	92	90
31	117	116	115	114	112	111	110	109	107	106	105	104	103	102	101	100	99	98	97	96	95	94
32	121	120	119	117	116	115	114	113	111	110	108	107	106	105	104	103	102	101	100	99	98	97
33	125	124	123	121	119	118	117	116	114	113	112	111	109	108	107	106	106	104	103	102	101	100
34	129	128	126	125	123	122	121	120	118	117	115	114	113	112	110	109	109	107	106	105	104	103
35	132	131	130	128	127	125	124	123	121	120	118	117	116	115	114	113	112	110	109	108	107	106
36	136	135	134	132	130	129	128	127	125	124	122	121	119	118	117	116	115	113	112	111	110	109
37	140	139	138	136	134	133	131	130	128	127	125	124	123	121	120	119	118	117	115	114	113	112
38	144	142	141	139	137	136	135	134	132	130	129	127	126	125	124	122	122	120	118	117	116	115
39	148	146	145	143	141	140	138	137	135	134	132	131	129	128	127	125	125	123	122	120	119	118
40	151	150	149	147	145	143	142	141	139	137	135	134	133	131	130	129	128	126	125	123	122	121
41	155	154	152	150	148	147	146	144	142	141	139	137	136	135	133	132	131	129	128	126	125	124
42	159	158	156	154	152	150	149	148	146	144	142	141	139	138	136	135	134	132	131	130	128	127
43	163	161	160	158	156	154	153	151	149	148	145	144	143	141	140	138	138	135	134	133	131	130
44	166	165	164	161	159	158	156	155	153	151	149	147	146	144	143	142	141	139	137	136	134	133
45	170	169	167	165	163	161	160	158	156	155	152	151	149	148	146	145	144	142	140	139	137	136
46	174	172	171	169	166	165	163	162	159	158	156	154	153	151	150	148	147	145	143	142	140	139
47	178	176	175	172	170	168	167	165	163	161	159	157	156	154	153	151	150	148	146	145	143	142
48	182	180	178	176	174	172	170	169	166	165	162	161	159	158	156	154	154	151	150	148	146	145
49	185	184	182	180	177	176	174	172	170	168	166	164	163	161	159	158	157	154	153	151	149	148
50	189	188	186	183	181	179	178	176	173	172	169	168	166	164	162	161	160	158	156	154	152	151
51	193	191	190	187	184	183	181	179	177	175	173	171	169	167	166	164	163	161	159	157	156	154
52	197	195	193	191	188	186	185	183	180	179	176	174	172	171	169	167	166	164	162	160	159	157
53	201	199	197	194	192	190	188	186	184	182	179	178	176	174	172	170	170	167	165	163	162	160
54	204	202	201	198	195	194	192	190	187	185	183	181	179	177	176	174	173	170	168	166	165	163
55	208	206	204	202	199	197	195	193	191	189	186	184	182	181	179	177	176	173	171	170	168	166
56	212	210	208	205	203	201	199	197	194	192	189	188	186	184	182	180	179	176	175	173	171	169
57	216	214	212	209	206	204	202	200	198	196	193	191	189	187	185	183	182	180	178	176	174	172
58	219	218	216	213	210	208	206	204	201	199	196	194	192	190	188	187	186	183	181	179	177	175
59	223	221	219	216	213	211	209	207	205	203	200	198	196	194	192	190	189	186	184	182	180	178
60	227	225	223	220	217	215	213	211	208	206	203	201	199	197	195	193	192	189	187	185	183	181
″	**227**	**225**	**223**	**220**	**217**	**215**	**213**	**211**	**208**	**206**	**203**	**201**	**199**	**197**	**195**	**193**	**192**	**189**	**187**	**185**	**183**	**181**

Proportional Parts

′	*l* sin	d	*l* csc	*l* tan	d	*l* cot	*l* sec	d	*l* cos	′
	8.	1′	11.	8.	1′	11.	10.	1′	9.	
0	84358	181	15642	84464	182	15536	00106	1	99894	60
1	539	179	461	646	180	354	107	1	893	59
2	718	179	282	826	180	174	108	1	892	58
3	897	178	103	85006	179	14994	109	0	891	57
4	85075	177	14925	185	178	815	109	1	891	56
5	252	177	748	363	177	637	110	1	890	55
6	429	176	571	540	177	460	111	1	889	54
7	605	175	395	717	176	283	112	1	888	53
8	780	175	220	893	176	107	113	1	887	52
9	955	173	045	86069	174	13931	114	1	886	51
10	86128	173	13872	243	174	757	115	1	885	50
11	301	173	699	417	174	583	116	1	884	49
12	474	171	526	591	172	409	117	1	883	48
13	645	171	355	763	172	237	118	1	882	47
14	816	171	184	935	171	065	119	1	881	46
15	987	169	013	87106	171	12894	120	1	880	45
16	87156	169	12844	277	170	723	121	0	879	44
17	325	169	675	447	169	553	121	1	879	43
18	494	167	506	616	169	384	122	1	878	42
19	661	168	339	785	168	215	123	1	877	41
20	829	166	171	953	167	047	124	1	876	40
21	995	166	005	88120	167	11880	125	1	875	39
22	88161	165	11839	287	166	713	126	1	874	38
23	326	164	674	453	165	547	127	1	873	37
24	490	164	510	618	165	382	128	1	872	36
25	654	163	346	783	165	217	129	1	871	35
26	817	163	183	948	163	052	130	1	870	34
27	980	162	020	89111	163	10889	131	1	869	33
28	89142	162	10858	274	163	726	132	1	868	32
29	304	160	696	437	161	563	133	1	867	31
30	89464	161	10536	89598	162	10402	00134	1	99866	30
31	625	159	375	760	160	240	135	1	865	29
32	784	159	216	920	160	080	136	1	864	28
33	943	159	057	90080	160	09920	137	1	863	27
34	90102	158	09898	240	159	760	138	1	862	26
35	260	157	740	399	158	601	139	1	861	25
36	417	157	583	557	158	443	140	1	860	24
37	574	156	426	715	157	285	141	1	859	23
38	730	155	270	872	157	128	142	1	858	22
39	885	155	115	91029	156	08971	143	1	857	21
40	91040	155	08960	185	155	815	144	1	856	20
41	195	154	805	340	155	660	145	1	855	19
42	349	153	651	495	155	505	146	1	854	18
43	502	153	498	650	153	350	147	1	853	17
44	655	152	345	803	154	197	148	1	852	16
45	807	152	193	957	153	043	149	1	851	15
46	959	151	041	92110	152	07890	150	2	850	14
47	92110	151	07890	262	152	738	152	1	848	13
48	261	150	739	414	151	586	153	1	847	12
49	411	150	589	565	151	435	154	1	846	11
50	561	149	439	716	150	284	155	1	845	10
51	710	149	290	866	150	134	156	1	844	9
52	859	148	141	93016	149	06984	157	1	843	8
53	93007	147	06993	165	148	835	158	1	842	7
54	154	147	846	313	149	687	159	1	841	6
55	301	147	699	462	147	538	160	1	840	5
56	448	146	552	609	147	391	161	1	839	4
57	594	146	406	756	147	244	162	1	838	3
58	740	145	260	903	146	097	163	1	837	2
59	885	145	115	94049	146	05951	164	2	836	1
60	94030		05970	94195		05805	00166		99834	0
′	8.	d	11.	8.	d	11.	10.	d	9.	′
	l cos	1′	*l* sec	*l* cot	1′	*l* tan	*l* csc	1′	*l* sin	

″	Proportional Parts						
	182	181	179	177	176	175	174
0	0	0	0	0	0	0	0
1	3	3	3	3	3	3	3
2	6	6	6	6	6	6	6
3	9	9	9	9	9	9	9
4	12	12	12	12	12	12	12
5	15	15	15	15	15	15	14
6	18	18	18	18	18	18	17
7	21	21	21	21	21	20	20
8	24	24	24	24	23	23	23
9	27	27	27	27	26	26	26
10	30	30	30	30	29	29	29
11	33	33	33	32	32	32	32
12	36	36	36	35	35	35	35
13	39	39	39	38	38	38	38
14	42	42	42	41	41	41	41
15	45	45	45	44	44	44	44
16	49	48	48	47	47	47	46
17	52	51	51	50	50	50	49
18	55	54	54	53	53	52	52
19	58	57	57	56	56	55	55
20	61	60	60	59	59	58	58
21	64	63	63	62	62	61	61
22	67	66	66	65	65	64	64
23	70	69	69	68	67	67	67
24	73	72	72	71	70	70	70
25	76	75	75	74	73	73	72
26	79	78	78	77	76	76	75
27	82	81	81	80	79	79	78
28	85	84	84	83	82	82	81
29	88	87	87	86	85	85	84
30	91	90	90	88	88	88	87
31	94	94	92	91	91	90	90
32	97	97	95	94	94	93	93
33	100	100	98	97	97	96	96
34	103	103	101	100	100	99	99
35	106	106	104	103	103	102	102
36	109	109	107	106	106	105	104
37	112	112	110	109	109	108	107
38	115	115	113	112	111	111	110
39	118	118	116	115	114	114	113
40	121	121	119	118	117	117	116
41	124	124	122	121	120	120	119
42	127	127	125	124	123	122	122
43	130	130	128	127	126	125	125
44	133	133	131	130	129	128	128
45	137	136	134	133	132	131	130
46	140	139	137	136	135	134	133
47	143	142	140	139	138	137	136
48	146	145	143	142	141	140	139
49	149	148	146	145	144	143	142
50	152	151	149	148	147	146	145
51	155	154	152	150	150	149	148
52	158	157	155	153	153	152	151
53	161	160	158	156	155	155	154
54	164	163	161	159	158	158	157
55	167	166	164	162	161	160	160
56	170	169	167	165	164	163	162
57	173	172	170	168	167	166	165
58	176	175	173	171	170	169	168
59	179	178	176	174	173	172	171
60	182	181	179	177	176	175	174
″	182	181	179	177	176	175	174
	Proportional Parts						

TABLE II

″	173	172	171	169	167	166	165	163	162	160	159	158	157	155	153	152	151	150	149	147	146	145
	Proportional Parts																					
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	2	2	2	2	2
2	6	6	6	6	6	6	6	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
3	9	9	9	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	7	7	7	7
4	12	11	11	11	11	11	11	11	11	11	11	11	10	10	10	10	10	10	10	10	10	10
5	14	14	14	14	14	14	14	14	14	13	13	13	13	13	13	13	13	12	12	12	12	12
6	17	17	17	17	17	17	16	16	16	16	16	16	16	16	15	15	15	15	15	15	15	14
7	20	20	20	20	19	19	19	19	19	19	19	18	18	18	18	18	18	18	17	17	17	17
8	23	23	23	23	22	22	22	22	22	21	21	21	21	21	20	20	20	20	20	20	19	19
9	26	26	26	25	25	25	25	24	24	24	24	24	24	23	23	23	23	22	22	22	22	22
10	29	29	28	28	28	28	28	27	27	27	[illegible]	26	26	26	26	25	25	25	25	24	24	24
11	32	32	31	31	31	30	30	30	30	29	29	29	29	28	28	28	28	28	27	27	27	27
12	35	34	34	34	33	33	33	33	32	32	32	32	31	31	31	30	30	30	30	29	29	29
13	37	37	37	37	36	36	36	35	35	35	34	34	34	34	33	33	33	32	32	32	32	31
14	40	40	40	39	39	39	38	38	38	37	37	37	37	36	36	35	35	35	35	34	34	34
15	43	43	43	42	42	42	41	41	40	40	40	40	39	39	38	38	38	38	37	37	36	36
16	46	46	46	45	45	44	44	43	43	43	42	42	42	41	41	41	40	40	40	39	39	39
17	49	49	48	48	47	47	47	46	46	45	45	45	44	44	43	43	43	42	42	42	41	41
18	52	52	51	51	50	50	50	49	49	48	48	47	47	46	46	46	45	45	45	44	44	44
19	55	54	54	54	53	53	52	52	51	51	50	50	50	49	48	48	48	48	47	47	46	46
20	58	57	57	56	56	55	55	54	54	53	53	53	52	52	51	51	50	50	50	49	49	48
21	61	60	60	59	58	58	58	57	57	56	56	55	55	54	54	53	53	52	52	51	51	51
22	63	63	63	62	61	61	60	60	59	59	58	58	58	57	56	56	55	55	55	54	54	53
23	66	66	66	65	64	64	63	62	62	61	61	61	60	59	59	58	58	58	57	56	56	56
24	69	69	68	68	67	66	66	65	65	64	64	63	63	62	61	61	60	60	60	59	58	58
25	72	72	71	70	70	69	69	68	68	67	66	66	65	65	64	63	63	62	62	61	61	60
26	75	75	74	73	72	72	72	71	70	69	69	68	68	67	66	66	65	65	65	64	63	63
27	78	77	77	76	75	75	74	73	73	72	72	71	71	70	69	68	68	68	67	66	66	65
28	81	80	80	79	78	77	77	76	76	75	74	74	73	72	71	71	70	70	70	69	68	68
29	84	83	83	82	81	80	80	79	78	77	77	76	76	75	74	73	73	72	72	71	71	70
30	86	86	86	84	84	83	82	82	81	80	80	79	78	78	76	76	76	75	74	74	73	72
31	89	89	88	87	86	86	85	84	84	83	82	82	81	80	79	79	78	78	77	76	75	75
32	92	92	91	90	89	89	88	87	86	85	85	84	84	83	82	81	81	80	79	78	78	77
33	95	95	94	93	92	91	91	90	89	88	87	87	86	85	84	84	83	82	82	81	80	80
34	98	97	97	96	95	94	94	92	92	91	90	90	89	88	87	86	86	85	84	83	83	82
35	101	100	100	99	97	97	96	95	94	93	93	92	92	90	89	89	88	88	87	86	85	85
36	104	103	103	101	100	100	99	98	97	96	95	95	94	93	92	91	91	90	89	88	88	87
37	107	106	105	104	103	102	102	101	100	99	98	97	97	96	94	94	93	92	92	91	90	89
38	110	109	108	107	106	105	104	103	103	101	101	100	99	98	97	96	96	95	94	93	92	92
39	112	112	111	110	109	108	107	106	105	104	103	103	102	101	99	99	98	98	97	96	95	94
40	115	115	114	113	111	111	110	109	108	107	106	105	105	103	102	101	101	100	99	98	97	97
41	118	118	117	115	114	113	113	111	111	109	109	108	107	106	105	104	103	102	102	100	100	99
42	121	120	120	118	117	116	116	114	113	112	111	111	110	108	107	106	106	105	104	103	102	102
43	124	123	123	121	120	119	118	117	116	115	114	113	113	111	110	109	108	108	107	105	105	104
44	127	126	125	124	122	122	121	120	119	117	117	116	115	114	112	111	111	110	109	108	107	106
45	130	129	128	127	125	124	124	122	122	120	119	118	118	116	115	114	113	112	112	110	110	109
46	133	132	131	130	128	127	126	125	124	123	122	121	120	119	117	117	116	115	114	113	112	111
47	136	135	134	132	131	130	129	128	127	125	125	124	123	121	120	119	118	118	117	115	114	114
48	138	138	137	135	134	133	132	130	130	128	127	126	126	124	122	122	121	120	119	118	117	116
49	141	140	140	138	136	136	135	133	132	131	130	129	128	127	125	124	123	122	122	120	119	118
50	144	143	142	141	139	138	138	136	135	133	132	132	131	129	128	127	126	125	124	122	122	121
51	147	146	145	144	142	141	140	139	138	136	135	134	133	132	130	129	128	128	127	125	124	123
52	150	149	148	146	145	144	143	141	140	139	138	137	136	134	133	132	131	130	129	127	127	126
53	153	152	151	149	148	147	146	144	143	141	140	140	139	137	135	134	133	132	132	130	129	128
54	156	155	154	152	150	149	148	147	146	144	143	142	141	140	138	137	136	135	134	132	131	130
55	159	158	157	155	153	152	151	149	148	147	146	145	144	142	140	139	138	138	137	135	134	133
56	161	161	160	158	156	155	154	152	151	149	148	147	147	145	143	142	141	140	139	137	136	135
57	164	163	162	161	159	158	157	155	154	152	151	150	149	147	145	144	143	142	142	140	139	138
58	167	166	165	163	161	160	160	158	157	155	154	153	152	150	148	147	146	145	144	142	141	140
59	170	169	168	166	164	163	162	160	159	157	156	155	154	152	150	149	148	148	147	145	144	143
60	173	172	171	169	167	166	165	163	162	160	159	158	157	155	153	152	151	150	149	147	146	145
″	**173**	**172**	**171**	**169**	**167**	**166**	**165**	**163**	**162**	**160**	**159**	**158**	**157**	**155**	**153**	**152**	**151**	**150**	**149**	**147**	**146**	**145**
	Proportional Parts																					

′	l sin 8.	d 1′	l csc 11.	l tan 8.	d 1′	l cot 11.	l sec 10.	d 1′	l cos 9.	′
0	94030	144	05970	94195	145	05805	00166	1	99834	60
1	174	143	826	340	145	660	167	1	833	59
2	317	144	683	485	145	515	168	1	832	58
3	461	142	539	630	143	370	169	1	831	57
4	603	143	397	773	144	227	170	1	830	56
5	746	141	254	917	143	083	171	1	829	55
6	887	142	113	95060	142	04940	172	1	828	54
7	95029	141	04971	202	142	798	173	2	827	53
8	170	140	830	344	142	656	175	1	825	52
9	310	140	690	486	141	514	176	1	824	51
10	450	139	550	627	140	373	177	1	823	50
11	589	139	411	767	141	233	178	1	822	49
12	728	139	272	908	139	092	179	1	821	48
13	867	138	133	96047	140	03953	180	1	820	47
14	96005	138	03995	187	138	813	181	2	819	46
15	143	137	857	325	139	675	183	1	817	45
16	280	137	720	464	138	536	184	1	816	44
17	417	136	583	602	137	398	185	1	815	43
18	553	136	447	739	138	261	186	1	814	42
19	689	136	311	877	136	123	187	1	813	41
20	825	135	175	97013	137	02987	188	2	812	40
21	960	135	040	150	135	850	190	1	810	39
22	97095	134	02905	285	136	715	191	1	809	38
23	229	134	771	421	135	579	192	1	808	37
24	363	133	637	556	135	444	193	1	807	36
25	496	133	504	691	134	309	194	2	806	35
26	629	133	371	825	134	175	196	1	804	34
27	762	132	238	959	133	041	197	1	803	33
28	894	132	106	98092	133	01908	198	1	802	32
29	98026	131	01974	225	133	775	199	1	801	31
30	98157	131	01843	98358	132	01642	00200	2	99800	30
31	288	131	712	490	132	510	202	1	798	29
32	419	130	581	622	131	378	203	1	797	28
33	549	130	451	753	131	247	204	1	796	27
34	679	129	321	884	131	116	205	2	795	26
35	808	129	192	99015	130	00985	207	1	793	25
36	937	129	063	145	130	855	208	1	792	24
37	99066	128	00934	275	130	725	209	1	791	23
38	194	128	806	405	129	595	210	2	790	22
39	322	128	678	534	128	466	212	1	788	21
40	450	127	550	662	129	338	213	1	787	20
41	577	127	423	791	128	209	214	1	786	19
42	704	126	296	919	127	081	215	2	785	18
43	830	126	170	00046	128	99954	217	1	783	17
44	956	126	044	174	127	826	218	1	782	16
45	00082	125	99918	301	126	699	219	1	781	15
46	207	125	793	427	126	573	220	2	780	14
47	332	124	668	553	126	447	222	1	778	13
48	456	125	544	679	126	321	223	1	777	12
49	581	123	419	805	125	195	224	1	776	11
50	704	124	296	930	125	070	225	2	775	10
51	828	123	172	01055	124	98945	227	1	773	9
52	951	123	049	179	124	821	228	1	772	8
53	01074	122	98926	303	124	697	229	2	771	7
54	196	122	804	427	123	573	231	1	769	6
55	318	122	682	550	123	450	232	1	768	5
56	440	121	560	673	123	327	233	2	767	4
57	561	121	439	796	122	204	235	1	765	3
58	682	121	318	918	122	082	236	1	764	2
59	803	120	197	02040	122	97960	237	2	763	1
60	01923		98077	02162		97838	00239		99761	0
′	9. l cos	d 1′	10. l sec	9. l cot	d 1′	10. l tan	10. l csc	d 1′	9. l sin	′

″	Proportional Parts 145	144	143	142	141	140	139
0	0	0	0	0	0	0	0
1	2	2	2	2	2	2	2
2	5	5	5	5	5	5	5
3	7	7	7	7	7	7	7
4	10	10	10	9	9	9	9
5	12	12	12	12	12	12	12
6	14	14	14	14	14	14	14
7	17	17	17	17	16	16	16
8	19	19	19	19	19	19	19
9	22	22	21	21	21	21	21
10	24	24	24	24	24	23	23
11	27	26	26	26	26	26	25
12	29	29	29	28	28	28	28
13	31	31	31	31	31	30	30
14	34	34	33	33	33	33	32
15	36	36	36	36	35	35	35
16	39	38	38	38	38	37	37
17	41	41	41	40	40	40	39
18	44	43	43	43	42	42	42
19	46	46	45	45	45	44	44
20	48	48	48	47	47	47	46
21	51	50	50	50	49	49	49
22	53	53	52	52	52	51	51
23	56	55	55	54	54	54	53
24	58	58	57	57	56	56	56
25	60	60	60	59	59	58	58
26	63	62	62	62	61	61	60
27	65	65	64	64	63	63	63
28	68	67	67	66	66	65	65
29	70	70	69	69	68	68	67
30	72	72	72	71	70	70	70
31	75	74	74	73	73	72	72
32	77	77	76	76	75	75	74
33	80	79	79	78	78	77	76
34	82	82	81	80	80	79	79
35	85	84	83	83	82	82	81
36	87	86	86	85	85	84	83
37	89	89	88	88	87	86	86
38	92	91	91	90	89	89	88
39	94	94	93	92	92	91	90
40	97	96	95	95	94	93	93
41	99	98	98	97	96	96	95
42	102	101	100	99	99	98	97
43	104	103	102	102	101	100	100
44	106	106	105	104	103	103	102
45	109	108	107	106	106	105	104
46	111	110	110	109	108	107	107
47	114	113	112	111	110	110	109
48	116	115	114	114	113	112	111
49	118	118	117	116	115	114	114
50	121	120	119	118	118	117	116
51	123	122	122	121	120	119	118
52	126	125	124	123	122	121	120
53	128	127	126	125	125	124	123
54	130	130	129	128	127	126	125
55	133	132	131	130	129	128	127
56	135	134	133	133	132	131	130
57	138	137	136	135	134	133	132
58	140	139	138	137	136	135	134
59	143	142	141	140	139	138	137
60	145	144	143	142	141	140	139
″	145 Proportional Parts	144	143	142	141	140	139

95° 84°

TABLE II

″	Proportional Parts																			
″	**138**	**137**	**136**	**135**	**134**	**133**	**132**	**131**	**130**	**129**	**128**	**127**	**126**	**125**	**124**	**123**	**122**	**121**	**120**	**2**	**1**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	0	0
2	5	5	5	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	0	0
3	7	7	7	7	7	7	7	7	6	6	6	6	6	6	6	6	6	6	6	0	0
4	9	9	9	9	9	9	9	9	9	9	9	8	8	8	8	8	8	8	8	0	0
5	12	11	11	11	11	11	11	11	11	11	11	11	10	10	10	10	10	10	10	0	0
6	14	14	14	14	13	13	13	13	13	13	13	13	13	12	12	12	12	12	12	0	0
7	16	16	16	16	16	16	15	15	15	15	15	15	15	15	14	14	14	14	14	0	0
8	18	18	18	18	18	18	18	17	17	17	17	17	17	17	17	16	16	16	16	0	0
9	21	21	20	20	20	20	20	20	20	19	19	19	19	19	19	18	18	18	18	0	0
10	23	23	23	22	22	22	22	22	22	22	21	21	21	21	21	20	20	20	20	0	0
11	25	25	25	25	25	24	24	24	24	24	23	23	23	23	23	23	22	22	22	0	0
12	28	27	27	27	27	27	26	26	26	26	26	25	25	25	25	25	24	24	24	0	0
13	30	30	29	29	29	29	29	28	28	28	28	28	27	27	27	27	26	26	26	0	0
14	32	32	32	32	31	31	31	31	30	30	30	30	29	29	29	29	28	28	28	0	0
15	34	34	34	34	34	33	33	33	32	32	32	32	32	31	31	31	30	30	30	0	0
16	37	37	36	36	36	35	35	35	35	34	34	34	34	33	33	33	33	32	32	1	0
17	39	39	39	38	38	38	37	37	37	37	36	36	36	35	35	35	35	34	34	1	0
18	41	41	41	40	40	40	40	39	39	39	38	38	38	38	37	37	37	36	36	1	0
19	44	43	43	43	42	42	42	41	41	41	41	40	40	40	39	39	39	38	38	1	0
20	46	46	45	45	45	44	44	44	43	43	43	42	42	42	41	41	41	40	40	1	0
21	48	48	48	47	47	47	46	46	46	45	45	44	44	44	43	43	43	42	42	1	0
22	51	50	50	50	49	49	48	48	48	47	47	47	46	46	45	45	45	44	44	1	0
23	53	53	52	52	51	51	51	50	50	49	49	49	48	48	48	47	47	46	46	1	0
24	55	55	54	54	54	53	53	52	52	52	51	51	50	50	50	49	49	48	48	1	0
25	58	57	57	56	56	55	55	55	54	54	53	53	52	52	52	51	51	50	50	1	0
26	60	59	59	58	58	58	57	57	56	56	55	55	55	54	54	53	53	52	52	1	0
27	62	62	61	61	60	60	59	59	58	58	58	57	57	56	56	55	55	54	54	1	0
28	64	64	63	63	63	62	62	61	61	60	60	59	59	58	58	57	57	56	56	1	0
29	67	66	66	65	65	64	64	63	63	62	62	61	61	60	60	59	59	58	58	1	0
30	69	68	68	68	67	66	66	66	65	64	64	64	63	62	62	62	61	60	60	**1**	0
31	71	71	70	70	69	69	68	68	67	67	66	66	65	65	64	64	63	63	62	1	1
32	74	73	73	72	71	71	70	70	69	69	68	68	67	67	66	66	65	65	64	1	1
33	76	75	75	74	74	73	73	72	72	71	70	70	69	69	68	68	67	67	66	1	1
34	78	78	77	76	76	75	75	74	74	73	73	72	71	71	70	70	69	69	68	1	1
35	80	80	79	79	78	78	77	76	76	75	75	74	74	73	72	72	71	71	70	1	1
36	83	82	82	81	80	80	79	79	78	77	77	76	76	75	74	74	73	73	72	1	1
37	85	84	84	83	83	82	81	81	80	80	79	78	78	77	76	76	75	75	74	1	1
38	87	87	86	86	85	84	84	83	82	82	81	80	80	79	79	78	77	77	76	1	1
39	90	89	88	88	87	86	86	85	84	84	83	83	82	81	81	80	79	79	78	1	1
40	92	91	91	90	89	89	88	87	87	86	85	85	84	83	83	82	81	81	80	1	1
41	94	94	93	92	92	91	90	90	89	88	87	87	86	85	85	84	83	83	82	1	1
42	97	96	95	94	94	93	92	92	91	90	90	89	88	88	87	86	85	85	84	1	1
43	99	98	97	97	96	95	95	94	93	92	92	91	90	90	89	88	87	87	86	1	1
44	101	100	100	99	98	98	97	96	95	95	94	93	92	92	91	90	89	89	88	1	1
45	104	103	102	101	100	100	99	98	98	97	96	95	94	94	93	92	92	91	90	2	1
46	106	105	104	104	103	102	101	100	100	99	98	97	97	96	95	94	94	93	92	2	1
47	108	107	107	106	105	104	103	103	102	101	100	99	99	98	97	96	96	95	94	2	1
48	110	110	109	108	107	106	106	105	104	103	102	102	101	100	99	98	98	97	96	2	1
49	113	112	111	110	109	109	108	107	106	105	105	104	103	102	101	100	100	99	98	2	1
50	115	114	113	112	112	111	110	109	108	108	107	106	105	104	103	102	102	101	100	2	1
51	117	116	116	115	114	113	112	111	110	110	109	108	107	106	105	105	104	103	102	2	1
52	120	119	118	117	116	115	114	114	113	112	111	110	109	108	107	107	106	105	104	2	1
53	122	121	120	119	118	117	117	116	115	114	113	112	111	110	110	109	108	107	106	2	1
54	124	123	122	122	121	120	119	118	117	116	115	114	113	112	112	111	110	109	108	2	1
55	126	126	125	124	123	122	121	120	119	118	117	116	116	115	114	113	112	111	110	2	1
56	129	128	127	126	125	124	123	122	121	120	119	119	118	117	116	115	114	113	112	2	1
57	131	130	129	128	127	126	125	124	124	123	122	121	120	119	118	117	116	115	114	2	1
58	133	132	131	130	130	129	128	127	126	125	124	123	122	121	120	119	118	117	116	2	1
59	136	135	134	133	132	131	130	129	128	127	126	125	124	123	122	121	120	119	118	2	1
60	138	137	136	135	134	133	132	131	130	129	128	127	126	125	124	123	122	121	120	**2**	**1**
″	**138**	**137**	**136**	**135**	**134**	**133**	**132**	**131**	**130**	**129**	**128**	**127**	**126**	**125**	**124**	**123**	**122**	**121**	**120**	**2**	**1**
	Proportional Parts																				

′	l sin	d 1′	l csc	l tan	d 1′	l cot	l sec	d 1′	l cos	′
	9.		10.	9.		10.	10.		9.	
0	01923	120	98077	02162	121	97838	00239	1	99761	60
1	02043	120	97957	283	121	717	240	1	760	59
2	163	120	837	404	121	596	241	2	759	58
3	283	119	717	525	120	475	243	1	757	57
4	402	118	598	645	121	355	244	1	756	56
5	520	119	480	766	119	234	245	2	755	55
6	639	118	361	885	120	115	247	1	753	54
7	757	117	243	03005	119	96995	248	1	752	53
8	874	118	126	124	118	876	249	2	751	52
9	992	117	008	242	119	758	251	1	749	51
10	03109	117	96891	361	118	639	252	1	748	50
11	226	116	774	479	118	521	253	2	747	49
12	342	116	658	597	117	403	255	1	745	48
13	458	116	542	714	118	286	256	2	744	47
14	574	116	426	832	116	168	258	1	742	46
15	690	115	310	948	117	052	259	1	741	45
16	805	115	195	04065	116	95935	260	2	740	44
17	920	114	080	181	116	819	262	1	738	43
18	04034	115	95966	297	116	703	263	1	737	42
19	149	113	851	413	115	587	264	2	736	41
20	262	114	738	528	115	472	266	1	734	40
21	376	114	624	643	115	357	267	2	733	39
22	490	113	510	758	115	242	269	1	731	38
23	603	112	397	873	114	127	270	2	730	37
24	715	113	285	987	114	013	272	1	728	36
25	828	112	172	05101	113	94899	273	1	727	35
26	940	112	060	214	114	786	274	2	726	34
27	05052	112	94948	328	113	672	276	1	724	33
28	164	111	836	441	112	559	277	2	723	32
29	275	111	725	553	113	447	279	1	721	31
30	05386	111	94614	05666	112	94334	00280	2	99720	30
31	497	110	503	778	112	222	282	1	718	29
32	607	110	393	890	112	110	283	1	717	28
33	717	110	283	06002	111	93998	284	2	716	27
34	827	110	173	113	111	887	286	1	714	26
35	937	109	063	224	111	776	287	2	713	25
36	06046	109	93954	335	110	665	289	1	711	24
37	155	109	845	445	111	555	290	2	710	23
38	264	108	736	556	110	444	292	1	708	22
39	372	109	628	666	109	334	293	2	707	21
40	481	108	519	775	110	225	295	1	705	20
41	589	107	411	885	109	115	296	2	704	19
42	696	108	304	994	109	006	298	1	702	18
43	804	107	196	07103	108	92897	299	2	701	17
44	911	107	089	211	109	789	301	1	699	16
45	07018	106	92982	320	108	680	302	2	698	15
46	124	107	876	428	108	572	304	1	696	14
47	231	106	769	536	107	464	305	2	695	13
48	337	105	663	643	108	357	307	1	693	12
49	442	106	558	751	107	249	308	2	692	11
50	548	105	452	858	106	142	310	1	690	10
51	653	105	347	964	107	036	311	2	689	9
52	758	105	242	08071	106	91929	313	1	687	8
53	863	105	137	177	106	823	314	2	686	7
54	968	104	032	283	106	717	316	1	684	6
55	08072	104	91928	389	106	611	317	2	683	5
56	176	104	824	495	105	505	319	1	681	4
57	280	103	720	600	105	400	320	2	680	3
58	383	103	617	705	105	295	322	1	678	2
59	486	103	514	810	104	190	323	2	677	1
60	08589		91411	08914		91086	00325		99675	0
′	9.	d	10.	9.	d	10.	10.	d	9.	′
	l cos	1′	l sec	l cot	1′	l tan	l csc	1′	l sin	

″	Proportional Parts 121	120	119	118	117
0	0	0	0	0	0
1	2	2	2	2	2
2	4	4	4	4	4
3	6	6	6	6	6
4	8	8	8	8	8
5	10	10	10	10	10
6	12	12	12	12	12
7	14	14	14	14	14
8	16	16	16	16	16
9	18	18	18	18	18
10	20	20	20	20	20
11	22	22	22	22	21
12	24	24	24	24	23
13	26	26	26	26	25
14	28	28	28	28	27
15	30	30	30	29	29
16	32	32	32	31	31
17	34	34	34	33	33
18	36	36	36	35	35
19	38	38	38	37	37
20	40	40	40	39	39
21	42	42	42	41	41
22	44	44	44	43	43
23	46	46	46	45	45
24	48	48	48	47	47
25	50	50	50	49	49
26	52	52	52	51	51
27	54	54	54	53	53
28	56	56	56	55	55
29	58	58	58	57	57
30	60	60	60	59	58
31	63	62	61	61	60
32	65	64	63	63	62
33	67	66	65	65	64
34	69	68	67	67	66
35	71	70	69	69	68
36	73	72	71	71	70
37	75	74	73	73	72
38	77	76	75	75	74
39	79	78	77	77	76
40	81	80	79	79	78
41	83	82	81	81	80
42	85	84	83	83	82
43	87	86	85	85	84
44	89	88	87	87	86
45	91	90	89	89	88
46	93	92	91	90	90
47	95	94	93	92	92
48	97	96	95	94	94
49	99	98	97	96	96
50	101	100	99	98	98
51	103	102	101	100	99
52	105	104	103	102	101
53	107	106	105	104	103
54	109	108	107	106	105
55	111	110	109	108	107
56	113	112	111	110	109
57	115	114	113	112	111
58	117	116	115	114	113
59	119	118	117	116	115
60	121	120	119	118	117
″	121	120	119	118	117
	Proportional Parts				

96° 83°

TABLE II

″	Proportional Parts														
	116	**115**	**114**	**113**	**112**	**111**	**110**	**109**	**108**	**107**	**106**	**105**	**104**	**2**	**1**
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	2	2	2	2	2	2	2	2	2	2	2	2	2	0	0
2	4	4	4	4	4	4	4	4	4	4	4	4	3	0	0
3	6	6	6	6	6	6	5	5	5	5	5	5	5	0	0
4	8	8	8	8	7	7	7	7	7	7	7	7	7	0	0
5	10	10	9	9	9	9	9	9	9	9	9	9	9	0	0
6	12	12	11	11	11	11	11	11	11	11	11	10	10	0	0
7	14	13	13	13	13	13	13	13	13	12	12	12	12	0	0
8	15	15	15	15	15	15	15	15	14	14	14	14	14	0	0
9	17	17	17	17	17	17	17	16	16	16	16	16	16	0	0
10	19	19	19	19	19	18	18	18	18	18	18	18	17	0	0
11	21	21	21	21	21	20	20	20	20	20	19	19	19	0	0
12	23	23	23	23	22	22	22	22	22	21	21	21	21	0	0
13	25	25	25	24	24	24	24	24	23	23	23	23	23	0	0
14	27	27	27	26	26	26	26	25	25	25	25	24	24	0	0
15	29	29	29	28	28	28	27	27	27	27	27	26	26	0	0
16	31	31	30	30	30	30	29	29	29	29	28	28	28	1	0
17	33	33	32	32	32	31	31	31	31	30	30	30	29	1	0
18	35	34	34	34	34	33	33	33	32	32	32	32	31	1	0
19	37	36	36	36	35	35	35	35	34	34	34	33	33	1	0
20	39	38	38	38	37	37	37	36	36	36	35	35	35	1	0
21	41	40	40	40	39	39	39	38	38	37	37	37	36	1	0
22	43	42	42	41	41	41	40	40	40	39	39	38	38	1	0
23	44	44	44	43	43	43	42	42	41	41	41	40	40	1	0
24	46	46	46	45	45	44	44	44	43	43	42	42	42	1	0
25	48	48	47	47	47	46	46	45	45	45	44	44	43	1	0
26	50	50	49	49	49	48	48	47	47	46	46	46	45	1	0
27	52	52	51	51	50	50	49	49	49	48	48	47	47	1	0
28	54	54	53	53	52	52	51	51	50	50	49	49	49	1	0
29	56	56	55	55	54	54	53	53	52	52	51	51	50	1	0
30	58	58	57	56	56	56	55	54	54	54	53	52	52	1	0
31	60	59	59	58	58	57	57	56	56	55	55	54	54	1	1
32	62	61	61	60	60	59	59	58	58	57	57	56	55	1	1
33	64	63	63	62	62	61	61	60	59	59	58	58	57	1	1
34	66	65	65	64	63	63	62	62	61	61	60	60	59	1	1
35	68	67	67	66	65	65	64	64	63	62	62	61	61	1	1
36	70	69	68	68	67	67	66	65	65	64	64	63	62	1	1
37	72	71	70	70	69	68	68	67	67	66	65	65	64	1	1
38	73	73	72	72	71	70	70	69	68	68	67	66	66	1	1
39	75	75	74	73	73	72	72	71	70	70	69	68	68	1	1
40	77	77	76	75	75	74	73	73	72	71	71	70	69	1	1
41	79	79	78	77	77	76	75	74	74	73	72	72	71	1	1
42	81	80	80	79	78	78	77	76	76	75	74	74	73	1	1
43	83	82	82	81	80	80	79	78	77	77	76	75	75	1	1
44	85	84	84	83	82	81	81	80	79	78	78	77	76	1	1
45	87	86	85	85	84	83	83	82	81	80	79	79	78	2	1
46	89	88	87	87	86	85	84	84	83	82	81	80	80	2	1
47	91	90	89	89	88	87	86	85	85	84	83	82	81	2	1
48	93	92	91	90	90	89	88	87	86	86	85	84	83	2	1
49	95	94	93	92	91	91	90	89	88	87	87	86	85	2	1
50	97	96	95	94	93	92	92	91	90	89	88	88	87	2	1
51	99	98	97	96	95	94	93	93	92	91	90	89	88	2	1
52	101	100	99	98	97	96	95	94	94	93	92	91	90	2	1
53	102	102	101	100	99	98	97	96	95	95	94	93	92	2	1
54	104	104	103	102	101	100	99	98	97	96	95	94	94	2	1
55	106	105	105	104	103	102	101	100	99	98	97	96	95	2	1
56	108	107	106	105	105	104	103	102	101	100	99	98	97	2	1
57	110	109	108	107	106	105	105	104	103	102	101	100	99	2	1
58	112	111	110	109	108	107	106	105	104	103	102	102	101	2	1
59	114	113	112	111	110	109	108	107	106	105	104	103	102	2	1
60	116	115	114	113	112	111	110	109	108	107	106	105	104	**2**	**1**
″	**116**	**115**	**114**	**113**	**112**	**111**	**110**	**109**	**108**	**107**	**106**	**105**	**104**	**2**	**1**
	Proportional Parts														

′	l sin 9.	d 1′	l csc 10.	l tan 9.	d 1′	l cot 10.	l sec 10.	d 1′	l cos 9.	′
0	**08589**	103	**91411**	**08914**	105	**91086**	**00325**	1	**99675**	**60**
1	692	103	308	**09019**	104	**90981**	326	2	674	59
2	795	102	205	123	104	877	328	2	672	58
3	897	102	103	227	103	773	330	1	670	57
4	999	102	001	330	104	670	331	2	669	56
5	**09101**	101	**90899**	434	103	566	333	1	667	**55**
6	202	102	798	537	103	463	334	2	666	54
7	304	101	696	640	102	360	336	1	664	53
8	405	101	595	742	103	258	337	2	663	52
9	506	100	494	845	102	155	339	2	661	51
10	606	101	394	947	102	053	341	1	659	**50**
11	707	100	293	**10049**	101	**89951**	342	2	658	49
12	807	100	193	150	102	850	344	1	656	48
13	907	99	093	252	101	748	345	2	655	47
14	**10006**	100	**89994**	353	101	647	347	2	653	46
15	106	99	894	454	101	546	349	1	651	**45**
16	205	99	795	555	101	445	350	2	650	44
17	304	98	696	656	100	344	352	1	648	43
18	402	99	598	756	100	244	353	2	647	42
19	501	98	499	856	100	144	355	2	645	41
20	599	98	401	956	100	044	357	1	643	**40**
21	697	98	303	**11056**	99	**88944**	358	2	642	39
22	795	98	205	155	99	845	360	2	640	38
23	893	97	107	254	99	746	362	1	638	37
24	990	97	010	353	99	647	363	2	637	36
25	**11087**	97	**88913**	452	99	548	365	2	635	**35**
26	184	97	816	551	98	449	367	1	633	34
27	281	96	719	649	98	351	368	2	632	33
28	377	97	623	747	98	253	370	1	630	32
29	474	96	526	845	98	155	371	2	629	31
30	**11570**	96	**88430**	**11943**	97	**88057**	**00373**	2	**99627**	**30**
31	666	95	334	**12040**	98	**87960**	375	1	625	29
32	761	96	239	138	97	862	376	2	624	28
33	857	95	143	235	97	765	378	2	622	27
34	952	95	048	332	96	668	380	2	620	26
35	**12047**	95	**87953**	428	97	572	382	1	618	**25**
36	142	94	858	525	96	475	383	2	617	24
37	236	95	764	621	96	379	385	2	615	23
38	331	94	669	717	96	283	387	1	613	22
39	425	94	575	813	96	187	388	2	612	21
40	519	93	481	909	95	091	390	2	610	**20**
41	612	94	388	**13004**	95	**86996**	392	1	608	19
42	706	93	294	099	95	901	393	2	607	18
43	799	93	201	194	95	806	395	2	605	17
44	892	93	108	289	95	711	397	2	603	16
45	985	93	015	384	94	616	399	1	601	**15**
46	**13078**	93	**86922**	478	95	522	400	2	600	14
47	171	92	829	573	94	427	402	2	598	13
48	263	92	737	667	94	333	404	1	596	12
49	355	92	645	761	93	239	405	2	595	11
50	447	92	553	854	94	146	407	2	593	**10**
51	539	91	461	948	93	052	409	2	591	9
52	630	92	370	**14041**	93	**85959**	411	1	589	8
53	722	91	278	134	93	866	412	2	588	7
54	813	91	187	227	93	773	414	2	586	6
55	904	90	096	320	92	680	416	2	584	**5**
56	994	91	006	412	92	588	418	1	582	4
57	**14085**	90	**85915**	504	93	496	419	2	581	3
58	175	91	825	597	91	403	421	2	579	2
59	266	90	734	688	92	312	423	2	577	1
60	**14356**		**85644**	**14780**		**85220**	**00425**		**99575**	**0**
′	9. l cos	d 1′	10. l sec	9. l cot	d 1′	10. l tan	10. l csc	d 1′	9. l sin	′

″	Proportional Parts 105	104	103	102
0	0	0	0	0
1	2	2	2	2
2	4	3	3	3
3	5	5	5	5
4	7	7	7	7
5	9	9	9	9
6	10	10	10	10
7	12	12	12	12
8	14	14	14	14
9	16	16	15	15
10	18	17	17	17
11	19	19	19	19
12	21	21	21	20
13	23	23	22	22
14	24	24	24	24
15	26	26	26	25
16	28	28	27	27
17	30	29	29	29
18	32	31	31	31
19	33	33	33	32
20	35	35	34	34
21	37	36	36	36
22	38	38	38	37
23	40	40	39	39
24	42	42	41	41
25	44	43	43	43
26	46	45	45	44
27	47	47	46	46
28	49	49	48	48
29	51	50	50	49
30	52	52	52	51
31	54	54	53	53
32	56	55	55	54
33	58	57	57	56
34	60	59	58	58
35	61	61	60	59
36	63	62	62	61
37	65	64	64	63
38	66	66	65	65
39	68	68	67	66
40	70	69	69	68
41	72	71	70	70
42	74	73	72	71
43	75	75	74	73
44	77	76	76	75
45	79	78	77	77
46	80	80	79	78
47	82	81	81	80
48	84	83	82	82
49	86	85	84	83
50	88	87	86	85
51	89	88	88	87
52	91	90	89	88
53	93	92	91	90
54	94	94	93	92
55	96	95	94	93
56	98	97	96	95
57	100	99	98	97
58	102	101	100	99
59	103	102	101	100
60	105	104	103	102
″	105 Proportional Parts	104	103	102

TABLE II

"	Proportional Parts													
	101	100	99	98	97	96	95	94	93	92	91	90	2	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	2	2	2	2	2	2	2	2	2	2	2	1	0	0
2	3	3	3	3	3	3	3	3	3	3	3	3	0	0
3	5	5	5	5	5	5	5	5	5	5	5	5	0	0
4	7	7	7	7	6	6	6	6	6	6	6	6	0	0
5	8	8	8	8	8	8	8	8	8	8	8	7	0	0
6	10	10	10	10	10	10	10	9	9	9	9	9	0	0
7	12	12	12	11	11	11	11	11	11	11	11	11	0	0
8	13	13	13	13	13	13	13	13	12	12	12	12	0	0
9	15	15	15	15	15	14	14	14	14	14	14	13	0	0
10	17	17	16	16	16	16	16	16	16	15	15	15	0	0
11	19	18	18	18	18	18	17	17	17	17	17	17	0	0
12	20	20	20	20	19	19	19	19	19	18	18	18	0	0
13	22	22	21	21	21	21	21	20	20	20	20	19	0	0
14	24	23	23	23	23	22	22	22	22	21	21	21	0	0
15	25	25	25	24	24	24	24	23	23	23	23	23	0	0
16	27	27	26	26	26	26	25	25	25	25	24	24	1	0
17	29	28	28	28	27	27	27	27	26	26	26	25	1	0
18	30	30	30	29	29	29	28	28	28	28	27	27	1	0
19	32	32	31	31	31	30	30	30	29	29	29	29	1	0
20	34	33	33	33	32	32	32	31	31	31	30	30	1	0
21	35	35	35	34	34	34	33	33	33	32	32	31	1	0
22	37	37	36	36	36	35	35	34	34	34	33	33	1	0
23	39	38	38	38	37	37	36	36	36	35	35	35	1	0
24	40	40	40	39	39	38	38	38	37	37	36	36	1	0
25	42	42	41	41	40	40	40	39	39	38	38	37	1	0
26	44	43	43	42	42	42	41	41	40	40	39	39	1	0
27	45	45	45	44	44	43	43	42	42	41	41	41	1	0
28	47	47	46	46	45	45	44	44	43	43	42	42	1	0
29	49	48	48	47	47	46	46	45	45	44	44	43	1	0
30	50	50	50	49	48	48	48	47	46	46	46	45	1	0
31	52	52	51	51	50	50	49	49	48	48	47	47	1	1
32	54	53	53	52	52	51	51	50	50	49	49	48	1	1
33	56	55	54	54	53	53	52	52	51	51	50	49	1	1
34	57	57	56	56	55	54	54	53	53	52	52	51	1	1
35	59	58	58	57	57	56	55	55	54	54	53	53	1	1
36	61	60	59	59	58	58	57	56	56	55	55	54	1	1
37	62	62	61	60	60	59	59	58	57	57	56	55	1	1
38	64	63	63	62	61	61	60	60	59	58	58	57	1	1
39	66	65	64	64	63	62	62	61	60	60	59	59	1	1
40	67	67	66	65	65	64	63	63	62	61	61	60	1	1
41	69	68	68	67	66	66	65	64	64	63	62	61	1	1
42	71	70	69	69	68	67	66	66	65	64	64	63	1	1
43	72	72	71	70	70	69	68	67	67	66	65	65	1	1
44	74	73	73	72	71	70	70	69	68	67	67	66	1	1
45	76	75	74	73	73	72	71	71	70	69	68	67	2	1
46	77	77	76	75	74	74	73	72	71	71	70	69	2	1
47	79	78	78	77	76	75	74	74	73	72	71	71	2	1
48	81	80	79	78	78	77	76	75	74	74	73	72	2	1
49	82	82	81	80	79	78	78	77	76	75	74	73	2	1
50	84	83	82	82	81	80	79	78	78	77	76	75	2	1
51	86	85	84	83	82	82	81	80	79	78	77	77	2	1
52	88	87	86	85	84	83	82	81	81	80	79	78	2	1
53	89	88	87	87	86	85	84	83	82	81	80	79	2	1
54	91	90	89	88	87	86	86	85	84	83	82	81	2	1
55	93	92	91	90	89	88	87	86	85	84	83	83	2	1
56	94	93	92	91	91	90	89	88	87	86	85	84	2	1
57	96	95	94	93	92	91	90	89	88	87	86	85	2	1
58	98	97	96	95	94	93	92	91	90	89	88	87	2	1
59	99	98	97	96	95	94	93	92	91	90	89	89	2	1
60	101	100	99	98	97	96	95	94	93	92	91	90	2	1
"	101	100	99	98	97	96	95	94	93	92	91	90	2	1
	Proportional Parts													

′	l sin 9.	d 1′	l csc 10.	l tan 9.	d 1′	l cot 10.	l sec 10.	d 1′	l cos 9.	′
0	14356	89	85644	14780	92	85220	00425	1	99575	60
1	445	90	555	872	91	128	426	2	574	59
2	535	89	465	963	91	037	428	2	572	58
3	624	90	376	15054	91	84946	430	2	570	57
4	714	89	286	145	91	855	432	2	568	56
5	803	88	197	236	91	764	434	1	566	55
6	891	89	109	327	90	673	435	2	565	54
7	980	89	020	417	91	583	437	2	563	53
8	15069	88	84931	508	90	492	439	2	561	52
9	157	88	843	598	90	402	441	2	559	51
10	245	88	755	688	89	312	443	1	557	50
11	333	88	667	777	90	223	444	2	556	49
12	421	87	579	867	89	133	446	2	554	48
13	508	88	492	956	90	044	448	2	552	47
14	596	87	404	16046	89	83954	450	2	550	46
15	683	87	317	135	89	865	452	2	548	45
16	770	87	230	224	88	776	454	1	546	44
17	857	87	143	312	89	688	455	2	545	43
18	944	86	056	401	88	599	457	2	543	42
19	16030	86	83970	489	88	511	459	2	541	41
20	116	87	884	577	88	423	461	2	539	40
21	203	86	797	665	88	335	463	2	537	39
22	289	85	711	753	88	247	465	2	535	38
23	374	86	626	841	87	159	467	1	533	37
24	460	85	540	928	88	072	468	2	532	36
25	545	86	455	17016	87	82984	470	2	530	35
26	631	85	369	103	87	897	472	2	528	34
27	716	85	284	190	87	810	474	2	526	33
28	801	85	199	277	86	723	476	2	524	32
29	886	84	114	363	87	637	478	2	522	31
30	16970	85	83030	17450	86	82550	00480	2	99520	30
31	17055	84	82945	536	86	464	482	1	518	29
32	139	84	861	622	86	378	483	2	517	28
33	223	84	777	708	86	292	485	2	515	27
34	307	84	693	794	86	206	487	2	513	26
35	391	83	609	880	85	120	489	2	511	25
36	474	84	526	965	86	035	491	2	509	24
37	558	83	442	18051	85	81949	493	2	507	23
38	641	83	359	136	85	864	495	2	505	22
39	724	83	276	221	85	779	497	2	503	21
40	807	83	193	306	85	694	499	2	501	20
41	890	83	110	391	84	609	501	2	499	19
42	973	82	027	475	85	525	503	2	497	18
43	18055	82	81945	560	84	440	505	1	495	17
44	137	83	863	644	84	356	506	2	494	16
45	220	82	780	728	84	272	508	2	492	15
46	302	81	698	812	84	188	510	2	490	14
47	383	82	617	896	83	104	512	2	488	13
48	465	82	535	979	84	021	514	2	486	12
49	547	81	453	19063	83	80937	516	2	484	11
50	628	81	372	146	83	854	518	2	482	10
51	709	81	291	229	83	771	520	2	480	9
52	790	81	210	312	83	688	522	2	478	8
53	871	81	129	395	83	605	524	2	476	7
54	952	81	048	478	83	522	526	2	474	6
55	19033	80	80967	561	82	439	528	2	472	5
56	113	80	887	643	82	357	530	2	470	4
57	193	80	807	725	82	275	532	2	468	3
58	273	80	727	807	82	193	534	2	466	2
59	353	80	647	889	82	111	.536	2	464	1
60	19433		80567	19971		80029	00538		99462	0
′	9 l cos	d 1	10. l sec	9. l cot	d 1′	10. l tan	10. l csc	d 1′	9. l sin	′

″	Proportional Parts 92	91	90
0	0	0	0
1	2	2	1
2	3	3	3
3	5	5	5
4	6	6	6
5	8	8	7
6	9	9	9
7	11	11	11
8	12	12	12
9	14	14	13
10	15	15	15
11	17	17	17
12	18	18	18
13	20	20	19
14	21	21	21
15	23	23	23
16	25	24	24
17	26	26	25
18	28	27	27
19	29	29	29
20	31	30	30
21	32	32	31
22	34	33	33
23	35	35	35
24	37	36	36
25	38	38	37
26	40	39	39
27	41	41	41
28	43	42	42
29	44	44	43
30	46	46	45
31	48	47	47
32	49	49	48
33	51	50	49
34	52	52	51
35	54	53	53
36	55	55	54
37	57	56	55
38	58	58	57
39	60	59	59
40	61	61	60
41	63	62	61
42	64	64	63
43	66	65	65
44	67	67	66
45	69	68	67
46	71	70	69
47	72	71	71
48	74	73	72
49	75	74	73
50	77	76	75
51	78	77	77
52	80	79	78
53	81	80	79
54	83	82	81
55	84	83	83
56	86	85	84
57	87	86	85
58	89	88	87
59	90	89	89
60	92	91	90
″	92 Proportional Parts	91	90

TABLE II

"	Proportional Parts											
	89	88	87	86	85	84	83	82	81	80	2	1
0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	0	0
2	3	3	3	3	3	3	3	3	3	3	0	0
3	4	4	4	4	4	4	4	4	4	4	0	0
4	6	6	6	6	6	6	6	5	5	5	0	0
5	7	7	7	7	7	7	7	7	7	7	0	0
6	9	9	9	9	8	8	8	8	8	8	0	0
7	10	10	10	10	10	10	10	10	9	9	0	0
8	12	12	12	11	11	11	11	11	11	11	0	0
9	13	13	13	13	13	13	12	12	12	12	0	0
10	15	15	14	14	14	14	14	14	14	13	0	0
11	16	16	16	16	16	15	15	15	15	15	0	0
12	18	18	17	17	17	17	17	16	16	16	0	0
13	19	19	19	19	18	18	18	18	18	17	0	0
14	21	21	20	20	20	20	19	19	19	19	0	0
15	22	22	22	21	21	21	21	21	20	20	0	0
16	24	23	23	23	23	22	22	22	22	21	1	0
17	25	25	25	24	24	24	24	23	23	23	1	0
18	27	26	26	26	26	25	25	25	24	24	1	0
19	28	28	28	27	27	27	26	26	26	25	1	0
20	30	29	29	29	28	28	28	27	27	27	1	0
21	31	31	30	30	30	29	29	29	28	28	1	0
22	33	32	32	32	31	31	30	30	30	29	1	0
23	34	34	33	33	33	32	32	31	31	31	1	0
24	36	35	35	34	34	34	33	33	32	32	1	0
25	37	37	36	36	35	35	35	34	34	33	1	0
26	39	38	38	37	37	36	36	36	35	35	1	0
27	40	40	39	39	38	38	37	37	36	36	1	0
28	42	41	41	40	40	39	39	38	38	37	1	0
29	43	43	42	42	41	41	40	40	39	39	1	0
30	44	44	44	43	42	42	42	41	40	40	1	0
31	46	45	45	44	44	43	43	42	42	41	1	1
32	47	47	46	46	45	45	44	44	43	43	1	1
33	49	48	48	47	47	46	46	45	45	44	1	1
34	50	50	49	49	48	48	47	46	46	45	1	1
35	52	51	51	50	50	49	48	48	47	47	1	1
36	53	53	52	52	51	50	50	49	49	48	1	1
37	55	54	54	53	52	52	51	51	50	49	1	1
38	56	56	55	54	54	53	53	52	51	51	1	1
39	58	57	57	56	55	55	54	53	53	52	1	1
40	59	59	58	57	57	56	55	55	54	53	1	1
41	61	60	59	59	58	57	57	56	55	55	1	1
42	62	62	61	60	60	59	58	57	57	56	1	1
43	64	63	62	62	61	60	59	59	58	57	1	1
44	65	65	64	63	62	62	61	60	59	59	1	1
45	67	66	65	65	64	63	62	61	61	60	2	1
46	68	67	67	66	65	64	64	63	62	61	2	1
47	70	69	68	67	67	66	65	64	63	63	2	1
48	71	70	70	69	68	67	66	66	65	64	2	1
49	73	72	71	70	69	69	68	67	66	65	2	1
50	74	73	72	72	71	70	69	68	68	67	2	1
51	76	75	74	73	72	71	71	70	69	68	2	1
52	77	76	75	75	74	73	72	71	70	69	2	1
53	79	78	77	76	75	74	73	72	72	71	2	1
54	80	79	78	77	76	76	75	74	73	72	2	1
55	82	81	80	79	78	77	76	75	74	73	2	1
56	83	82	81	80	79	78	77	77	76	75	2	1
57	85	84	83	82	81	80	79	78	77	76	2	1
58	86	85	84	83	82	81	80	79	78	77	2	1
59	88	87	86	85	84	83	82	81	80	79	2	1
60	89	88	87	86	85	84	83	82	81	80	2	1
"	89	88	87	86	85	84	83	82	81	80	2	1
	Proportional Parts											

′	*l* sin 9.	d 1′	*l* csc 10.	*l* tan 9.	d 1′	*l* cot 10.	*l* sec 10.	d 1′	*l* cos 9.	′
0	19433	80	80567	19971	82	80029	00538	2	99462	60
1	513	79	487	20053	81	79947	540	2	460	59
2	592	80	408	134	82	866	542	2	458	58
3	672	79	328	216	81	784	544	2	456	57
4	751	79	249	297	81	703	546	2	454	56
5	830	79	170	378	81	622	548	2	452	55
6	909	79	091	459	81	541	550	2	450	54
7	988	79	012	540	81	460	552	2	448	53
8	20067	78	79933	621	80	379	554	2	446	52
9	145	78	855	701	81	299	556	2	444	51
10	223	79	777	782	80	218	558	2	442	50
11	302	78	698	862	80	138	560	2	440	49
12	380	78	620	942	80	058	562	2	438	48
13	458	77	542	21022	80	78978	564	2	436	47
14	535	78	465	102	80	898	566	2	434	46
15	613	78	387	182	79	818	568	3	432	45
16	691	77	309	261	80	739	571	2	429	44
17	768	77	232	341	79	659	573	2	427	43
18	845	77	155	420	79	580	575	2	425	42
19	922	77	078	499	79	501	577	2	423	41
20	999	77	001	578	79	422	579	2	421	40
21	21076	77	78924	657	79	343	581	2	419	39
22	153	76	847	736	78	264	583	2	417	38
23	229	77	771	814	79	186	585	2	415	37
24	306	76	694	893	78	107	587	2	413	36
25	382	76	618	971	78	029	589	2	411	35
26	458	76	542	22049	78	77951	591	2	409	34
27	534	76	466	127	78	873	593	3	407	33
28	610	75	390	205	78	795	596	2	404	32
29	685	76	315	283	78	717	598	2	402	31
30	21761	75	78239	22361	77	77639	00600	2	99400	30
31	836	76	164	438	78	562	602	2	398	29
32	912	75	088	516	77	484	604	2	396	28
33	987	75	013	593	77	407	606	2	394	27
34	22062	75	77938	670	77	330	608	2	392	26
35	137	74	863	747	77	253	610	2	390	25
36	211	75	789	824	77	176	612	3	388	24
37	286	75	714	901	76	099	615	2	385	23
38	361	74	639	977	77	023	617	2	383	22
39	435	74	565	23054	76	76946	619	2	381	21
40	509	74	491	130	76	870	621	2	379	20
41	583	74	417	206	77	794	623	2	377	19
42	657	74	343	283	76	717	625	3	375	18
43	731	74	269	359	76	641	628	2	372	17
44	805	73	195	435	75	565	630	2	370	16
45	878	74	122	510	76	490	632	2	368	15
46	952	73	048	586	75	414	634	2	366	14
47	23025	73	76975	661	76	339	636	2	364	13
48	098	73	902	737	75	263	638	3	362	12
49	171	73	829	812	75	188	641	2	359	11
50	244	73	756	887	75	113	643	2	357	10
51	317	73	683	962	75	038	645	2	355	9
52	390	72	610	24037	75	75963	647	2	353	8
53	462	73	538	112	74	888	649	3	351	7
54	535	72	465	186	75	814	652	2	348	6
55	607	72	393	261	74	739	654	2	346	5
56	679	73	321	335	75	665	656	2	344	4
57	752	71	248	410	74	590	658	2	342	3
58	823	72	177	484	74	516	660	3	340	2
59	895	72	105	558	74	442	663	2	337	1
60	23967		76033	24632		75368	00665		99335	0
′	9. *l* cos	d 1′	10. *l* sec	9. *l* cot	d 1′	10. *l* tan	10. *l* csc	d 1′	9. *l* sin	′

Proportional Parts

″	82	81	80	79	78	77	76	75	74	73	72	71	3	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	0	0
2	3	3	3	3	3	3	3	2	2	2	2	2	0	0
3	4	4	4	4	4	4	4	4	4	4	4	4	0	0
4	5	5	5	5	5	5	5	5	5	5	5	5	0	0
5	7	7	7	7	7	6	6	6	6	6	6	6	0	0
6	8	8	8	8	8	8	8	8	7	7	7	7	0	0
7	10	9	9	9	9	9	9	9	9	9	8	8	0	0
8	11	11	11	11	10	10	10	10	10	10	10	9	0	0
9	12	12	12	12	12	12	11	11	11	11	11	11	0	0
10	14	14	13	13	13	13	13	12	12	12	12	12	0	0
11	15	15	15	14	14	14	14	14	14	13	13	13	1	0
12	16	16	16	16	16	15	15	15	15	15	14	14	1	0
13	18	18	17	17	17	17	16	16	16	16	16	15	1	0
14	19	19	19	18	18	18	18	18	17	17	17	17	1	0
15	21	20	20	20	19	19	19	19	19	18	18	18	1	0
16	22	22	21	21	21	21	20	20	20	19	19	19	1	1
17	23	23	23	22	22	22	22	21	21	21	20	20	1	1
18	25	24	24	24	23	23	23	22	22	22	22	21	1	1
19	26	26	25	25	25	24	24	24	23	23	23	22	1	1
20	27	27	27	26	26	26	25	25	25	24	24	24	1	1
21	29	28	28	28	27	27	27	26	26	26	25	25	1	1
22	30	30	29	29	29	28	28	28	27	27	26	26	1	1
23	31	31	31	30	30	30	29	29	28	28	28	27	1	1
24	33	32	32	32	31	31	30	30	30	29	29	28	1	1
25	34	34	33	33	33	32	32	31	31	30	30	30	1	1
26	36	35	35	34	34	33	33	32	32	32	31	31	1	1
27	37	36	36	36	35	35	34	34	33	33	32	32	1	1
28	38	38	37	37	36	36	35	35	35	34	34	33	1	1
29	40	39	39	38	38	37	37	36	36	35	35	34	1	1
30	41	40	40	40	39	38	38	38	37	36	36	36	2	1
31	42	42	41	41	40	40	39	39	38	38	37	37	2	1
32	44	43	43	42	42	41	41	40	39	39	38	38	2	1
33	45	45	44	43	43	42	42	41	41	40	40	39	2	1
34	46	46	45	45	44	44	43	42	42	41	41	40	2	1
35	48	47	47	46	45	45	44	44	43	43	42	41	2	1
36	49	49	48	47	47	46	46	45	44	44	43	43	2	1
37	51	50	49	49	48	47	47	46	46	45	44	44	2	1
38	52	51	51	50	49	49	48	48	47	46	46	45	2	1
39	53	53	52	51	51	50	49	49	48	47	47	46	2	1
40	55	54	53	53	52	51	51	50	49	49	48	47	2	1
41	56	55	55	54	53	53	52	51	51	50	49	49	2	1
42	57	57	56	55	55	54	53	52	52	51	50	50	2	1
43	59	58	57	57	56	55	54	54	53	52	52	51	2	1
44	60	59	59	58	57	56	56	55	54	54	53	52	2	1
45	61	61	60	59	59	58	57	56	55	55	54	53	2	2
46	63	62	61	61	60	59	58	58	57	56	55	54	2	2
47	64	63	63	62	61	60	60	59	58	57	56	56	2	2
48	66	65	64	63	62	62	61	60	59	58	58	57	2	2
49	67	66	65	65	64	63	62	61	60	60	59	58	2	2
50	68	68	67	66	65	64	63	62	62	61	60	59	2	2
51	70	69	68	67	66	65	65	64	63	62	61	60	3	2
52	71	70	69	68	68	67	66	65	64	63	62	62	3	2
53	72	72	71	70	69	68	67	66	65	64	64	63	3	2
54	74	73	72	71	70	69	68	68	67	66	65	64	3	2
55	75	74	73	72	71	71	70	69	68	67	66	65	3	2
56	77	76	75	74	73	72	71	70	69	68	67	66	3	2
57	78	77	76	75	74	73	72	71	70	69	68	67	3	2
58	79	78	77	76	75	74	73	72	72	71	70	69	3	2
59	81	80	79	78	77	76	75	74	73	72	71	70	3	2
60	82	81	80	79	78	77	76	75	74	73	72	71	3	2
″	82	81	80	79	78	77	76	75	74	73	72	71	3	2

Proportional Parts

99° 80°

′	l sin 9.	d 1′	l csc 10.	l tan 9.	d 1′	l cot 10.	l sec 10.	d 1′	l cos 9.	′
0	23967		76033	24632		75368	00665		99335	60
1	24039	72	75961	706	74	294	667	2	333	59
2	110	71	890	779	73	221	669	2	331	58
3	181	71	819	853	74	147	672	3	328	57
4	253	72	747	926	73	074	674	2	326	56
5	324	71	676	25000	74	000	676	2	324	55
6	395	71	605	073	73	74927	678	2	322	54
7	466	71	534	146	73	854	681	3	319	53
8	536	70	464	219	73	781	683	2	317	52
9	607	71	393	292	73	708	685	2	315	51
10	677	70	323	365	73	635	687	2	313	50
11	748	71	252	437	72	563	690	3	310	49
12	818	70	182	510	73	490	692	2	308	48
13	888	70	112	582	72	418	694	2	306	47
14	958	70	042	655	73	345	696	2	304	46
15	25028	70	74972	727	72	273	699	3	301	45
16	098	70	902	799	72	201	701	2	299	44
17	168	70	832	871	72	129	703	2	297	43
18	237	69	763	943	72	057	706	3	294	42
19	307	70	693	26015	72	73985	708	2	292	41
20	376	69	624	086	71	914	710	2	290	40
21	445	69	555	158	72	842	712	2	288	39
22	514	69	486	229	71	771	715	3	285	38
23	583	69	417	301	72	699	717	2	283	37
24	652	69	348	372	71	628	719	2	281	36
25	721	69	279	443	71	557	722	3	278	35
26	790	69	210	514	71	486	724	2	276	34
27	858	68	142	585	71	415	726	2	274	33
28	927	69	073	655	70	345	729	3	271	32
29	995	68	005	726	71	274	731	2	269	31
30	26063	68	73937	26797	71	73203	00733	2	99267	30
31	131	68	869	867	70	133	736	3	264	29
32	199	68	801	937	70	063	738	2	262	28
33	267	68	733	27008	71	72992	740	2	260	27
34	335	68	665	078	70	922	743	3	257	26
35	403	68	597	148	70	852	745	2	255	25
36	470	67	530	218	70	782	748	3	252	24
37	538	68	462	288	70	712	750	2	250	23
38	605	67	395	357	69	643	752	2	248	22
39	672	67	328	427	70	573	755	3	245	21
40	739	67	261	496	69	504	757	2	243	20
41	806	67	194	566	70	434	759	2	241	19
42	873	67	127	635	69	365	762	3	238	18
43	940	67	060	704	69	296	764	2	236	17
44	27007	67	72993	773	69	227	767	3	233	16
45	073	66	927	842	69	158	769	2	231	15
46	140	67	860	911	69	089	771	2	229	14
47	206	66	794	980	69	020	774	3	226	13
48	273	67	727	28049	69	71951	776	2	224	12
49	339	66	661	117	68	883	779	3	221	11
50	405	66	595	186	69	814	781	2	219	10
51	471	66	529	254	68	746	783	2	217	9
52	537	66	463	323	69	677	786	3	214	8
53	602	65	398	391	68	609	788	2	212	7
54	668	66	332	459	68	541	791	3	209	6
55	734	66	266	527	68	473	793	2	207	5
56	799	65	201	595	68	405	796	3	204	4
57	864	65	136	662	67	338	798	2	202	3
58	930	66	070	730	68	270	800	2	200	2
59	995	65	005	798	68	202	803	3	197	1
60	28060	65	71940	28865	67	71135	00805	2	99195	0
′	9. l cos	d 1′	10. l sec	9. l cot	d 1′	10. l tan	10. l csc	d 1′	9. l sin	′

″	Proportional Parts											
″	74	73	72	71	70	69	68	67	66	65	3	2
0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	0	0
2	2	2	2	2	2	2	2	2	2	2	0	0
3	4	4	4	4	3	3	3	3	3	3	0	0
4	5	5	5	5	5	5	5	4	4	4	0	0
5	6	6	6	6	6	6	6	6	5	5	0	0
6	7	7	7	7	7	7	7	7	7	6	0	0
7	9	9	8	8	8	8	8	8	8	8	0	0
8	10	10	10	9	9	9	9	9	9	9	0	0
9	11	11	11	11	11	10	10	10	10	10	0	0
10	12	12	12	12	12	12	11	11	11	11	0	0
11	14	13	13	13	13	13	12	12	12	12	1	0
12	15	15	14	14	14	14	14	13	13	13	1	0
13	16	16	16	15	15	15	15	15	14	14	1	0
14	17	17	17	17	16	16	16	16	15	15	1	0
15	19	18	18	18	17	17	17	17	17	16	1	0
16	20	19	19	19	19	18	18	18	18	17	1	1
17	21	21	20	20	20	20	19	19	19	18	1	1
18	22	22	22	21	21	21	20	20	20	20	1	1
19	23	23	23	22	22	22	22	21	21	21	1	1
20	25	24	24	24	23	23	23	22	22	22	1	1
21	26	26	25	25	25	24	24	23	23	23	1	1
22	27	27	26	26	26	25	25	25	24	24	1	1
23	28	28	28	27	27	26	26	26	25	25	1	1
24	30	29	29	28	28	28	27	27	26	26	1	1
25	31	30	30	30	29	29	28	28	27	27	1	1
26	32	32	31	31	30	30	29	29	29	28	1	1
27	33	33	32	32	31	31	31	30	30	29	1	1
28	35	34	34	33	33	32	32	31	31	30	1	1
29	36	35	35	34	34	33	33	32	32	31	1	1
30	37	36	36	36	35	34	34	34	33	32	2	1
31	38	38	37	37	36	36	35	35	34	34	2	1
32	39	39	38	38	37	37	36	36	35	35	2	1
33	41	40	40	39	39	38	37	37	36	36	2	1
34	42	41	41	40	40	39	39	38	37	37	2	1
35	43	43	42	41	41	40	40	39	39	38	2	1
36	44	44	43	43	42	41	41	40	40	39	2	1
37	46	45	44	44	43	43	42	41	41	40	2	1
38	47	46	46	45	44	44	43	42	42	41	2	1
39	48	47	47	46	45	45	44	44	43	42	2	1
40	49	49	48	47	47	46	45	45	44	43	2	1
41	51	50	49	49	48	47	46	46	45	44	2	1
42	52	51	50	50	49	48	48	47	46	46	2	1
43	53	52	52	51	50	49	49	48	47	47	2	1
44	54	54	53	52	51	51	50	49	48	48	2	1
45	55	55	54	53	53	52	51	50	49	49	2	2
46	57	56	55	54	54	53	52	51	51	50	2	2
47	58	57	56	56	55	54	53	52	52	51	2	2
48	59	58	58	57	56	55	54	54	53	52	2	2
49	60	60	59	58	57	56	56	55	54	53	2	2
50	62	61	60	59	58	58	57	56	55	54	2	2
51	63	62	61	60	59	59	58	57	56	55	3	2
52	64	63	62	62	61	60	59	58	57	56	3	2
53	65	64	64	63	62	61	60	59	58	57	3	2
54	67	66	65	64	63	62	61	60	59	58	3	2
55	68	67	66	65	64	63	62	61	61	60	3	2
56	69	68	67	66	65	64	63	63	62	61	3	2
57	70	69	68	67	67	66	65	64	63	62	3	2
58	72	71	70	69	68	67	66	65	64	63	3	2
59	73	72	71	70	69	68	67	66	65	64	3	2
60	74	73	72	71	70	69	68	67	66	65	3	2
″	74	73	72	71	70	69	68	67	66	65	3	2
	Proportional Parts											

′	*l* sin 9.	d 1′	*l* csc 10.	*l* tan 9.	d 1′	*l* cot 10.	*l* sec 10.	d 1′	*l* cos 9.	′
0	28060		71940	28865		71135	00805		99195	60
1	125	65	875	933	68	067	808	3	192	59
2	190	65	810	29000	67	000	810	2	190	58
3	254	64	746	067	67	70933	813	3	187	57
4	319	65	681	134	67	866	815	2	185	56
5	384	65	616	201	67	799	818	3	182	55
6	448	64	552	268	67	732	820	2	180	54
7	512	64	488	335	67	665	823	3	177	53
8	577	65	423	402	67	598	825	2	175	52
9	641	64	359	468	66	532	828	3	172	51
10	705	64	295	535	67	465	830	2	170	50
11	769	64	231	601	66	399	833	3	167	49
12	833	64	167	668	67	332	835	2	165	48
13	896	63	104	734	66	266	838	3	162	47
14	960	64	040	800	66	200	840	2	160	46
15	29024	64	70976	866	66	134	843	3	157	45
16	087	63	913	932	66	068	845	2	155	44
17	150	63	850	998	66	002	848	3	152	43
18	214	64	786	30064	66	69936	850	2	150	42
19	277	63	723	130	66	870	853	3	147	41
20	340	63	660	195	65	805	855	2	145	40
21	403	63	597	261	66	739	858	3	142	39
22	466	63	534	326	65	674	860	2	140	38
23	529	63	471	391	65	609	863	3	137	37
24	591	62	409	457	66	543	865	2	135	36
25	654	63	346	522	65	478	868	3	132	35
26	716	62	284	587	65	413	870	2	130	34
27	779	63	221	652	65	348	873	3	127	33
28	841	62	159	717	65	283	876	3	124	32
29	903	62	097	782	65	218	878	2	122	31
30	29966	63	70034	30846	64	69154	00881	3	99119	30
31	30028	62	69972	911	65	089	883	2	117	29
32	090	62	910	975	64	025	886	3	114	28
33	151	61	849	31040	65	68960	888	2	112	27
34	213	62	787	104	64	896	891	3	109	26
35	275	62	725	168	64	832	894	3	106	25
36	336	61	664	233	65	767	896	2	104	24
37	398	62	602	297	64	703	899	3	101	23
38	459	61	541	361	64	639	901	2	099	22
39	521	62	479	425	64	575	904	3	096	21
40	582	61	418	489	64	511	907	3	093	20
41	643	61	357	552	63	448	909	2	091	19
42	704	61	296	616	64	384	912	3	088	18
43	765	61	235	679	63	321	914	2	086	17
44	826	61	174	743	64	257	917	3	083	16
45	887	61	113	806	63	194	920	3	080	15
46	947	60	053	870	64	130	922	2	078	14
47	31008	61	68992	933	63	067	925	3	075	13
48	068	60	932	996	63	004	928	3	072	12
49	129	61	871	32059	63	67941	930	2	070	11
50	189	60	811	122	63	878	933	3	067	10
51	250	61	750	185	63	815	936	3	064	9
52	310	60	690	248	63	752	938	2	062	8
53	370	60	630	311	63	689	941	3	059	7
54	430	60	570	373	62	627	944	3	056	6
55	490	60	510	436	63	564	946	2	054	5
56	549	59	451	498	62	502	949	3	051	4
57	609	60	391	561	63	439	952	3	048	3
58	669	60	331	623	62	377	954	2	046	2
59	728	59	272	685	62	315	957	3	043	1
60	31788	60	68212	32747	62	67253	00960	3	99040	0
′	9. *l* cos	d 1′	10. *l* sec	9. *l* cot	d 1′	10. *l* tan	10. *l* csc	d 1′	9. *l* sin	′

Proportional Parts

″	68	67	66	65	64	63	62	61	60	59	3	2
0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	0	0
2	2	2	2	2	2	2	2	2	2	2	0	0
3	3	3	3	3	3	3	3	3	3	3	0	0
4	5	4	4	4	4	4	4	4	4	4	0	0
5	6	6	6	5	5	5	5	5	5	5	0	0
6	7	7	7	6	6	6	6	6	6	6	0	0
7	8	8	8	8	7	7	7	7	7	7	0	0
8	9	9	9	9	9	8	8	8	8	8	0	0
9	10	10	10	10	10	9	9	9	9	9	0	0
10	11	11	11	11	11	10	10	10	10	10	0	0
11	12	12	12	12	12	12	11	11	11	11	1	0
12	14	13	13	13	13	13	12	12	12	12	1	0
13	15	15	14	14	14	14	13	13	13	13	1	0
14	16	16	15	15	15	15	14	14	14	14	1	0
15	17	17	17	16	16	16	15	15	15	15	1	0
16	18	18	18	17	17	17	17	16	16	16	1	1
17	19	19	19	18	18	18	18	17	17	17	1	1
18	20	20	20	20	19	19	19	18	18	18	1	1
19	22	21	21	21	20	20	20	19	19	19	1	1
20	23	22	22	22	21	21	21	20	20	20	1	1
21	24	23	23	23	22	22	22	21	21	21	1	1
22	25	25	24	24	23	23	23	22	22	22	1	1
23	26	26	25	25	25	24	24	23	23	23	1	1
24	27	27	26	26	26	25	25	24	24	24	1	1
25	28	28	27	27	27	26	26	25	25	25	1	1
26	29	29	29	28	28	27	27	26	26	26	1	1
27	31	30	30	29	29	28	28	27	27	27	1	1
28	32	31	31	30	30	29	29	28	28	28	1	1
29	33	32	32	31	31	30	30	29	29	29	1	1
30	34	34	33	32	32	32	31	30	30	30	2	1
31	35	35	34	34	33	33	32	32	31	30	2	1
32	36	36	35	35	34	34	33	33	32	31	2	1
33	37	37	36	36	35	35	34	34	33	32	2	1
34	39	38	37	37	36	36	35	35	34	33	2	1
35	40	39	39	38	37	37	36	36	35	34	2	1
36	41	40	40	39	38	38	37	37	36	35	2	1
37	42	41	41	40	39	39	38	38	37	36	2	1
38	43	42	42	41	41	40	39	39	38	37	2	1
39	44	44	43	42	42	41	40	40	39	38	2	1
40	45	45	44	43	43	42	41	41	40	39	2	1
41	46	46	45	44	44	43	42	42	41	40	2	1
42	48	47	46	46	45	44	43	43	42	41	2	1
43	49	48	47	47	46	45	44	44	43	42	2	1
44	50	49	48	48	47	46	45	45	44	43	2	1
45	51	50	49	49	48	47	47	46	45	44	2	2
46	52	51	51	50	49	48	48	47	46	45	2	2
47	53	52	52	51	50	49	49	48	47	46	2	2
48	54	54	53	52	51	50	50	49	48	47	2	2
49	56	55	54	53	52	51	51	50	49	48	2	2
50	57	56	55	54	53	52	52	51	50	49	2	2
51	58	57	56	55	54	54	53	52	51	50	3	2
52	59	58	57	56	55	55	54	53	52	51	3	2
53	60	59	58	57	57	56	55	54	53	52	3	2
54	61	60	59	58	58	57	56	55	54	53	3	2
55	62	61	61	60	59	58	57	56	55	54	3	2
56	63	63	62	61	60	59	58	57	56	55	3	2
57	65	64	63	62	61	60	59	58	57	56	3	2
58	66	65	64	63	62	61	60	59	58	57	3	2
59	67	66	65	64	63	62	61	60	59	58	3	2
60	68	67	66	65	64	63	62	61	60	59	3	2
″	68	67	66	65	64	63	62	61	60	59	3	2

Proportional Parts

′	*l* sin 9.	d 1′	*l* csc 10.	*l* tan 9.	d 1′	*l* cot 10.	*l* sec 10.	d 1′	*l* cos 9.	′
0	31788	59	68212	32747	63	67253	00960	2	99040	60
1	847	60	153	810	62	190	962	3	038	59
2	907	59	093	872	61	128	965	3	035	58
3	966	59	034	933	62	067	968	2	032	57
4	32025	59	67975	995	62	005	970	3	030	56
5	084	59	916	33057	62	66943	973	3	027	55
6	143	59	857	119	61	881	976	2	024	54
7	202	59	798	180	62	820	978	3	022	53
8	261	58	739	242	61	758	981	3	019	52
9	319	59	681	303	62	697	984	3	016	51
10	378	59	622	365	61	635	987	2	013	50
11	437	58	563	426	61	574	989	3	011	49
12	495	58	505	487	61	513	992	3	008	48
13	553	59	447	548	61	452	995	3	005	47
14	612	58	388	609	61	391	998	2	002	46
15	670	58	330	670	61	330	01000	3	000	45
16	728	58	272	731	61	269	003	3	98997	44
17	786	58	214	792	61	208	006	3	994	43
18	844	58	156	853	60	147	009	2	991	42
19	902	58	098	913	61	087	011	3	989	41
20	960	58	040	974	60	026	014	3	986	40
21	33018	57	66982	34034	61	65966	017	3	983	39
22	075	58	925	095	60	905	020	2	980	38
23	133	57	867	155	60	845	022	3	978	37
24	190	58	810	215	61	785	025	3	975	36
25	248	57	752	276	60	724	028	3	972	35
26	305	57	695	336	60	664	031	2	969	34
27	362	58	638	396	60	604	033	3	967	33
28	420	57	580	456	60	544	036	3	964	32
29	477	57	523	516	60	484	039	3	961	31
30	33534	57	66466	34576	59	65424	01042	3	98958	30
31	591	56	409	635	60	365	045	2	955	29
32	647	57	353	695	60	305	047	3	953	28
33	704	57	296	755	59	245	050	3	950	27
34	761	57	239	814	60	186	053	3	947	26
35	818	56	182	874	59	126	056	3	944	25
36	874	57	126	933	59	067	059	3	941	24
37	931	56	069	992	59	008	062	2	938	23
38	987	56	013	35051	60	64949	064	3	936	22
39	34043	57	65957	111	59	889	067	3	933	21
40	100	56	900	170	59	830	070	3	930	20
41	156	56	844	229	59	771	073	3	927	19
42	212	56	788	288	59	712	076	3	924	18
43	268	56	732	347	58	653	079	2	921	17
44	324	56	676	405	59	595	081	3	919	16
45	380	56	620	464	59	536	084	3	916	15
46	436	55	564	523	58	477	087	3	913	14
47	491	56	509	581	59	419	090	3	910	13
48	547	55	453	640	58	360	093	3	907	12
49	602	56	398	698	59	302	096	3	904	11
50	658	55	342	757	58	243	099	3	901	10
51	713	56	287	815	58	185	102	2	898	9
52	769	55	231	873	58	127	104	3	896	8
53	824	55	176	931	58	069	107	3	893	7
54	879	55	121	989	58	011	110	3	890	6
55	934	55	066	36047	58	63953	113	3	887	5
56	989	55	011	105	58	895	116	3	884	4
57	35044	55	64956	163	58	837	119	3	881	3
58	099	55	901	221	58	779	122	3	878	2
59	154	55	846	279	57	721	125	3	875	1
60	35209		64791	36336		63664	01128		98872	0
′	9. *l* cos	d 1′	10. *l* sec	9. *l* cot	d 1′	10. *l* tan	10. *l* csc	d 1′	9. *l* sin	′

″	Proportional Parts 63	62	61	60	59	58	57	56	55	3	2
0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	0	0
2	2	2	2	2	2	2	2	2	2	0	0
3	3	3	3	3	3	3	3	3	3	0	0
4	4	4	4	4	4	4	4	4	4	0	0
5	5	5	5	5	5	5	5	5	5	0	0
6	6	6	6	6	6	6	6	6	6	0	0
7	7	7	7	7	7	7	7	7	6	0	0
8	8	8	8	8	8	8	8	7	7	0	0
9	9	9	9	9	9	9	9	8	8	0	0
10	10	10	10	10	10	10	10	9	9	0	0
11	12	11	11	11	11	11	10	10	10	1	0
12	13	12	12	12	12	12	11	11	11	1	0
13	14	13	13	13	13	13	12	12	12	1	0
14	15	14	14	14	14	14	13	13	13	1	0
15	16	15	15	15	15	14	14	14	14	1	0
16	17	17	16	16	16	15	15	15	15	1	1
17	18	18	17	17	17	16	16	16	16	1	1
18	19	19	18	18	18	17	17	17	16	1	1
19	20	20	19	19	19	18	18	18	17	1	1
20	21	21	20	20	20	19	19	19	18	1	1
21	22	22	21	21	21	20	20	20	19	1	1
22	23	23	22	22	22	21	21	21	20	1	1
23	24	24	23	23	23	22	22	21	21	1	1
24	25	25	24	24	24	23	23	22	22	1	1
25	26	26	25	25	25	24	24	23	23	1	1
26	27	27	26	26	26	25	25	24	24	1	1
27	28	28	27	27	27	26	26	25	25	1	1
28	29	29	28	28	28	27	27	26	26	1	1
29	30	30	29	29	29	28	28	27	27	1	1
30	32	31	30	30	30	29	28	28	28	2	1
31	33	32	32	31	30	30	29	29	28	2	1
32	34	33	33	32	31	31	30	30	29	2	1
33	35	34	34	33	32	32	31	31	30	2	1
34	36	35	35	34	33	33	32	32	31	2	1
35	37	36	36	35	34	34	33	33	32	2	1
36	38	37	37	36	35	35	34	34	33	2	1
37	39	38	38	37	36	36	35	35	34	2	1
38	40	39	39	38	37	37	36	35	35	2	1
39	41	40	40	39	38	38	37	36	36	2	1
40	42	41	41	40	39	39	38	37	37	2	1
41	43	42	42	41	40	40	39	38	38	2	1
42	44	43	43	42	41	41	40	39	38	2	1
43	45	44	44	43	42	42	41	40	39	2	1
44	46	45	45	44	43	43	42	41	40	2	1
45	47	47	46	45	44	44	43	42	41	2	2
46	48	48	47	46	45	44	44	43	42	2	2
47	49	49	48	47	46	45	45	44	43	2	2
48	50	50	49	48	47	46	46	45	44	2	2
49	51	51	50	49	48	47	47	46	45	2	2
50	52	52	51	50	49	48	48	47	46	2	2
51	54	53	52	51	50	49	48	48	47	3	2
52	55	54	53	52	51	50	49	49	48	3	2
53	56	55	54	53	52	51	50	49	49	3	2
54	57	56	55	54	53	52	51	50	50	3	2
55	58	57	56	55	54	53	52	51	50	3	2
56	59	58	57	56	55	54	53	52	51	3	2
57	60	59	58	57	56	55	54	53	52	3	2
58	61	60	59	58	57	56	55	54	53	3	2
59	62	61	60	59	58	57	56	55	54	3	2
60	63	62	61	60	59	58	57	56	55	3	2
″	63	62	61	60	59	58	57	56	55	3	2

Proportional Parts

′	l sin 9.	d 1′	l csc 10.	l tan 9.	d 1′	l cot 10.	l sec 10.	d 1′	l cos 9.	′
0	35209	54	64791	36336	58	63664	01128	3	98872	60
1	263	55	737	394	58	606	131	2	869	59
2	318	55	682	452	57	548	133	3	867	58
3	373	54	627	509	57	491	136	3	864	57
4	427	54	573	566	58	434	139	3	861	56
5	481	55	519	624	57	376	142	3	858	55
6	536	54	464	681	57	319	145	3	855	54
7	590	54	410	738	57	262	148	3	852	53
8	644	54	356	795	57	205	151	3	849	52
9	698	54	302	852	57	148	154	3	846	51
10	752	54	248	909	57	091	157	3	843	50
11	806	54	194	966	57	034	160	3	840	49
12	860	54	140	37023	57	62977	163	3	837	48
13	914	54	086	080	57	920	166	3	834	47
14	968	54	032	137	56	863	169	3	831	46
15	36022	53	63978	193	57	807	172	3	828	45
16	075	54	925	250	56	750	175	3	825	44
17	129	53	871	306	57	694	178	3	822	43
18	182	54	818	363	56	637	181	3	819	42
19	236	53	764	419	57	581	184	3	816	41
20	289	53	711	476	56	524	187	3	813	40
21	342	53	658	532	56	468	190	3	810	39
22	395	54	605	588	56	412	193	3	807	38
23	449	53	551	644	56	356	196	3	804	37
24	502	53	498	700	56	300	199	3	801	36
25	555	53	445	756	56	244	202	3	798	35
26	608	52	392	812	56	188	205	3	795	34
27	660	53	340	868	56	132	208	3	792	33
28	713	53	287	924	56	076	211	3	789	32
29	766	53	234	980	55	020	214	3	786	31
30	36819	52	63181	38035	56	61965	01217	3	98783	30
31	871	53	129	091	56	909	220	3	780	29
32	924	52	076	147	55	853	223	3	777	28
33	976	52	024	202	55	798	226	3	774	27
34	37028	53	62972	257	56	743	229	3	771	26
35	081	52	919	313	55	687	232	3	768	25
36	133	52	867	368	55	632	235	3	765	24
37	185	52	815	423	56	577	238	3	762	23
38	237	52	763	479	55	521	241	3	759	22
39	289	52	711	534	55	466	244	3	756	21
40	341	52	659	589	55	411	247	3	753	20
41	393	52	607	644	55	356	250	4	750	19
42	445	52	555	699	55	301	254	3	746	18
43	497	52	503	754	54	246	257	3	743	17
44	549	51	451	808	55	192	260	3	740	16
45	600	52	400	863	55	137	263	3	737	15
46	652	51	348	918	54	082	266	3	734	14
47	703	52	297	972	55	028	269	3	731	13
48	755	51	245	39027	55	60973	272	3	728	12
49	806	52	194	082	54	918	275	3	725	11
50	858	51	142	136	54	864	278	3	722	10
51	909	51	091	190	55	810	281	4	719	9
52	960	51	040	245	54	755	285	3	715	8
53	38011	51	61989	299	54	701	288	3	712	7
54	062	51	938	353	54	647	291	3	709	6
55	113	51	887	407	54	593	294	3	706	5
56	164	51	836	461	54	539	297	3	703	4
57	215	51	785	515	54	485	300	3	700	3
58	266	51	734	569	54	431	303	3	697	2
59	317	51	683	623	54	377	306	4	694	1
60	38368		61632	39677		60323	01310		98690	0
′	9. l cos	d 1′	10. l sec	9. l cot	d 1′	10. l tan	10. l csc	d 1′	9. l sin	′

Proportional Parts

″	58	57	56	55	54	53	52	51	4	3	2
0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	0	0	0
2	2	2	2	2	2	2	2	2	0	0	0
3	3	3	3	3	3	3	3	3	0	0	0
4	4	4	4	4	4	4	3	3	0	0	0
5	5	5	5	5	4	4	4	4	0	0	0
6	6	6	6	6	5	5	5	5	0	0	0
7	7	7	7	6	6	6	6	6	0	0	0
8	8	8	7	7	7	7	7	7	1	0	0
9	9	9	8	8	8	8	8	8	1	0	0
10	10	10	9	9	9	9	9	8	1	0	0
11	11	10	10	10	10	10	10	9	1	1	0
12	12	11	11	11	11	11	10	10	1	1	0
13	13	12	12	12	12	11	11	11	1	1	0
14	14	13	13	13	13	12	12	12	1	1	0
15	14	14	14	14	14	13	13	13	1	1	0
16	15	15	15	15	14	14	14	14	1	1	1
17	16	16	16	16	15	15	15	14	1	1	1
18	17	17	17	16	16	16	16	15	1	1	1
19	18	18	18	17	17	17	16	16	1	1	1
20	19	19	19	18	18	18	17	17	1	1	1
21	20	20	20	19	19	19	18	18	1	1	1
22	21	21	21	20	20	19	19	19	1	1	1
23	22	22	21	21	21	20	20	20	2	1	1
24	23	23	22	22	22	21	21	20	2	1	1
25	24	24	23	23	22	22	22	21	2	1	1
26	25	25	24	24	23	23	23	22	2	1	1
27	26	26	25	25	24	24	23	23	2	1	1
28	27	27	26	26	25	25	24	24	2	1	1
29	28	28	27	27	26	26	25	25	2	1	1
30	29	28	28	28	27	26	26	26	2	2	1
31	30	29	29	28	28	27	27	26	2	2	1
32	31	30	30	29	29	28	28	27	2	2	1
33	32	31	31	30	30	29	29	28	2	2	1
34	33	32	32	31	31	30	29	29	2	2	1
35	34	33	33	32	32	31	30	30	2	2	1
36	35	34	34	33	32	32	31	31	2	2	1
37	36	35	35	34	33	33	32	31	2	2	1
38	37	36	35	35	34	34	33	32	3	2	1
39	38	37	36	36	35	34	34	33	3	2	1
40	39	38	37	37	36	35	35	34	3	2	1
41	40	39	38	38	37	36	36	35	3	2	1
42	41	40	39	38	38	37	36	36	3	2	1
43	42	41	40	39	39	38	37	37	3	2	1
44	43	42	41	40	40	39	38	37	3	2	1
45	44	43	42	41	40	40	39	38	3	2	2
46	44	44	43	42	41	41	40	39	3	2	2
47	45	45	44	43	42	42	41	40	3	2	2
48	46	46	45	44	43	42	42	41	3	2	2
49	47	47	46	45	44	43	42	42	3	2	2
50	48	48	47	46	45	44	43	42	3	2	2
51	49	48	48	47	46	45	44	43	3	3	2
52	50	49	49	48	47	46	45	44	3	3	2
53	51	50	49	49	48	47	46	45	4	3	2
54	52	51	50	50	49	48	47	46	4	3	2
55	53	52	51	50	50	49	48	47	4	3	2
56	54	53	52	51	50	49	49	48	4	3	2
57	55	54	53	52	51	50	49	48	4	3	2
58	56	55	54	53	52	51	50	49	4	3	2
59	57	56	55	54	53	52	51	50	4	3	2
60	58	57	56	55	54	53	52	51	4	3	2
″	58	57	56	55	54	53	52	51	4	3	2

Proportional Parts

103° 76°

′	l sin 9.	d 1′	l csc 10.	l tan 9.	d 1′	l cot 10.	l sec 10.	d 1′	l cos 9.	′
0	38368		61632	39677		60323	01310		98690	60
1	418	50	582	731	54	269	313	3	687	59
2	469	51	531	785	54	215	316	3	684	58
3	519	50	481	838	53	162	319	3	681	57
4	570	51	430	892	54	108	322	3	678	56
5	620	50	380	945	53	055	325	3	675	55
6	670	50	330	999	54	001	329	4	671	54
7	721	51	279	40052	53	59948	332	3	668	53
8	771	50	229	106	54	894	335	3	665	52
9	821	50	179	159	53	841	338	3	662	51
10	871	50	129	212	53	788	341	3	659	50
11	921	50	079	266	54	734	344	3	656	49
12	971	50	029	319	53	681	348	4	652	48
13	39021	50	60979	372	53	628	351	3	649	47
14	071	50	929	425	53	575	354	3	646	46
15	121	50	879	478	53	522	357	3	643	45
16	170	49	830	531	53	469	360	3	640	44
17	220	50	780	584	53	416	364	4	636	43
18	270	50	730	636	52	364	367	3	633	42
19	319	49	681	689	53	311	370	3	630	41
20	369	50	631	742	53	258	373	3	627	40
21	418	49	582	795	53	205	377	4	623	39
22	467	49	533	847	52	153	380	3	620	38
23	517	50	483	900	53	100	383	3	617	37
24	566	49	434	952	52	048	386	3	614	36
25	615	49	385	41005	53	58995	390	4	610	35
26	664	49	336	057	52	943	393	3	607	34
27	713	49	287	109	52	891	396	3	604	33
28	762	49	238	161	52	839	399	3	601	32
29	811	49	189	214	53	786	403	4	597	31
30	39860	49	60140	41266	52	58734	01406	3	98594	30
31	909	49	091	318	52	682	409	3	591	29
32	958	49	042	370	52	630	412	3	588	28
33	40006	48	59994	422	52	578	416	4	584	27
34	055	49	945	474	52	526	419	3	581	26
35	103	48	897	526	52	474	422	3	578	25
36	152	49	848	578	52	422	426	4	574	24
37	200	48	800	629	51	371	429	3	571	23
38	249	49	751	681	52	319	432	3	568	22
39	297	48	703	733	52	267	435	3	565	21
40	346	49	654	784	51	216	439	4	561	20
41	394	48	606	836	52	164	442	3	558	19
42	442	48	558	887	51	113	445	3	555	18
43	490	48	510	939	52	061	449	4	551	17
44	538	48	462	990	51	010	452	3	548	16
45	586	48	414	42041	51	57959	455	3	545	15
46	634	48	366	093	52	907	459	4	541	14
47	682	48	318	144	51	856	462	3	538	13
48	730	48	270	195	51	805	465	3	535	12
49	778	48	222	246	51	754	469	4	531	11
50	825	47	175	297	51	703	472	3	528	10
51	873	48	127	348	51	652	475	3	525	9
52	921	48	079	399	51	601	479	4	521	8
53	968	47	032	450	51	550	482	3	518	7
54	41016	48	58984	501	51	499	485	3	515	6
55	063	47	937	552	51	448	489	4	511	5
56	111	48	889	603	51	397	492	3	508	4
57	158	47	842	653	50	347	495	3	505	3
58	205	47	795	704	51	296	499	4	501	2
59	252	47	748	755	51	245	502	3	498	1
60	41300	48	58700	42805	50	57195	01506	4	98494	0
′	9. l cos	d 1′	10. l sec	9. l cot	d 1′	10. l tan	10. l csc	d 1′	9. l sin	′

Proportional Parts										
″	54	53	52	51	50	49	48	47	4	3
0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	0	0
2	2	2	2	2	2	2	2	2	0	0
3	3	3	3	3	2	2	2	2	0	0
4	4	4	3	3	3	3	3	3	0	0
5	4	4	4	4	4	4	4	4	0	0
6	5	5	5	5	5	5	5	5	0	0
7	6	6	6	6	6	6	6	5	0	0
8	7	7	7	7	7	7	6	6	1	0
9	8	8	8	8	8	7	7	7	1	0
10	9	9	9	8	8	8	8	8	1	0
11	10	10	10	9	9	9	9	9	1	1
12	11	11	10	10	10	10	10	9	1	1
13	12	11	11	11	11	11	10	10	1	1
14	13	12	12	12	12	11	11	11	1	1
15	14	13	13	13	12	12	12	12	1	1
16	14	14	14	14	13	13	13	13	1	1
17	15	15	15	14	14	14	14	13	1	1
18	16	16	16	15	15	15	14	14	1	1
19	17	17	16	16	16	16	15	15	1	1
20	18	18	17	17	17	16	16	16	1	1
21	19	19	18	18	18	17	17	16	1	1
22	20	19	19	19	18	18	18	17	1	1
23	21	20	20	20	19	19	18	18	2	1
24	22	21	21	20	20	20	19	19	2	1
25	22	22	22	21	21	20	20	20	2	1
26	23	23	23	22	22	21	21	20	2	1
27	24	24	23	23	22	22	22	21	2	1
28	25	25	24	24	23	23	22	22	2	1
29	26	26	25	25	24	24	23	23	2	1
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31	28	27	27	26	26	25	25	24	2	2
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54	49	48	47	46	45	44	43	42	4	3
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58	52	51	50	49	48	47	46	45	4	3
59	53	52	51	50	49	48	47	46	4	3
60	54	53	52	51	50	49	48	47	4	3
″	54	53	52	51	50	49	48	47	4	3
Proportional Parts										

104° 75°

′	l sin 9.	d 1′	l csc 10.	l tan 9.	d 1′	l cot 10.	l sec 10.	d 1′	l cos 9.	′
0	41300	47	58700	42805	51	57195	01506	3	98494	60
1	347	47	653	856	50	144	509	3	491	59
2	394	47	606	906	51	094	512	4	488	58
3	441	47	559	957	50	043	516	3	484	57
4	488	47	512	43007	50	56993	519	4	481	56
5	535	47	465	057	51	943	523	3	477	55
6	582	46	418	108	50	892	526	3	474	54
7	628	47	372	158	50	842	529	4	471	53
8	675	47	325	208	50	792	533	3	467	52
9	722	46	278	258	50	742	536	4	464	51
10	768	47	232	308	50	692	540	3	460	50
11	815	46	185	358	50	642	543	4	457	49
12	861	47	139	408	50	592	547	3	453	48
13	908	46	092	458	50	542	550	3	450	47
14	954	47	046	508	50	492	553	4	447	46
15	42001	46	57999	558	49	442	557	3	443	45
16	047	46	953	607	50	393	560	4	440	44
17	093	47	907	657	50	343	564	3	436	43
18	140	46	860	707	49	293	567	4	433	42
19	186	46	814	756	50	244	571	3	429	41
20	232	46	768	806	49	194	574	4	426	40
21	278	46	722	855	50	145	578	3	422	39
22	324	46	676	905	49	095	581	4	419	38
23	370	46	630	954	50	046	585	3	415	37
24	416	45	584	44004	49	55996	588	3	412	36
25	461	46	539	053	49	947	591	4	409	35
26	507	46	493	102	49	898	595	3	405	34
27	553	46	447	151	50	849	598	4	402	33
28	599	45	401	201	49	799	602	3	398	32
29	644	46	356	250	49	750	605	4	395	31
30	42690	45	57310	44299	49	55701	01609	3	98391	30
31	735	46	265	348	49	652	612	4	388	29
32	781	45	219	397	49	603	616	3	384	28
33	826	46	174	446	49	554	619	4	381	27
34	872	45	128	495	49	505	623	4	377	26
35	917	45	083	544	48	456	627	3	373	25
36	962	46	038	592	49	408	630	4	370	24
37	43008	45	56992	641	49	359	634	3	366	23
38	053	45	947	690	48	310	637	4	363	22
39	098	45	902	738	49	262	641	3	359	21
40	143	45	857	787	49	213	644	4	356	20
41	188	45	812	836	48	164	648	3	352	19
42	233	45	767	884	49	116	651	4	349	18
43	278	45	722	933	48	067	655	3	345	17
44	323	44	677	981	48	019	658	4	342	16
45	367	45	633	45029	49	54971	662	4	338	15
46	412	45	588	078	48	922	666	3	334	14
47	457	45	543	126	48	874	669	4	331	13
48	502	44	498	174	48	826	673	3	327	12
49	546	45	454	222	49	778	676	4	324	11
50	591	44	409	271	48	729	680	3	320	10
51	635	45	365	319	48	681	683	4	317	9
52	680	44	320	367	48	633	687	4	313	8
53	724	45	276	415	48	585	691	3	309	7
54	769	44	231	463	48	537	694	4	306	6
55	813	44	187	511	48	489	698	3	302	5
56	857	44	143	559	47	441	701	4	299	4
57	901	45	099	606	48	394	705	4	295	3
58	946	44	054	654	48	346	709	3	291	2
59	990	44	010	702	48	298	712	4	288	1
60	44034		55966	45750		54250	01716		98284	0
′	9. l cos	d 1′	10. l sec	9. l cot	d 1′	10. l tan	10. l csc	d 1′	9. l sin	′

″	51	50	49	48	47	46	45	44	4	3
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3	3	2	2	2	2	2	2	2	0	0
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8	7	7	7	6	6	6	6	6	1	0
9	8	8	7	7	7	7	7	7	1	0
10	8	8	8	8	8	8	8	7	1	0
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12	10	10	10	10	9	9	9	9	1	1
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58	49	48	47	46	45	44	44	43	4	3
59	50	49	48	47	46	45	44	43	4	3
60	51	50	49	48	47	46	45	44	4	3
″	51	50	49	48	47	46	45	44	4	3

Proportional Parts

′	l sin 9.	d 1′	l csc 10.	l tan 9.	d 1′	l cot 10.	l sec 10.	d 1′	l cos 9.	′
0	44034	44	55966	45750	47	54250	01716	3	98284	60
1	078	44	922	797	48	203	719	4	281	59
2	122	44	878	845	47	155	723	4	277	58
3	166	44	834	892	48	108	727	3	273	57
4	210	43	790	940	47	060	730	4	270	56
5	253	44	747	987	48	013	734	4	266	55
6	297	44	703	46035	47	53965	738	3	262	54
7	341	44	659	082	48	918	741	4	259	53
8	385	43	615	130	47	870	745	4	255	52
9	428	44	572	177	47	823	749	3	251	51
10	472	44	528	224	47	776	752	4	248	50
11	516	43	484	271	48	729	756	4	244	49
12	559	43	441	319	47	681	760	3	240	48
13	602	44	398	366	47	634	763	4	237	47
14	646	43	354	413	47	587	767	4	233	46
15	689	44	311	460	47	540	771	3	229	45
16	733	43	267	507	47	493	774	4	226	44
17	776	43	224	554	47	446	778	4	222	43
18	819	43	181	601	47	399	782	3	218	42
19	862	43	138	648	46	352	785	4	215	41
20	905	43	095	694	47	306	789	4	211	40
21	948	44	052	741	47	259	793	3	207	39
22	992	43	008	788	47	212	796	4	204	38
23	45035	42	54965	835	46	165	800	4	200	37
24	077	43	923	881	47	119	804	4	196	36
25	120	43	880	928	47	072	808	3	192	35
26	163	43	837	975	46	025	811	4	189	34
27	206	43	794	47021	47	52979	815	4	185	33
28	249	43	751	068	46	932	819	4	181	32
29	292	42	708	114	46	886	823	3	177	31
30	45334	43	54666	47160	47	52840	01826	4	98174	30
31	377	42	623	207	46	793	830	4	170	29
32	419	43	581	253	46	747	834	4	166	28
33	462	42	538	299	47	701	838	3	162	27
34	504	43	496	346	46	654	841	4	159	26
35	547	42	453	392	46	608	845	4	155	25
36	589	43	411	438	46	562	849	4	151	24
37	632	42	368	484	46	516	853	3	147	23
38	674	42	326	530	46	470	856	4	144	22
39	716	42	284	576	46	424	860	4	140	21
40	758	43	242	622	46	378	864	4	136	20
41	801	42	199	668	46	332	868	3	132	19
42	843	42	157	714	46	286	871	4	129	18
43	885	42	115	760	46	240	875	4	125	17
44	927	42	073	806	46	194	879	4	121	16
45	969	42	031	852	45	148	883	4	117	15
46	46011	42	53989	897	46	103	887	3	113	14
47	053	42	947	943	46	057	890	4	110	13
48	095	41	905	989	46	011	894	4	106	12
49	136	42	864	48035	45	51965	898	4	102	11
50	178	42	822	080	46	920	902	4	098	10
51	220	42	780	126	45	874	906	4	094	9
52	262	41	738	171	46	829	910	3	090	8
53	303	42	697	217	45	783	913	4	087	7
54	345	41	655	262	45	738	917	4	083	6
55	386	42	614	307	46	693	921	4	079	5
56	428	41	572	353	45	647	925	4	075	4
57	469	42	531	398	45	602	929	4	071	3
58	511	41	489	443	46	557	933	4	067	2
59	552	42	448	489	45	511	937	3	063	1
60	46594		53406	48534		51466	01940		98060	0
′	9. l cos	d 1′	10. l sec	9. l cot	d 1′	10. l tan	10. l csc	d 1′	9. l sin	′

Proportional Parts

″	48	47	46	45	44	43	42	41	4	3
0	0	0	0	0	0	0	0	0	0	0
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3	2	2	2	2	2	2	2	2	0	0
4	3	3	3	3	3	3	3	3	0	0
5	4	4	4	4	4	4	4	3	0	0
6	5	5	5	4	4	4	4	4	0	0
7	6	5	5	5	5	5	5	5	0	0
8	6	6	6	6	6	6	6	5	1	0
9	7	7	7	7	7	6	6	6	1	0
10	8	8	8	8	7	7	7	7	1	0
11	9	9	8	8	8	8	8	8	1	1
12	10	9	9	9	9	9	8	8	1	1
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16	13	13	12	12	12	11	11	11	1	1
17	14	13	13	13	12	12	12	12	1	1
18	14	14	14	14	13	13	13	12	1	1
19	15	15	15	14	14	14	13	13	1	1
20	16	16	15	15	15	14	14	14	1	1
21	17	16	16	16	15	15	15	14	1	1
22	18	17	17	16	16	16	15	15	1	1
23	18	18	18	17	17	16	16	16	2	1
24	19	19	18	18	18	17	17	16	2	1
25	20	20	19	19	18	18	18	17	2	1
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45	36	35	34	34	33	32	32	31	3	2
46	37	36	35	34	34	33	32	31	3	2
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53	42	42	41	40	39	38	37	36	4	3
54	43	42	41	40	40	39	38	37	4	3
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56	45	44	43	42	41	40	39	38	4	3
57	46	45	44	43	42	41	40	39	4	3
58	46	45	44	44	43	42	41	40	4	3
59	47	46	45	44	43	42	41	40	4	3
60	48	47	46	45	44	43	42	41	4	3
″	48	47	46	45	44	43	42	41	4	3

Proportional Parts

′	l sin 9.	d 1′	l csc 10.	l tan 9.	d 1′	l cot 10.	l sec 10.	d 1′	l cos 9.	′
0	46594	41	53406	48534	45	51466	01940	4	98060	60
1	635	41	365	579	45	421	944	4	056	59
2	676	41	324	624	45	376	948	4	052	58
3	717	41	283	669	45	331	952	4	048	57
4	758	42	242	714	45	286	956	4	044	56
5	800	41	200	759	45	241	960	4	040	55
6	841	41	159	804	45	196	964	4	036	54
7	882	41	118	849	45	151	968	3	032	53
8	923	41	077	894	45	106	971	4	029	52
9	964	41	036	939	45	061	975	4	025	51
10	47005	40	52995	984	45	016	979	4	021	50
11	045	41	955	49029	44	50971	983	4	017	49
12	086	41	914	073	45	927	987	4	013	48
13	127	41	873	118	45	882	991	4	009	47
14	168	41	832	163	44	837	995	4	005	46
15	209	40	791	207	45	793	999	4	001	45
16	249	41	751	252	44	748	02003	4	97997	44
17	290	40	710	296	45	704	007	4	993	43
18	330	41	670	341	44	659	011	3	989	42
19	371	40	629	385	45	615	014	4	986	41
20	411	41	589	430	44	570	018	4	982	40
21	452	40	548	474	45	526	022	4	978	39
22	492	41	508	519	44	481	026	4	974	38
23	533	40	467	563	44	437	030	4	970	37
24	573	40	427	607	45	393	034	4	966	36
25	613	41	387	652	44	348	038	4	962	35
26	654	40	346	696	44	304	042	4	958	34
27	694	40	306	740	44	260	046	4	954	33
28	734	40	266	784	44	216	050	4	950	32
29	774	40	226	828	44	172	054	4	946	31
30	47814	40	52186	49872	44	50128	02058	4	97942	30
31	854	40	146	916	44	084	062	4	938	29
32	894	40	106	960	44	040	066	4	934	28
33	934	40	066	50004	44	49996	070	4	930	27
34	974	40	026	048	44	952	074	4	926	26
35	48014	40	51986	092	44	908	078	4	922	25
36	054	40	946	136	44	864	082	4	918	24
37	094	39	906	180	43	820	086	4	914	23
38	133	40	867	223	44	777	090	4	910	22
39	173	40	827	267	44	733	094	4	906	21
40	213	39	787	311	44	689	098	4	902	20
41	252	40	748	355	43	645	102	4	898	19
42	292	40	708	398	44	602	106	4	894	18
43	332	39	668	442	43	558	110	4	890	17
44	371	40	629	485	44	515	114	4	886	16
45	411	39	589	529	43	471	118	4	882	15
46	450	40	550	572	44	428	122	4	878	14
47	490	39	510	616	43	384	126	4	874	13
48	529	39	471	659	44	341	130	4	870	12
49	568	39	432	703	43	297	134	5	866	11
50	607	40	393	746	43	254	139	4	861	10
51	647	39	353	789	44	211	143	4	857	9
52	686	39	314	833	43	167	147	4	853	8
53	725	39	275	876	43	124	151	4	849	7
54	764	39	236	919	43	081	155	4	845	6
55	803	39	197	962	43	038	159	4	841	5
56	842	39	158	51005	43	48995	163	4	837	4
57	881	39	119	048	44	952	167	4	833	3
58	920	39	080	092	43	908	171	4	829	2
59	959	39	041	135	43	865	175	4	825	1
60	48998		51002	51178		48822	02179		97821	0
′	9. l cos	d 1′	10. l sec	9. l cot	d 1′	10. l tan	10. l csc	d 1′	9. l sin	′

Proportional Parts

″	45	44	43	42	41	40	39	5	4	3
0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	0	0	0
2	2	1	1	1	1	1	1	0	0	0
3	2	2	2	2	2	2	2	0	0	0
4	3	3	3	3	3	3	3	0	0	0
5	4	4	4	4	3	3	3	0	0	0
6	4	4	4	4	4	4	4	0	0	0
7	5	5	5	5	5	5	5	1	0	0
8	6	6	6	6	5	5	5	1	1	0
9	7	7	6	6	6	6	6	1	1	0
10	8	7	7	7	7	7	6	1	1	0
11	8	8	8	8	8	7	7	1	1	1
12	9	9	9	8	8	8	8	1	1	1
13	10	10	9	9	9	9	8	1	1	1
14	10	10	10	10	10	9	9	1	1	1
15	11	11	11	10	10	10	10	1	1	1
16	12	12	11	11	11	11	10	1	1	1
17	13	12	12	12	12	11	11	1	1	1
18	14	13	13	13	12	12	12	2	1	1
19	14	14	14	13	13	13	12	2	1	1
20	15	15	14	14	14	13	13	2	1	1
21	16	15	15	15	14	14	14	2	1	1
22	16	16	16	15	15	15	14	2	1	1
23	17	17	16	16	16	15	15	2	2	1
24	18	18	17	17	16	16	16	2	2	1
25	19	18	18	18	17	17	16	2	2	1
26	20	19	19	18	18	17	17	2	2	1
27	20	20	19	19	18	18	18	2	2	1
28	21	21	20	20	19	19	18	2	2	1
29	22	21	21	20	20	19	19	2	2	1
30	22	22	22	21	20	20	20	2	2	2
31	23	23	22	22	21	21	20	3	2	2
32	24	23	23	22	22	21	21	3	2	2
33	25	24	24	23	23	22	21	3	2	2
34	26	25	24	24	23	23	22	3	2	2
35	26	26	25	24	24	23	23	3	2	2
36	27	26	26	25	25	24	23	3	2	2
37	28	27	27	26	25	25	24	3	2	2
38	28	28	27	27	26	25	25	3	3	2
39	29	29	28	27	27	26	25	3	3	2
40	30	29	29	28	27	27	26	3	3	2
41	31	30	29	29	28	27	27	3	3	2
42	32	31	30	29	29	28	27	4	3	2
43	32	32	31	30	29	29	28	4	3	2
44	33	32	32	31	30	29	29	4	3	2
45	34	33	32	32	31	30	29	4	3	2
46	34	34	33	32	31	31	30	4	3	2
47	35	34	34	33	32	31	31	4	3	2
48	36	35	34	34	33	32	31	4	3	2
49	37	36	35	34	33	33	32	4	3	2
50	38	37	36	35	34	33	32	4	3	2
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52	39	38	37	36	36	35	34	4	3	3
53	40	39	38	37	36	35	34	4	4	3
54	40	40	39	38	37	36	35	4	4	3
55	41	40	39	38	38	37	36	5	4	3
56	42	41	40	39	38	37	36	5	4	3
57	43	42	41	40	39	38	37	5	4	3
58	44	43	42	41	40	39	38	5	4	3
59	44	43	42	41	40	39	38	5	4	3
60	45	44	43	42	41	40	39	5	4	3
″	45	44	43	42	41	40	39	5	4	3

Proportional Parts

′	*l* sin	d	*l* csc	*l* tan	d	*l* cot	*l* sec	d	*l* cos	′
	9.	1′	10.	9.	1′	10.	10.	1′	9.	
0	48998	39	51002	51178	43	48822	02179	4	97821	60
1	49037	39	50963	221	43	779	183	5	817	59
2	076	39	924	264	42	736	188	4	812	58
3	115	38	885	306	43	694	192	4	808	57
4	153	39	847	349	43	651	196	4	804	56
5	192	39	808	392	43	608	200	4	800	55
6	231	38	769	435	43	565	204	4	796	54
7	269	39	731	478	42	522	208	4	792	53
8	308	39	692	520	43	480	212	4	788	52
9	347	38	653	563	43	437	216	5	784	51
10	385	39	615	606	42	394	221	4	779	50
11	424	38	576	648	43	352	225	4	775	49
12	462	38	538	691	43	309	229	4	771	48
13	500	39	500	734	42	266	233	4	767	47
14	539	38	461	776	43	224	237	4	763	46
15	577	38	423	819	42	181	241	5	759	45
16	615	39	385	861	42	139	246	4	754	44
17	654	38	346	903	43	097	250	4	750	43
18	692	38	308	946	42	054	254	4	746	42
19	730	38	270	988	43	012	258	4	742	41
20	768	38	232	52031	42	47969	262	4	738	40
21	806	38	194	073	42	927	266	5	734	39
22	844	38	156	115	42	885	271	4	729	38
23	882	38	118	157	43	843	275	4	725	37
24	920	38	080	200	42	800	279	4	721	36
25	958	38	042	242	42	758	283	4	717	35
26	996	38	004	284	42	716	287	5	713	34
27	50034	38	49966	326	42	674	292	4	708	33
28	072	38	928	368	42	632	296	4	704	32
29	110	38	890	410	42	590	300	4	700	31
30	50148	37	49852	52452	42	47548	02304	5	97696	30
31	185	38	815	494	42	506	309	4	691	29
32	223	38	777	536	42	464	313	4	687	28
33	261	37	739	578	42	422	317	4	683	27
34	298	38	702	620	41	380	321	5	679	26
35	336	38	664	661	42	339	326	4	674	25
36	374	37	626	703	42	297	330	4	670	24
37	411	38	589	745	42	255	334	4	666	23
38	449	37	551	787	42	213	338	5	662	22
39	486	37	514	829	41	171	343	4	657	21
40	523	38	477	870	42	130	347	4	653	20
41	561	37	439	912	41	088	351	4	649	19
42	598	37	402	953	42	047	355	5	645	18
43	635	38	365	995	42	005	360	4	640	17
44	673	37	327	53037	41	46963	364	4	636	16
45	710	37	290	078	42	922	368	4	632	15
46	747	37	253	120	41	880	372	5	628	14
47	784	37	216	161	41	839	377	4	623	13
48	821	37	179	202	42	798	381	4	619	12
49	858	38	142	244	41	756	385	5	615	11
50	896	37	104	285	42	715	390	4	610	10
51	933	37	067	327	41	673	394	4	606	9
52	970	37	030	368	41	632	398	5	602	8
53	51007	36	48993	409	41	591	403	4	597	7
54	043	37	957	450	42	550	407	4	593	6
55	080	37	920	492	41	508	411	5	589	5
56	117	37	883	533	41	467	416	4	584	4
57	154	37	846	574	41	426	420	4	580	3
58	191	36	809	615	41	385	424	5	576	2
59	227	37	773	656	41	344	429	4	571	1
60	51264		48736	53697		46303	02433		97567	0
′	9.	d	10.	9.	d	10.	10.	d	9.	′
	l cos	1′	*l* sec	*l* cot	1′	*l* tan	*l* csc	1′	*l* sin	

108° 71°

″	Proportional Parts								
	43	42	41	39	38	37	36	5	4
0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	0	0
2	1	1	1	1	1	1	1	0	0
3	2	2	2	2	2	2	2	0	0
4	3	3	3	3	3	2	2	0	0
5	4	4	3	3	3	3	3	0	0
6	4	4	4	4	4	4	4	0	0
7	5	5	5	5	4	4	4	1	0
8	6	6	5	5	5	5	5	1	1
9	6	6	6	6	6	6	5	1	1
10	7	7	7	6	6	6	6	1	1
11	8	8	8	7	7	7	7	1	1
12	9	8	8	8	8	7	7	1	1
13	9	9	9	8	8	8	8	1	1
14	10	10	10	9	9	9	8	1	1
15	11	10	10	10	10	9	9	1	1
16	11	11	11	10	10	10	10	1	1
17	12	12	12	11	11	10	10	1	1
18	13	13	12	12	11	11	11	2	1
19	14	13	13	12	12	12	11	2	1
20	14	14	14	13	13	12	12	2	1
21	15	15	14	14	13	13	13	2	1
22	16	15	15	14	14	14	13	2	1
23	16	16	16	15	15	14	14	2	2
24	17	17	16	16	15	15	14	2	2
25	18	18	17	16	16	15	15	2	2
26	19	18	18	17	16	16	16	2	2
27	19	19	18	18	17	17	16	2	2
28	20	20	19	18	18	17	17	2	2
29	21	20	20	19	18	18	17	2	2
30	22	21	20	20	19	18	18	2	2
31	22	22	21	20	20	19	19	3	2
32	23	22	22	21	20	20	19	3	2
33	24	23	23	21	21	20	20	3	2
34	24	24	23	22	22	21	20	3	2
35	25	24	24	23	22	22	21	3	2
36	26	25	25	23	23	22	22	3	2
37	27	26	25	24	23	23	22	3	2
38	27	27	26	25	24	23	23	3	3
39	28	27	27	25	25	24	23	3	3
40	29	28	27	26	25	25	24	3	3
41	29	29	28	27	26	25	25	3	3
42	30	29	29	27	27	26	25	4	3
43	31	30	29	28	27	27	26	4	3
44	32	31	30	29	28	27	26	4	3
45	32	32	31	29	28	28	27	4	3
46	33	32	31	30	29	28	28	4	3
47	34	33	32	31	30	29	28	4	3
48	34	34	33	31	30	30	29	4	3
49	35	34	33	32	31	30	29	4	3
50	36	35	34	32	32	31	30	4	3
51	37	36	35	33	32	31	31	4	3
52	37	36	36	34	33	32	31	4	3
53	38	37	36	34	34	33	32	4	4
54	39	38	37	35	34	33	32	4	4
55	39	38	38	36	35	34	33	5	4
56	40	39	38	36	35	35	34	5	4
57	41	40	39	37	36	35	34	5	4
58	42	41	40	38	37	36	35	5	4
59	42	41	40	38	37	36	35	5	4
60	43	42	41	39	38	37	36	5	4
″	43	42	41	39	38	37	36	5	4
	Proportional Parts								

′	*l* sin 9.	d 1′	*l* csc 10.	*l* tan 9.	d 1′	*l* cot 10.	*l* sec 10.	d 1′	*l* cos 9.	′
0	51264	37	48736	53697	41	46303	02433	4	97567	60
1	301	37	699	738	41	262	437	5	563	59
2	338	36	662	779	41	221	442	4	558	58
3	374	37	626	820	41	180	446	4	554	57
4	411	36	589	861	41	139	450	5	550	56
5	447	37	553	902	41	098	455	4	545	55
6	484	36	516	943	41	057	459	5	541	54
7	520	37	480	984	41	016	464	4	536	53
8	557	36	443	54025	40	45975	468	4	532	52
9	593	36	407	065	41	935	472	5	528	51
10	629	37	371	106	41	894	477	4	523	50
11	666	36	334	147	40	853	481	4	519	49
12	702	36	298	187	41	813	485	5	515	48
13	738	36	262	228	41	772	490	4	510	47
14	774	37	226	269	40	731	494	5	506	46
15	811	36	189	309	41	691	499	4	501	45
16	847	36	153	350	40	650	503	5	497	44
17	883	36	117	390	41	610	508	4	492	43
18	919	36	081	431	40	569	512	4	488	42
19	955	36	045	471	41	529	516	5	484	41
20	991	36	009	512	40	488	521	4	479	40
21	52027	36	47973	552	41	448	525	5	475	39
22	063	36	937	593	40	407	530	4	470	38
23	099	36	901	633	40	367	534	5	466	37
24	135	36	865	673	41	327	539	4	461	36
25	171	36	829	714	40	286	543	4	457	35
26	207	35	793	754	40	246	547	5	453	34
27	242	36	758	794	41	206	552	4	448	33
28	278	36	722	835	40	165	556	5	444	32
29	314	36	686	875	40	125	561	4	439	31
30	52350	35	47650	54915	40	45085	02565	5	97435	30
31	385	36	615	955	40	045	570	4	430	29
32	421	35	579	995	40	005	574	5	426	28
33	456	36	544	55035	40	44965	579	4	421	27
34	492	35	508	075	40	925	583	5	417	26
35	527	36	473	115	40	885	588	4	412	25
36	563	35	437	155	40	845	592	5	408	24
37	598	36	402	195	40	805	597	4	403	23
38	634	35	366	235	40	765	601	5	399	22
39	669	36	331	275	40	725	606	4	394	21
40	705	35	295	315	40	685	610	5	390	20
41	740	35	260	355	40	645	615	4	385	19
42	775	36	225	395	39	605	619	5	381	18
43	811	35	189	434	40	566	624	4	376	17
44	846	35	154	474	40	526	628	5	372	16
45	881	35	119	514	40	486	633	4	367	15
46	916	35	084	554	39	446	637	5	363	14
47	951	35	049	593	40	407	642	5	358	13
48	986	35	014	633	40	367	647	4	353	12
49	53021	35	46979	673	39	327	651	5	349	11
50	056	36	944	712	40	288	656	4	344	10
51	092	34	908	752	39	248	660	5	340	9
52	126	35	874	791	40	209	665	4	335	8
53	161	35	839	831	39	169	669	5	331	7
54	196	35	804	870	40	130	674	4	326	6
55	231	35	769	910	39	090	678	5	322	5
56	266	35	734	949	40	051	683	5	317	4
57	301	35	699	989	39	011	688	4	312	3
58	336	34	664	56028	39	43972	692	5	308	2
59	370	35	630	067	40	933	697	4	303	1
60	53405		46595	56107		43893	02701		97299	0
′	9. *l* cos	d 1′	10. *l* sec	9. *l* cot	d 1′	10. *l* tan	10. *l* csc	d 1′	9. *l* sin	′

Proportional Parts

″	41	40	39	37	36	35	34	5	4
0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	0	0
2	1	1	1	1	1	1	1	0	0
3	2	2	2	2	2	2	2	0	0
4	3	3	3	2	2	2	2	0	0
5	3	3	3	3	3	3	3	0	0
6	4	4	4	4	4	4	3	0	0
7	5	5	5	4	4	4	4	1	0
8	5	5	5	5	5	5	5	1	1
9	6	6	6	6	5	5	5	1	1
10	7	7	6	6	6	6	6	1	1
11	8	7	7	7	7	6	6	1	1
12	8	8	8	7	7	7	7	1	1
13	9	9	8	8	8	8	7	1	1
14	10	9	9	9	8	8	8	1	1
15	10	10	10	9	9	9	8	1	1
16	11	11	10	10	10	9	9	1	1
17	12	11	11	10	10	10	10	1	1
18	12	12	12	11	11	10	10	2	1
19	13	13	12	12	11	11	11	2	1
20	14	13	13	12	12	12	11	2	1
21	14	14	14	13	13	12	12	2	1
22	15	15	14	14	13	13	12	2	1
23	16	15	15	14	14	13	13	2	2
24	16	16	16	15	14	14	14	2	2
25	17	17	16	15	15	15	14	2	2
26	18	17	17	16	16	15	15	2	2
27	18	18	18	17	16	16	15	2	2
28	19	19	18	17	17	16	16	2	2
29	20	19	19	18	17	17	16	2	2
30	20	20	20	18	18	18	17	2	2
31	21	21	20	19	19	18	18	3	2
32	22	21	21	20	19	19	18	3	2
33	23	22	21	20	20	19	19	3	2
34	23	23	22	21	20	20	19	3	2
35	24	23	23	22	21	20	20	3	2
36	25	24	23	22	22	21	20	3	2
37	25	25	24	23	22	22	21	3	2
38	26	25	25	23	23	22	22	3	3
39	27	26	25	24	23	23	22	3	3
40	27	27	26	25	24	23	23	3	3
41	28	27	27	25	25	24	23	3	3
42	29	28	27	26	25	24	24	4	3
43	29	29	28	27	26	25	24	4	3
44	30	29	29	27	26	26	25	4	3
45	31	30	29	28	27	26	26	4	3
46	31	31	30	28	28	27	26	4	3
47	32	31	31	29	28	27	27	4	3
48	33	32	31	30	29	28	27	4	3
49	33	33	32	30	29	29	28	4	3
50	34	33	32	31	30	29	28	4	3
51	35	34	33	31	31	30	29	4	3
52	36	35	34	32	31	30	29	4	3
53	36	35	34	33	32	31	30	4	4
54	37	36	35	33	32	32	31	4	4
55	38	37	36	34	33	32	31	5	4
56	38	37	36	35	34	33	32	5	4
57	39	38	37	35	34	33	32	5	4
58	40	39	38	36	35	34	33	5	4
59	40	39	38	36	35	34	33	5	4
60	41	40	39	37	36	35	34	5	4
″	41	40	39	37	36	35	34	5	4

Proportional Parts

′	*l* sin	d	*l* csc	*l* tan	d	*l* cot	*l* sec	d	*l* cos	′
	9.	1′	10.	9.	1′	10.	10.	1′	9.	
0	53405	35	46595	56107	39	43893	02701	5	97299	60
1	440	35	560	146	39	854	706	5	294	59
2	475	34	525	185	39	815	711	4	289	58
3	509	35	491	224	40	776	715	5	285	57
4	544	34	456	264	39	736	720	4	280	56
5	578	35	422	303	39	697	724	5	276	55
6	613	34	387	342	39	658	729	5	271	54
7	647	35	353	381	39	619	734	4	266	53
8	682	34	318	420	39	580	738	5	262	52
9	716	35	284	459	39	541	743	5	257	51
10	751	34	249	498	39	502	748	4	252	50
11	785	34	215	537	39	463	752	5	248	49
12	819	35	181	576	39	424	757	5	243	48
13	854	34	146	615	39	385	762	4	238	47
14	888	34	112	654	39	346	766	5	234	46
15	922	35	078	693	39	307	771	5	229	45
16	957	34	043	732	39	268	776	4	224	44
17	991	34	009	771	39	229	780	5	220	43
18	54025	34	45975	810	39	190	785	5	215	42
19	059	34	941	849	38	151	790	4	210	41
20	093	34	907	887	39	113	794	5	206	40
21	127	34	873	926	39	074	799	5	201	39
22	161	34	839	965	39	035	804	4	196	38
23	195	34	805	57004	38	42996	808	5	192	37
24	229	34	771	042	39	958	813	5	187	36
25	263	34	737	081	39	919	818	4	182	35
26	297	34	703	120	38	880	822	5	178	34
27	331	34	669	158	39	842	827	5	173	33
28	365	34	635	197	38	803	832	5	168	32
29	399	34	601	235	39	765	837	4	163	31
30	54433	33	45567	57274	38	42726	02841	5	97159	30
31	466	34	534	312	39	688	846	5	154	29
32	500	34	500	351	38	649	851	4	149	28
33	534	33	466	389	39	611	855	5	145	27
34	567	34	433	428	38	572	860	5	140	26
35	601	34	399	466	38	534	865	5	135	25
36	635	33	365	504	39	496	870	4	130	24
37	668	34	332	543	38	457	874	5	126	23
38	702	33	298	581	38	419	879	5	121	22
39	735	34	265	619	39	381	884	5	116	21
40	769	33	231	658	38	342	889	4	111	20
41	802	34	198	696	38	304	893	5	107	19
42	836	33	164	734	38	266	898	5	102	18
43	869	34	131	772	38	228	903	5	097	17
44	903	33	097	810	39	190	908	5	092	16
45	936	33	064	849	38	151	913	4	087	15
46	969	34	031	887	38	113	917	5	083	14
47	55003	33	44997	925	38	075	922	5	078	13
48	036	33	964	963	38	037	927	5	073	12
49	069	33	931	58001	38	41999	932	5	068	11
50	102	34	898	039	38	961	937	4	063	10
51	136	33	864	077	38	923	941	5	059	9
52	169	33	831	115	38	885	946	5	054	8
53	202	33	798	153	38	847	951	5	049	7
54	235	33	765	191	38	809	956	5	044	6
55	268	33	732	229	38	771	961	4	039	5
56	301	33	699	267	37	733	965	5	035	4
57	334	33	666	304	38	696	970	5	030	3
58	367	33	633	342	38	658	975	5	025	2
59	400	33	600	380	38	620	980	5	020	1
60	55433		44567	58418		41582	02985		97015	0
′	9.	d	10.	9.	d	10.	10.	d	9.	′
	l cos	1′	*l* sec	*l* cot	1′	*l* tan	*l* csc	1′	*l* sin	

″	Proportional Parts								
	40	39	38	37	35	34	33	5	4
0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	0	0
2	1	1	1	1	1	1	1	0	0
3	2	2	2	2	2	2	2	0	0
4	3	3	3	2	2	2	2	0	0
5	3	3	3	3	3	3	3	0	0
6	4	4	4	4	4	3	3	0	0
7	5	5	4	4	4	4	4	1	0
8	5	5	5	5	5	5	4	1	1
9	6	6	6	6	5	5	5	1	1
10	7	6	6	6	6	6	6	1	1
11	7	7	7	7	6	6	6	1	1
12	8	8	8	7	7	7	7	1	1
13	9	8	8	8	8	7	7	1	1
14	9	9	9	9	8	8	8	1	1
15	10	10	10	9	9	8	8	1	1
16	11	10	10	10	9	9	9	1	1
17	11	11	11	10	10	10	9	1	1
18	12	12	11	11	10	10	10	2	1
19	13	12	12	12	11	11	10	2	1
20	13	13	13	12	12	11	11	2	1
21	14	14	13	13	12	12	12	2	1
22	15	14	14	14	13	12	12	2	1
23	15	15	15	14	13	13	13	2	2
24	16	16	15	15	14	14	13	2	2
25	17	16	16	15	15	14	14	2	2
26	17	17	16	16	15	15	14	2	2
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33	22	21	21	20	19	19	18	3	2
34	23	22	22	21	20	19	19	3	2
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36	24	23	23	22	21	20	20	3	2
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42	28	27	27	26	24	24	23	4	3
43	29	28	27	27	25	24	24	4	3
44	29	29	28	27	26	25	24	4	3
45	30	29	28	28	26	26	25	4	3
46	31	30	29	28	27	26	25	4	3
47	31	31	30	29	27	27	26	4	3
48	32	31	30	30	28	27	26	4	3
49	33	32	31	30	29	28	27	4	3
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51	34	33	32	31	30	29	28	4	3
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53	35	34	34	33	31	30	29	4	4
54	36	35	34	33	32	31	30	4	4
55	37	36	35	34	32	31	30	5	4
56	37	36	35	35	33	32	31	5	4
57	38	37	36	35	33	32	31	5	4
58	39	38	37	36	34	33	32	5	4
59	39	38	37	36	34	33	32	5	4
60	40	39	38	37	35	34	33	5	4
″	40	39	38	37	35	34	33	5	4
	Proportional Parts								

′	*l* sin	d 1′	*l* csc	*l* tan	d 1′	*l* cot	*l* sec	d 1′	*l* cos	′
	9.		10.	9.		10.	10.		9.	
0	55433	33	44567	58418	37	41582	02985	5	97015	60
1	466	33	534	455	38	545	990	5	010	59
2	499	33	501	493	38	507	995	4	005	58
3	532	32	468	531	38	469	999	5	001	57
4	564	33	436	569	37	431	03004	5	96996	56
5	597	33	403	606	38	394	009	5	991	55
6	630	33	370	644	37	356	014	5	986	54
7	663	32	337	681	38	319	019	5	981	53
8	695	33	305	719	38	281	024	5	976	52
9	728	33	272	757	37	243	029	5	971	51
10	761	32	239	794	38	206	034	4	966	50
11	793	33	207	832	37	168	038	5	962	49
12	826	32	174	869	38	131	043	5	957	48
13	858	33	142	907	37	093	048	5	952	47
14	891	32	109	944	37	056	053	5	947	46
15	923	33	077	981	38	019	058	5	942	45
16	956	32	044	59019	37	40981	063	5	937	44
17	988	33	012	056	38	944	068	5	932	43
18	56021	32	43979	094	37	906	073	5	927	42
19	053	32	947	131	37	869	078	5	922	41
20	085	33	915	168	37	832	083	5	917	40
21	118	32	882	205	38	795	088	5	912	39
22	150	32	850	243	37	757	093	4	907	38
23	182	33	818	280	37	720	097	5	903	37
24	215	32	785	317	37	683	102	5	898	36
25	247	32	753	354	37	646	107	5	893	35
26	279	32	721	391	38	609	112	5	888	34
27	311	32	689	429	37	571	117	5	883	33
28	343	32	657	466	37	534	122	5	878	32
29	375	33	625	503	37	497	127	5	873	31
30	56408	32	43592	59540	37	40460	03132	5	96868	30
31	440	32	560	577	37	423	137	5	863	29
32	472	32	528	614	37	386	142	5	858	28
33	504	32	496	651	37	349	147	5	853	27
34	536	32	464	688	37	312	152	5	848	26
35	568	31	432	725	37	275	157	5	843	25
36	599	32	401	762	37	238	162	5	838	24
37	631	32	369	799	36	201	167	5	833	23
38	663	32	337	835	37	165	172	5	828	22
39	695	32	305	872	37	128	177	5	823	21
40	727	32	273	909	37	091	182	5	818	20
41	759	31	241	946	37	054	187	5	813	19
42	790	32	210	983	36	017	192	5	808	18
43	822	32	178	60019	37	39981	197	5	803	17
44	854	32	146	056	37	944	202	5	798	16
45	886	31	114	093	37	907	207	5	793	15
46	917	32	083	130	36	870	212	5	788	14
47	949	31	051	166	37	834	217	5	783	13
48	980	32	020	203	37	797	222	6	778	12
49	57012	32	42988	240	36	760	228	5	772	11
50	044	31	956	276	37	724	233	5	767	10
51	075	32	925	313	36	687	238	5	762	9
52	107	31	893	349	37	651	243	5	757	8
53	138	31	862	386	36	614	248	5	752	7
54	169	32	831	422	37	578	253	5	747	6
55	201	31	799	459	36	541	258	5	742	5
56	232	32	768	495	37	505	263	5	737	4
57	264	31	736	532	36	468	268	5	732	3
58	295	31	705	568	37	432	273	5	727	2
59	326	32	674	605	36	395	278	5	722	1
60	57358		42642	60641		39359	03283		96717	0
′	9.	d	10.	9.	d	10.	10.	d	9.	′
	l cos	1′	*l* sec	*l* cot	1′	*l* tan	*l* csc	1′	*l* sin	

″	Proportional Parts								
	38	37	36	33	32	31	6	5	4
0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	0	0	0
2	1	1	1	1	1	1	0	0	0
3	2	2	2	2	2	2	0	0	0
4	3	2	2	2	2	2	0	0	0
5	3	3	3	3	3	3	0	0	0
6	4	4	4	3	3	3	1	0	0
7	4	4	4	4	4	4	1	1	0
8	5	5	5	4	4	4	1	1	1
9	6	6	5	5	5	5	1	1	1
10	6	6	6	6	5	5	1	1	1
11	7	7	7	6	6	6	1	1	1
12	8	7	7	7	6	6	1	1	1
13	8	8	8	7	7	7	1	1	1
14	9	9	8	8	7	7	1	1	1
15	10	9	9	8	8	8	2	1	1
16	10	10	10	9	9	8	2	1	1
17	11	10	10	9	9	9	2	1	1
18	11	11	11	10	10	9	2	2	1
19	12	12	11	10	10	10	2	2	1
20	13	12	12	11	11	10	2	2	1
21	13	13	13	12	11	11	2	2	1
22	14	14	13	12	12	11	2	2	1
23	15	14	14	13	12	12	2	2	2
24	15	15	14	13	13	12	2	2	2
25	16	15	15	14	13	13	2	2	2
26	16	16	16	14	14	13	3	2	2
27	17	17	16	15	14	14	3	2	2
28	18	17	17	15	15	14	3	2	2
29	18	18	17	16	15	15	3	2	2
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34	22	21	20	19	18	18	3	3	2
35	22	22	21	19	19	18	4	3	2
36	23	22	22	20	19	19	4	3	2
37	23	23	22	20	20	19	4	3	2
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42	27	26	25	23	22	22	4	4	3
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54	34	33	32	30	29	28	5	4	4
55	35	34	33	30	29	28	6	5	4
56	35	35	34	31	30	29	6	5	4
57	36	35	34	31	30	29	6	5	4
58	37	36	35	32	31	30	6	5	4
59	37	36	35	32	31	30	6	5	4
60	38	37	36	33	32	31	6	5	4
″	38	37	36	33	32	31	6	5	4
	Proportional Parts								

′	*l* sin 9.	d 1′	*l* csc 10.	*l* tan 9.	d 1′	*l* cot 10.	*l* sec 10.	d 1′	*l* cos 9.	′
0	57358	31	42642	60641	36	39359	03283	6	96717	60
1	389	31	611	677	37	323	289	5	711	59
2	420	31	580	714	36	286	294	5	706	58
3	451	31	549	750	36	250	299	5	701	57
4	482	32	518	786	37	214	304	5	696	56
5	514	31	486	823	36	177	309	5	691	55
6	545	31	455	859	36	141	314	5	686	54
7	576	31	424	895	36	105	319	5	681	53
8	607	31	393	931	36	069	324	6	676	52
9	638	31	362	967	37	033	330	5	670	51
10	669	31	331	61004	36	38996	335	5	665	50
11	700	31	300	040	36	960	340	5	660	49
12	731	31	269	076	36	924	345	5	655	48
13	762	31	238	112	36	888	350	5	650	47
14	793	31	207	148	36	852	355	5	645	46
15	824	31	176	184	36	816	360	6	640	45
16	855	30	145	220	36	780	366	5	634	44
17	885	31	115	256	36	744	371	5	629	43
18	916	31	084	292	36	708	376	5	624	42
19	947	31	053	328	36	672	381	5	619	41
20	978	30	022	364	36	636	386	6	614	40
21	58008	31	41992	400	36	600	392	5	608	39
22	039	31	961	436	36	564	397	5	603	38
23	070	31	930	472	36	528	402	5	598	37
24	101	30	899	508	36	492	407	5	593	36
25	131	31	869	544	35	456	412	6	588	35
26	162	30	838	579	36	421	418	5	582	34
27	192	31	808	615	36	385	423	5	577	33
28	223	30	777	651	36	349	428	5	572	32
29	253	31	747	687	35	313	433	5	567	31
30	58284	30	41716	61722	36	38278	03438	6	96562	30
31	314	31	686	758	36	242	444	5	556	29
32	345	30	655	794	36	206	449	5	551	28
33	375	31	625	830	35	170	454	5	546	27
34	406	30	594	865	36	135	459	6	541	26
35	436	31	564	901	35	099	465	5	535	25
36	467	30	533	936	36	064	470	5	530	24
37	497	30	503	972	36	028	475	5	525	23
38	527	30	473	62008	35	37992	480	6	520	22
39	557	31	443	043	36	957	486	5	514	21
40	588	30	412	079	35	921	491	5	509	20
41	618	30	382	114	36	886	496	6	504	19
42	648	30	352	150	35	850	502	5	498	18
43	678	31	322	185	36	815	507	5	493	17
44	709	30	291	221	35	779	512	5	488	16
45	739	30	261	256	36	744	517	6	483	15
46	769	30	231	292	35	708	523	5	477	14
47	799	30	201	327	35	673	528	5	472	13
48	829	30	171	362	36	638	533	6	467	12
49	859	30	141	398	35	602	539	5	461	11
50	889	30	111	433	35	567	544	5	456	10
51	919	30	081	468	36	532	549	6	451	9
52	949	30	051	504	35	496	555	5	445	8
53	979	30	021	539	35	461	560	5	440	7
54	59009	30	40991	574	35	426	565	6	435	6
55	039	30	961	609	36	391	571	5	429	5
56	069	29	931	645	35	355	576	5	424	4
57	098	30	902	680	35	320	581	6	419	3
58	128	30	872	715	35	285	587	5	413	2
59	158	30	842	750	35	250	592	5	408	1
60	59188		40812	62785		37215	03597		96403	0
′	9. *l* cos	d 1′	10. *l* sec	9. *l* cot	d 1′	10. *l* tan	10. *l* csc	d 1′	9. *l* sin	′

″	37	36	35	32	31	30	29	6	5
0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	0	0	0	0
2	1	1	1	1	1	1	1	0	0
3	2	2	2	2	2	2	1	0	0
4	2	2	2	2	2	2	2	0	0
5	3	3	3	3	3	2	2	0	0
6	4	4	4	3	3	3	3	1	0
7	4	4	4	4	4	4	3	1	1
8	5	5	5	4	4	4	4	1	1
9	6	5	5	5	5	4	4	1	1
10	6	6	6	5	5	5	5	1	1
11	7	7	6	6	6	6	5	1	1
12	7	7	7	6	6	6	6	1	1
13	8	8	8	7	7	6	6	1	1
14	9	8	8	7	7	7	7	1	1
15	9	9	9	8	8	8	7	2	1
16	10	10	9	9	8	8	8	2	1
17	10	10	10	9	9	8	8	2	1
18	11	11	10	10	9	9	9	2	2
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21	13	13	12	11	11	10	10	2	2
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23	14	14	13	12	12	12	11	2	2
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27	17	16	16	14	14	14	13	3	2
28	17	17	16	15	14	14	14	3	2
29	18	17	17	15	15	14	14	3	2
30	18	18	18	16	16	15	14	3	2
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32	20	19	19	17	17	16	15	3	3
33	20	20	19	18	17	16	16	3	3
34	21	20	20	18	18	17	16	3	3
35	22	21	20	19	18	18	17	4	3
36	22	22	21	19	19	18	17	4	3
37	23	22	22	20	19	18	18	4	3
38	23	23	22	20	20	19	18	4	3
39	24	23	23	21	20	20	19	4	3
40	25	24	23	21	21	20	19	4	3
41	25	25	24	22	21	20	20	4	3
42	26	25	24	22	22	21	20	4	4
43	27	26	25	23	22	22	21	4	4
44	27	26	26	23	23	22	21	4	4
45	28	27	26	24	23	22	22	4	4
46	28	28	27	25	24	23	22	5	4
47	29	28	27	25	24	24	23	5	4
48	30	29	28	26	25	24	23	5	4
49	30	29	29	26	25	24	24	5	4
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54	33	32	32	29	28	27	26	5	4
55	34	33	32	29	28	28	27	6	5
56	35	34	33	30	29	28	27	6	5
57	35	34	33	30	29	28	28	6	5
58	36	35	34	31	30	29	28	6	5
59	36	35	34	31	30	30	29	6	5
60	37	36	35	32	31	30	29	6	5
″	37	36	35	32	31	30	29	6	5

Proportional Parts

112° 67°

′	l sin 9.	d 1′	l csc 10.	l tan 9.	d 1′	l cot 10.	l sec 10.	d 1′	l cos 9.	′
0	59188	30	40812	62785	35	37215	03597	6	96403	60
1	218	29	782	820	35	180	603	5	397	59
2	247	30	753	855	35	145	608	5	392	58
3	277	30	723	890	36	110	613	6	387	57
4	307	29	693	926	35	074	619	5	381	56
5	336	30	664	961	35	039	624	6	376	55
6	366	30	634	996	35	004	630	5	370	54
7	396	29	604	63031	35	36969	635	5	365	53
8	425	30	575	066	35	934	640	6	360	52
9	455	29	545	101	34	899	646	5	354	51
10	484	30	516	135	35	865	651	6	349	50
11	514	29	486	170	35	830	657	5	343	49
12	543	30	457	205	35	795	662	5	338	48
13	573	29	427	240	35	760	667	6	333	47
14	602	30	398	275	35	725	673	5	327	46
15	632	29	368	310	35	690	678	6	322	45
16	661	29	339	345	34	655	684	5	316	44
17	690	30	310	379	35	621	689	6	311	43
18	720	29	280	414	35	586	695	5	305	42
19	749	29	251	449	35	551	700	6	300	41
20	778	30	222	484	35	516	706	5	294	40
21	808	29	192	519	34	481	711	5	289	39
22	837	29	163	553	35	447	716	6	284	38
23	866	29	134	588	35	412	722	5	278	37
24	895	29	105	623	34	377	727	6	273	36
25	924	30	076	657	35	343	733	5	267	35
26	954	29	046	692	34	308	738	6	262	34
27	983	29	017	726	35	274	744	5	256	33
28	60012	29	39988	761	35	239	749	6	251	32
29	041	29	959	796	34	204	755	5	245	31
30	60070	29	39930	63830	35	36170	03760	6	96240	30
31	099	29	901	865	34	135	766	5	234	29
32	128	29	872	899	35	101	771	6	229	28
33	157	29	843	934	34	066	777	5	223	27
34	186	29	814	968	35	032	782	6	218	26
35	215	29	785	64003	34	35997	788	5	212	25
36	244	29	756	037	35	963	793	6	207	24
37	273	29	727	072	34	928	799	5	201	23
38	302	29	698	106	34	894	804	6	196	22
39	331	28	669	140	35	860	810	5	190	21
40	359	29	641	175	34	825	815	6	185	20
41	388	29	612	209	34	791	821	5	179	19
42	417	29	583	243	35	757	826	6	174	18
43	446	28	554	278	34	722	832	6	168	17
44	474	29	526	312	34	688	838	5	162	16
45	503	29	497	346	35	654	843	6	157	15
46	532	29	468	381	34	619	849	5	151	14
47	561	28	439	415	34	585	854	6	146	13
48	589	29	411	449	34	551	860	5	140	12
49	618	28	382	483	34	517	865	6	135	11
50	646	29	354	517	35	483	871	6	129	10
51	675	29	325	552	34	448	877	5	123	9
52	704	28	296	586	34	414	882	6	118	8
53	732	29	268	620	34	380	888	5	112	7
54	761	28	239	654	34	346	893	6	107	6
55	789	29	211	688	34	312	899	6	101	5
56	818	28	182	722	34	278	905	5	095	4
57	846	29	154	756	34	244	910	6	090	3
58	875	28	125	790	34	210	916	5	084	2
59	903	28	097	824	34	176	921	6	079	1
60	60931		39069	64858		35142	03927		96073	0
′	9. l cos	d 1′	10. l sec	9. l cot	d 1′	10. l tan	10. l csc	d 1′	9. l sin	′

″	Proportional Parts 36	35	34	30	29	28	6	5
0	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0
2	1	1	1	1	1	1	0	0
3	2	2	2	2	1	1	0	0
4	2	2	2	2	2	2	0	0
5	3	3	3	2	2	2	0	0
6	4	4	3	3	3	3	1	0
7	4	4	4	4	3	3	1	1
8	5	5	5	4	4	4	1	1
9	5	5	5	4	4	4	1	1
10	6	6	6	5	5	5	1	1
11	7	6	6	6	5	5	1	1
12	7	7	7	6	6	6	1	1
13	8	8	7	6	6	6	1	1
14	8	8	8	7	7	7	1	1
15	9	9	8	8	7	7	2	1
16	10	9	9	8	8	7	2	1
17	10	10	10	8	8	8	2	1
18	11	10	10	9	9	8	2	2
19	11	11	11	10	9	9	2	2
20	12	12	11	10	10	9	2	2
21	13	12	12	10	10	10	2	2
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39	23	23	22	20	19	18	4	3
40	24	23	23	20	19	19	4	3
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56	34	33	32	28	27	26	6	5
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58	35	34	33	29	28	27	6	5
59	35	34	33	30	29	28	6	5
60	36	35	34	30	29	28	6	5
″	36	35	34	30	29	28	6	5
	Proportional Parts							

′	*l* sin	d	*l* csc	*l* tan	d	*l* cot	*l* sec	d	*l* cos	′
	9.	1′	10.	9.	1′	10.	10.	1′	9.	
0	60931	29	39069	64858	34	35142	03927	6	96073	60
1	960	28	040	892	34	108	933	5	067	59
2	988	28	012	926	34	074	938	6	062	58
3	61016	29	38984	960	34	040	944	6	056	57
4	045	28	955	994	34	006	950	5	050	56
5	073	28	927	65028	34	34972	955	6	045	55
6	101	28	899	062	34	938	961	5	039	54
7	129	29	871	096	34	904	966	6	034	53
8	158	28	842	130	34	870	972	6	028	52
9	186	28	814	164	33	836	978	5	022	51
10	214	28	786	197	34	803	983	6	017	50
11	242	28	758	231	34	769	989	6	011	49
12	270	28	730	265	34	735	995	5	005	48
13	298	28	702	299	34	701	04000	6	000	47
14	326	28	674	333	33	667	006	6	95994	46
15	354	28	646	366	34	634	012	6	988	45
16	382	29	618	400	34	600	018	5	982	44
17	411	27	589	434	33	566	023	6	977	43
18	438	28	562	467	34	533	029	6	971	42
19	466	28	534	501	34	499	035	5	965	41
20	494	28	506	535	33	465	040	6	960	40
21	522	28	478	568	34	432	046	6	954	39
22	550	28	450	602	34	398	052	6	948	38
23	578	28	422	636	33	364	058	5	942	37
24	606	28	394	669	34	331	063	6	937	36
25	634	28	366	703	33	297	069	6	931	35
26	662	27	338	736	34	264	075	5	925	34
27	689	28	311	770	33	230	080	6	920	33
28	717	28	283	803	34	197	086	6	914	32
29	745	28	255	837	33	163	092	6	908	31
30	61773	27	38227	65870	34	34130	04098	5	95902	30
31	800	28	200	904	33	096	103	6	897	29
32	828	28	172	937	34	063	109	6	891	28
33	856	27	144	971	33	029	115	6	885	27
34	883	28	117	66004	34	33996	121	6	879	26
35	911	28	089	038	33	962	127	5	873	25
36	939	27	061	071	33	929	132	6	868	24
37	966	28	034	104	34	896	138	6	862	23
38	994	27	006	138	33	862	144	6	856	22
39	62021	28	37979	171	33	829	150	6	850	21
40	049	27	951	204	34	796	156	5	844	20
41	076	28	924	238	33	762	161	6	839	19
42	104	27	896	271	33	729	167	6	833	18
43	131	28	869	304	33	696	173	6	827	17
44	159	27	841	337	34	663	179	6	821	16
45	186	28	814	371	33	629	185	5	815	15
46	214	27	786	404	33	596	190	6	810	14
47	241	27	759	437	33	563	196	6	804	13
48	268	28	732	470	33	530	202	6	798	12
49	296	27	704	503	34	497	208	6	792	11
50	323	27	677	537	33	463	214	6	786	10
51	350	27	650	570	33	430	220	5	780	9
52	377	28	623	603	33	397	225	6	775	8
53	405	27	595	636	33	364	231	6	769	7
54	432	27	568	669	33	331	237	6	763	6
55	459	27	541	702	33	298	243	6	757	5
56	486	27	514	735	33	265	249	6	751	4
57	513	28	487	768	33	232	255	6	745	3
58	541	27	459	801	33	199	261	6	739	2
59	568	27	432	834	33	166	267	5	733	1
60	62595		37405	66867		33133	04272		95728	0
′	9.	d	10.	9.	d	10.	10.	d	9.	′
	l cos	1′	*l* sec	*l* cot	1′	*l* tan	*l* csc	1′	*l* sin	

″	Proportional Parts						
	34	33	29	28	27	6	5
0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0
2	1	1	1	1	1	0	0
3	2	2	1	1	1	0	0
4	2	2	2	2	2	0	0
5	3	3	2	2	2	0	0
6	3	3	3	3	3	1	0
7	4	4	3	3	3	1	1
8	5	4	4	4	4	1	1
9	5	5	4	4	4	1	1
10	6	6	5	5	4	1	1
11	6	6	5	5	5	1	1
12	7	7	6	6	5	1	1
13	7	7	6	6	6	1	1
14	8	8	7	7	6	1	1
15	8	8	7	7	7	2	1
16	9	9	8	7	7	2	1
17	10	9	8	8	8	2	1
18	10	10	9	8	8	2	2
19	11	10	9	9	9	2	2
20	11	11	10	9	9	2	2
21	12	12	10	10	9	2	2
22	12	12	11	10	10	2	2
23	13	13	11	11	10	2	2
24	14	13	12	11	11	2	2
25	14	14	12	12	11	2	2
26	15	14	13	12	12	3	2
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34	19	19	16	16	15	3	3
35	20	19	17	16	16	4	3
36	20	20	17	17	16	4	3
37	21	20	18	17	17	4	3
38	22	21	18	18	17	4	3
39	22	21	19	18	18	4	3
40	23	22	19	19	18	4	3
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42	24	23	20	20	19	4	4
43	24	24	21	20	19	4	4
44	25	24	21	21	20	4	4
45	26	25	22	21	20	4	4
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50	28	28	24	23	22	5	4
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54	31	30	26	25	24	5	4
55	31	30	27	26	25	6	5
56	32	31	27	26	25	6	5
57	32	31	28	27	26	6	5
58	33	32	28	27	26	6	5
59	33	32	29	28	27	6	5
60	34	33	29	28	27	6	5
″	34	33	29	28	27	6	5
	Proportional Parts						

′	l sin 9.	d 1′	l csc 10.	l tan 9.	d 1′	l cot 10.	l sec 10.	d 1′	l cos 9.	′
0	62595	27	37405	66867	33	33133	04272	6	95728	60
1	622	27	378	900	33	100	278	6	722	59
2	649	27	351	933	33	067	284	6	716	58
3	676	27	324	966	33	034	290	6	710	57
4	703	27	297	999	33	001	296	6	704	56
5	730	27	270	67032	33	32968	302	6	698	55
6	757	27	243	065	33	935	308	6	692	54
7	784	27	216	098	33	902	314	6	686	53
8	811	27	189	131	32	869	320	6	680	52
9	838	27	162	163	33	837	326	6	674	51
10	865	27	135	196	33	804	332	5	668	50
11	892	26	108	229	33	771	337	6	663	49
12	918	27	082	262	33	738	343	6	657	48
13	945	27	055	295	32	705	349	6	651	47
14	972	27	028	327	33	673	355	6	645	46
15	999	27	001	360	33	640	361	6	639	45
16	63026	26	36974	393	33	607	367	6	633	44
17	052	27	948	426	32	574	373	6	627	43
18	079	27	921	458	33	542	379	6	621	42
19	106	27	894	491	33	509	385	6	615	41
20	133	26	867	524	32	476	391	6	609	40
21	159	27	841	556	33	444	397	6	603	39
22	186	27	814	589	33	411	403	6	597	38
23	213	26	787	622	32	378	409	6	591	37
24	239	27	761	654	33	346	415	6	585	36
25	266	26	734	687	32	313	421	6	579	35
26	292	27	708	719	33	281	427	6	573	34
27	319	26	681	752	33	248	433	6	567	33
28	345	27	655	785	32	215	439	6	561	32
29	372	26	628	817	33	183	445	6	555	31
30	63398	27	36602	67850	32	32150	04451	6	95549	30
31	425	26	575	882	33	118	457	6	543	29
32	451	27	549	915	32	085	463	6	537	28
33	478	26	522	947	33	053	469	6	531	27
34	504	27	496	980	32	020	475	6	525	26
35	531	26	469	68012	32	31988	481	6	519	25
36	557	26	443	044	33	956	487	6	513	24
37	583	27	417	077	32	923	493	7	507	23
38	610	26	390	109	33	891	500	6	500	22
39	636	26	364	142	32	858	506	6	494	21
40	662	27	338	174	32	826	512	6	488	20
41	689	26	311	206	33	794	518	6	482	19
42	715	26	285	239	32	761	524	6	476	18
43	741	26	259	271	32	729	530	6	470	17
44	767	27	233	303	33	697	536	6	464	16
45	794	26	206	336	32	664	542	6	458	15
46	820	26	180	368	32	632	548	6	452	14
47	846	26	154	400	32	600	554	6	446	13
48	872	26	128	432	33	568	560	6	440	12
49	898	26	102	465	32	535	566	7	434	11
50	924	26	076	497	32	503	573	6	427	10
51	950	26	050	529	32	471	579	6	421	9
52	976	26	024	561	32	439	585	6	415	8
53	64002	26	35998	593	33	407	591	6	409	7
54	028	26	972	626	32	374	597	6	403	6
55	054	26	946	658	32	342	603	6	397	5
56	080	26	920	690	32	310	609	7	391	4
57	106	26	894	722	32	278	616	6	384	3
58	132	26	868	754	32	246	622	6	378	2
59	158	26	842	786	32	214	628	6	372	1
60	64184		35816	68818		31182	04634		95366	0
′	9. l cos	d 1′	10. l sec	9. l cot	d 1′	10. l tan	10. l csc	d 1′	9. l sin	′

″	Proportional Parts 33	32	27	26	7	6	5
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4	2	2	2	2	0	0	0
5	3	3	2	2	1	0	0
6	3	3	3	3	1	1	0
7	4	4	3	3	1	1	1
8	4	4	4	3	1	1	1
9	5	5	4	4	1	1	1
10	6	5	4	4	1	1	1
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15	8	8	7	6	2	2	1
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27	15	14	12	12	3	3	2
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34	19	18	15	15	4	3	3
35	19	19	16	15	4	4	3
36	20	19	16	16	4	4	3
37	20	20	17	16	4	4	3
38	21	20	17	16	4	4	3
39	21	21	18	17	5	4	3
40	22	21	18	17	5	4	3
41	23	22	18	18	5	4	3
42	23	22	19	18	5	4	4
43	24	23	19	19	5	4	4
44	24	23	20	19	5	4	4
45	25	24	20	20	5	4	4
46	25	25	21	20	5	5	4
47	26	25	21	20	5	5	4
48	26	26	22	21	6	5	4
49	27	26	22	21	6	5	4
50	28	27	22	22	6	5	4
51	28	27	23	22	6	5	4
52	29	28	23	23	6	5	4
53	29	28	24	23	6	5	4
54	30	29	24	23	6	5	4
55	30	29	25	24	6	6	5
56	31	30	25	24	7	6	5
57	31	30	26	25	7	6	5
58	32	31	26	25	7	6	5
59	32	31	27	26	7	6	5
60	33	32	27	26	7	6	5
″	33 Proportional Parts	32	27	26	7	6	5

′	l sin 9.	d 1′	l csc 10.	l tan 9.	d 1′	l cot 10.	l sec 10.	d 1′	l cos 9.	′
0	64184	26	35816	68818	32	31182	04634	6	95366	60
1	210	26	790	850	32	150	640	6	360	59
2	236	26	764	882	32	118	646	6	354	58
3	262	26	738	914	32	086	652	7	348	57
4	288	25	712	946	32	054	659	6	341	56
5	313	26	687	978	32	022	665	6	335	55
6	339	26	661	69010	32	30990	671	6	329	54
7	365	26	635	042	32	958	677	6	323	53
8	391	26	609	074	32	926	683	7	317	52
9	417	25	583	106	32	894	690	6	310	51
10	442	26	558	138	32	862	696	6	304	50
11	468	26	532	170	32	830	702	6	298	49
12	494	25	506	202	32	798	708	6	292	48
13	519	26	481	234	32	766	714	7	286	47
14	545	26	455	266	32	734	721	6	279	46
15	571	25	429	298	31	702	727	6	273	45
16	596	26	404	329	32	671	733	6	267	44
17	622	25	378	361	32	639	739	7	261	43
18	647	26	353	393	32	607	746	6	254	42
19	673	25	327	425	32	575	752	6	248	41
20	698	26	302	457	31	543	758	6	242	40
21	724	25	276	488	32	512	764	7	236	39
22	749	26	251	520	32	480	771	6	229	38
23	775	25	225	552	32	448	777	6	223	37
24	800	26	200	584	31	416	783	6	217	36
25	826	25	174	615	32	385	789	7	211	35
26	851	26	149	647	32	353	796	6	204	34
27	877	25	123	679	31	321	802	6	198	33
28	902	25	098	710	32	290	808	7	192	32
29	927	26	073	742	32	258	815	6	185	31
30	64953	25	35047	69774	31	30226	04821	6	95179	30
31	978	25	022	805	32	195	827	6	173	29
32	65003	26	34997	837	31	163	833	7	167	28
33	029	25	971	868	32	132	840	6	160	27
34	054	25	946	900	32	100	846	6	154	26
35	079	25	921	932	31	068	852	7	148	25
36	104	26	896	963	32	037	859	6	141	24
37	130	25	870	995	31	005	865	6	135	23
38	155	25	845	70026	32	29974	871	7	129	22
39	180	25	820	058	31	942	878	6	122	21
40	205	25	795	089	32	911	884	6	116	20
41	230	25	770	121	31	879	890	7	110	19
42	255	26	745	152	32	848	897	6	103	18
43	281	25	719	184	31	816	903	7	097	17
44	306	25	694	215	32	785	910	6	090	16
45	331	25	669	247	31	753	916	6	084	15
46	356	25	644	278	31	722	922	7	078	14
47	381	25	619	309	32	691	929	6	071	13
48	406	25	594	341	31	659	935	6	065	12
49	431	25	569	372	32	628	941	7	059	11
50	456	25	544	404	31	596	948	6	052	10
51	481	25	519	435	31	565	954	7	046	9
52	506	25	494	466	32	534	961	6	039	8
53	531	25	469	498	31	502	967	6	033	7
54	556	24	444	529	31	471	973	7	027	6
55	580	25	420	560	32	440	980	6	020	5
56	605	25	395	592	31	408	986	7	014	4
57	630	25	370	623	31	377	993	6	007	3
58	655	25	345	654	31	346	999	6	001	2
59	680	25	320	685	32	315	05005	7	94995	1
60	65705		34295	70717		29283	05012		94988	0
′	9. l cos	d 1′	10. l sec	9. l cot	d 1′	10. l tan	10. l csc	d 1′	9. l sin	′

Proportional Parts

″	32	31	26	25	24	7	6
0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0
2	1	1	1	1	1	0	0
3	2	2	1	1	1	0	0
4	2	2	2	2	2	0	0
5	3	3	2	2	2	1	0
6	3	3	3	2	2	1	1
7	4	4	3	3	3	1	1
8	4	4	3	3	3	1	1
9	5	5	4	4	4	1	1
10	5	5	4	4	4	1	1
11	6	6	5	5	4	1	1
12	6	6	5	5	5	1	1
13	7	7	6	5	5	2	1
14	7	7	6	6	6	2	1
15	8	8	6	6	6	2	2
16	9	8	7	7	6	2	2
17	9	9	7	7	7	2	2
18	10	9	8	8	7	2	2
19	10	10	8	8	8	2	2
20	11	10	9	8	8	2	2
21	11	11	9	9	8	2	2
22	12	11	10	9	9	3	2
23	12	12	10	10	9	3	2
24	13	12	10	10	10	3	2
25	13	13	11	10	10	3	2
26	14	13	11	11	10	3	3
27	14	14	12	11	11	3	3
28	15	14	12	12	11	3	3
29	15	15	13	12	12	3	3
30	16	16	13	12	12	4	3
31	17	16	13	13	12	4	3
32	17	17	14	13	13	4	3
33	18	17	14	14	13	4	3
34	18	18	15	14	14	4	3
35	19	18	15	15	14	4	4
36	19	19	16	15	14	4	4
37	20	19	16	15	15	4	4
38	20	20	16	16	15	4	4
39	21	20	17	16	16	5	4
40	21	21	17	17	16	5	4
41	22	21	18	17	16	5	4
42	22	22	18	18	17	5	4
43	23	22	19	18	17	5	4
44	23	23	19	18	18	5	4
45	24	23	20	19	18	5	4
46	25	24	20	19	18	5	5
47	25	24	20	20	19	5	5
48	26	25	21	20	19	6	5
49	26	25	21	20	20	6	5
50	27	26	22	21	20	6	5
51	27	26	22	21	20	6	5
52	28	27	23	22	21	6	5
53	28	27	23	22	21	6	5
54	29	28	23	22	22	6	5
55	29	28	24	23	22	6	6
56	30	29	24	23	22	7	6
57	30	29	25	24	23	7	6
58	31	30	25	24	23	7	6
59	31	30	26	25	24	7	6
60	32	31	26	25	24	7	6
″	32	31	26	25	24	7	6

Proportional Parts

′	*l* sin 9.	d 1′	*l* csc 10.	*l* tan 9.	d 1′	*l* cot 10.	*l* sec 10.	d 1′	*l* cos 9.	′
0	65705	24	34295	70717	31	29283	05012	6	94988	60
1	729	25	271	748	31	252	018	7	982	59
2	754	25	246	779	31	221	025	6	975	58
3	779	25	221	810	31	190	031	7	969	57
4	804	24	196	841	32	159	038	6	962	56
5	828	25	172	873	31	127	044	7	956	55
6	853	25	147	904	31	096	051	6	949	54
7	878	24	122	935	31	065	057	7	943	53
8	902	25	098	966	31	034	064	6	936	52
9	927	25	073	997	31	003	070	7	930	51
10	952	24	048	71028	31	28972	077	6	923	50
11	976	25	024	059	31	941	083	6	917	49
12	66001	24	33999	090	31	910	089	7	911	48
13	025	25	975	121	32	879	096	6	904	47
14	050	25	950	153	31	847	102	7	898	46
15	075	24	925	184	31	816	109	6	891	45
16	099	25	901	215	31	785	115	7	885	44
17	124	24	876	246	31	754	122	7	878	43
18	148	25	852	277	31	723	129	6	871	42
19	173	24	827	308	31	692	135	7	865	41
20	197	24	803	339	31	661	142	6	858	40
21	221	25	779	370	31	630	148	7	852	39
22	246	24	754	401	30	599	155	6	845	38
23	270	25	730	431	31	569	161	7	839	37
24	295	24	705	462	31	538	168	6	832	36
25	319	24	681	493	31	507	174	7	826	35
26	343	25	657	524	31	476	181	6	819	34
27	368	24	632	555	31	445	187	7	813	33
28	392	24	608	586	31	414	194	7	806	32
29	416	25	584	617	31	383	201	6	799	31
30	66441	24	33559	71648	31	28352	05207	7	94793	30
31	465	24	535	679	30	321	214	6	786	29
32	489	24	511	709	31	291	220	7	780	28
33	513	24	487	740	31	260	227	6	773	27
34	537	25	463	771	31	229	233	7	767	26
35	562	24	438	802	31	198	240	7	760	25
36	586	24	414	833	30	167	247	6	753	24
37	610	24	390	863	31	137	253	7	747	23
38	634	24	366	894	31	106	260	6	740	22
39	658	24	342	925	30	075	266	7	734	21
40	682	24	318	955	31	045	273	7	727	20
41	706	25	294	986	31	014	280	6	720	19
42	731	24	269	72017	31	27983	286	7	714	18
43	755	24	245	048	30	952	293	7	707	17
44	779	24	221	078	31	922	300	6	700	16
45	803	24	197	109	31	891	306	7	694	15
46	827	24	173	140	30	860	313	7	687	14
47	851	24	149	170	31	830	320	6	680	13
48	875	24	125	201	30	799	326	7	674	12
49	899	23	101	231	31	769	333	7	667	11
50	922	24	078	262	31	738	340	6	660	10
51	946	24	054	293	30	707	346	7	654	9
52	970	24	030	323	31	677	353	7	647	8
53	994	24	006	354	30	646	360	6	640	7
54	67018	24	32982	384	31	616	366	7	634	6
55	042	24	958	415	30	585	373	7	627	5
56	066	24	934	445	31	555	380	6	620	4
57	090	23	910	476	30	524	386	7	614	3
58	113	24	887	506	31	494	393	7	607	2
59	137	24	863	537	30	463	400	7	600	1
60	67161		32839	72567		27433	05407		94593	0
′	9. *l* cos	d 1′	10. *l* sec	9. *l* cot	d 1′	10. *l* tan	10. *l* csc	d 1′	9. *l* sin	′

Proportional Parts

″	32	31	30	25	24	23	7	6
0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0
2	1	1	1	1	1	1	0	0
3	2	2	2	1	1	1	0	0
4	2	2	2	2	2	2	0	0
5	3	3	2	2	2	2	1	0
6	3	3	3	2	2	2	1	1
7	4	4	4	3	3	3	1	1
8	4	4	4	3	3	3	1	1
9	5	5	4	4	4	3	1	1
10	5	5	5	4	4	4	1	1
11	6	6	6	5	4	4	1	1
12	6	6	6	5	5	5	1	1
13	7	7	6	5	5	5	2	1
14	7	7	7	6	6	5	2	1
15	8	8	8	6	6	6	2	2
16	9	8	8	7	6	6	2	2
17	9	9	8	7	7	7	2	2
18	10	9	9	8	7	7	2	2
19	10	10	10	8	8	7	2	2
20	11	10	10	8	8	8	2	2
21	11	11	10	9	8	8	2	2
22	12	11	11	9	9	8	3	2
23	12	12	12	10	9	9	3	2
24	13	12	12	10	10	9	3	2
25	13	13	12	10	10	10	3	2
26	14	13	13	11	10	10	3	3
27	14	14	14	11	11	10	3	3
28	15	14	14	12	11	11	3	3
29	15	15	14	12	12	11	3	3
30	16	16	15	12	12	12	4	3
31	17	16	16	13	12	12	4	3
32	17	17	16	13	13	12	4	3
33	18	17	16	14	13	13	4	3
34	18	18	17	14	14	13	4	3
35	19	18	18	15	14	13	4	4
36	19	19	18	15	14	14	4	4
37	20	19	18	15	15	14	4	4
38	20	20	19	16	15	15	4	4
39	21	20	20	16	16	15	5	4
40	21	21	20	17	16	15	5	4
41	22	21	20	17	16	16	5	4
42	22	22	21	18	17	16	5	4
43	23	22	22	18	17	16	5	4
44	23	23	22	18	18	17	5	4
45	24	23	22	19	18	17	5	4
46	25	24	23	19	18	18	5	5
47	25	24	24	20	19	18	5	5
48	26	25	24	20	19	18	6	5
49	26	25	24	20	20	19	6	5
50	27	26	25	21	20	19	6	5
51	27	26	26	21	20	20	6	5
52	28	27	26	22	21	20	6	5
53	28	27	26	22	21	20	6	5
54	29	28	27	22	22	21	6	5
55	29	28	28	23	22	21	6	6
56	30	29	28	23	22	21	7	6
57	30	29	28	24	23	22	7	6
58	31	30	29	24	23	22	7	6
59	31	30	30	25	24	23	7	6
60	32	31	30	25	24	23	7	6
″	32	31	30	25	24	23	7	6

Proportional Parts

′	*l* sin 9.	d 1′	*l* csc 10.	*l* tan 9.	d 1′	*l* cot 10.	*l* sec 10.	d 1′	*l* cos 9.	′
0	67161	24	32839	72567	31	27433	05407	6	94593	60
1	185	23	815	598	30	402	413	7	587	59
2	208	24	792	628	31	372	420	7	580	58
3	232	24	768	659	30	341	427	6	573	57
4	256	24	744	689	31	311	433	7	567	56
5	280	23	720	720	30	280	440	7	560	55
6	303	24	697	750	30	250	447	7	553	54
7	327	23	673	780	31	220	454	6	546	53
8	350	24	650	811	30	189	460	7	540	52
9	374	24	626	841	31	159	467	7	533	51
10	398	23	602	872	30	128	474	7	526	50
11	421	24	579	902	30	098	481	6	519	49
12	445	23	555	932	31	068	487	7	513	48
13	468	24	532	963	30	037	494	7	506	47
14	492	23	508	993	30	007	501	7	499	46
15	515	24	485	73023	31	26977	508	7	492	45
16	539	23	461	054	30	946	515	6	485	44
17	562	24	438	084	30	916	521	7	479	43
18	586	23	414	114	30	886	528	7	472	42
19	609	24	391	144	31	856	535	7	465	41
20	633	23	367	175	30	825	542	7	458	40
21	656	24	344	205	30	795	549	6	451	39
22	680	23	320	235	30	765	555	7	445	38
23	703	23	297	265	30	735	562	7	438	37
24	726	24	274	295	31	705	569	7	431	36
25	750	23	250	326	30	674	576	7	424	35
26	773	23	227	356	30	644	583	7	417	34
27	796	24	204	386	30	614	590	6	410	33
28	820	23	180	416	30	584	596	7	404	32
29	843	23	157	446	30	554	603	7	397	31
30	67866	24	32134	73476	31	26524	05610	7	94390	30
31	890	23	110	507	30	493	617	7	383	29
32	913	23	087	537	30	463	624	7	376	28
33	936	23	064	567	30	433	631	7	369	27
34	959	23	041	597	30	403	638	7	362	26
35	982	24	018	627	30	373	645	6	355	25
36	68006	23	31994	657	30	343	651	7	349	24
37	029	23	971	687	30	313	658	7	342	23
38	052	23	948	717	30	283	665	7	335	22
39	075	23	925	747	30	253	672	7	328	21
40	098	23	902	777	30	223	679	7	321	20
41	121	23	879	807	30	193	686	7	314	19
42	144	23	856	837	30	163	693	7	307	18
43	167	23	833	867	30	133	700	7	300	17
44	190	23	810	897	30	103	707	7	293	16
45	213	24	787	927	30	073	714	7	286	15
46	237	23	763	957	30	043	721	6	279	14
47	260	23	740	987	30	013	727	7	273	13
48	283	22	717	74017	30	25983	734	7	266	12
49	305	23	695	047	30	953	741	7	259	11
50	328	23	672	077	30	923	748	7	252	10
51	351	23	649	107	30	893	755	7	245	9
52	374	23	626	137	29	863	762	7	238	8
53	397	23	603	166	30	834	769	7	231	7
54	420	23	580	196	30	804	776	7	224	6
55	443	23	557	226	30	774	783	7	217	5
56	466	23	534	256	30	744	790	7	210	4
57	489	23	511	286	30	714	797	7	203	3
58	512	22	488	316	29	684	804	7	196	2
59	534	23	466	345	30	655	811	7	189	1
60	68557		31443	74375		25625	05818		94182	0
′	9. *l* cos	d 1′	10. *l* sec	9. *l* cot	d 1′	10. *l* tan	10. *l* csc	d 1′	9. *l* sin	′

″	Proportional Parts 31	30	29	24	23	22	7	6
0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0
2	1	1	1	1	1	1	0	0
3	2	2	1	1	1	1	0	0
4	2	2	2	2	2	1	0	0
5	3	2	2	2	2	2	1	0
6	3	3	3	2	2	2	1	1
7	4	4	3	3	3	3	1	1
8	4	4	4	3	3	3	1	1
9	5	4	4	4	3	3	1	1
10	5	5	5	4	4	4	1	1
11	6	6	5	4	4	4	1	1
12	6	6	6	5	5	4	1	1
13	7	6	6	5	5	5	2	1
14	7	7	7	6	5	5	2	1
15	8	8	7	6	6	6	2	2
16	8	8	8	6	6	6	2	2
17	9	8	8	7	7	6	2	2
18	9	9	9	7	7	7	2	2
19	10	10	9	8	7	7	2	2
20	10	10	10	8	8	7	2	2
21	11	10	10	8	8	8	2	2
22	11	11	11	9	8	8	3	2
23	12	12	11	9	9	8	3	2
24	12	12	12	10	9	9	3	2
25	13	12	12	10	10	9	3	2
26	13	13	13	10	10	10	3	3
27	14	14	13	11	10	10	3	3
28	14	14	14	11	11	10	3	3
29	15	14	14	12	11	11	3	3
30	16	15	14	12	12	11	4	3
31	16	16	15	12	12	11	4	3
32	17	16	15	13	12	12	4	3
33	17	16	16	13	13	12	4	3
34	18	17	16	14	13	12	4	3
35	18	18	17	14	13	13	4	4
36	19	18	17	14	14	13	4	4
37	19	18	18	15	14	14	4	4
38	20	19	18	15	15	14	4	4
39	20	20	19	16	15	14	5	4
40	21	20	19	16	15	15	5	4
41	21	20	20	16	16	15	5	4
42	22	21	20	17	16	15	5	4
43	22	22	21	17	16	16	5	4
44	23	22	21	18	17	16	5	4
45	23	22	22	18	17	16	5	4
46	24	23	22	18	18	17	5	5
47	24	24	23	19	18	17	5	5
48	25	24	23	19	18	18	6	5
49	25	24	24	20	19	18	6	5
50	26	25	24	20	19	18	6	5
51	26	26	25	20	20	19	6	5
52	27	26	25	21	20	19	6	5
53	27	26	26	21	20	19	6	5
54	28	27	26	22	21	20	6	5
55	28	28	27	22	21	20	6	6
56	29	28	27	22	21	21	7	6
57	29	28	28	23	22	21	7	6
58	30	29	28	23	22	21	7	6
59	30	30	29	24	23	22	7	6
60	31	30	29	24	23	22	7	6
″	31	30	29	24	23	22	7	6
	Proportional Parts							

′	*l* sin 9.	d 1′	*l* csc 10.	*l* tan 9.	d 1′	*l* cot 10.	*l* sec 10.	d 1′	*l* cos 9.	′
0	68557	23	31443	74375	30	25625	05818	7	94182	60
1	580	23	420	405	30	595	825	7	175	59
2	603	22	397	435	30	565	832	7	168	58
3	625	23	375	465	29	535	839	7	161	57
4	648	23	352	494	30	506	846	7	154	56
5	671	23	329	524	30	476	853	7	147	55
6	694	22	306	554	29	446	860	7	140	54
7	716	23	284	583	30	417	867	7	133	53
8	739	23	261	613	30	387	874	7	126	52
9	762	22	238	643	30	357	881	7	119	51
10	784	23	216	673	29	327	888	7	112	50
11	807	22	193	702	30	298	895	7	105	49
12	829	23	171	732	30	268	902	8	098	48
13	852	23	148	762	29	238	910	7	090	47
14	875	22	125	791	30	209	917	7	083	46
15	897	23	103	821	30	179	924	7	076	45
16	920	22	080	851	29	149	931	7	069	44
17	942	23	058	880	30	120	938	7	062	43
18	965	22	035	910	29	090	945	7	055	42
19	987	23	013	939	30	061	952	7	048	41
20	69010	22	30990	969	29	031	959	7	041	40
21	032	23	968	998	30	002	966	7	034	39
22	055	22	945	75028	30	24972	973	7	027	38
23	077	23	923	058	29	942	980	8	020	37
24	100	22	900	087	30	913	988	7	012	36
25	122	22	878	117	29	883	995	7	005	35
26	144	23	856	146	30	854	06002	7	93998	34
27	167	22	833	176	29	824	009	7	991	33
28	189	23	811	205	30	795	016	7	984	32
29	212	22	788	235	29	765	023	7	977	31
30	69234	22	30766	75264	30	24736	06030	7	93970	30
31	256	23	744	294	29	706	037	8	963	29
32	279	22	721	323	30	677	045	7	955	28
33	301	22	699	353	29	647	052	7	948	27
34	323	22	677	382	29	618	059	7	941	26
35	345	23	655	411	30	589	066	7	934	25
36	368	22	632	441	29	559	073	7	927	24
37	390	22	610	470	30	530	080	8	920	23
38	412	22	588	500	29	500	088	7	912	22
39	434	22	566	529	29	471	095	7	905	21
40	456	23	544	558	30	442	102	7	898	20
41	479	22	521	588	29	412	109	7	891	19
42	501	22	499	617	30	383	116	8	884	18
43	523	22	477	647	29	353	124	7	876	17
44	545	22	455	676	29	324	131	7	869	16
45	567	22	433	705	30	295	138	7	862	15
46	589	22	411	735	29	265	145	8	855	14
47	611	22	389	764	29	236	153	7	847	13
48	633	22	367	793	29	207	160	7	840	12
49	655	22	345	822	30	178	167	7	833	11
50	677	22	323	852	29	148	174	7	826	10
51	699	22	301	881	29	119	181	8	819	9
52	721	22	279	910	29	090	189	7	811	8
53	743	22	257	939	30	061	196	7	804	7
54	765	22	235	969	29	031	203	8	797	6
55	787	22	213	998	29	002	211	7	789	5
56	809	22	191	76027	29	23973	218	7	782	4
57	831	22	169	056	30	944	225	7	775	3
58	853	22	147	086	29	914	232	8	768	2
59	875	22	125	115	29	885	240	7	760	1
60	69897		30103	76144		23856	06247		93753	0
′	9. *l* cos	d 1′	10. *l* sec	9. *l* cot	d 1′	10. *l* tan	10. *l* csc	d 1′	9. *l* sin	′

Proportional Parts ″	30	29	23	22	8	7
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	1	1	1	1	0	0
3	2	1	1	1	0	0
4	2	2	2	1	1	0
5	2	2	2	2	1	1
6	3	3	2	2	1	1
7	4	3	3	3	1	1
8	4	4	3	3	1	1
9	4	4	3	3	1	1
10	5	5	4	4	1	1
11	6	5	4	4	1	1
12	6	6	5	4	2	1
13	6	6	5	5	2	2
14	7	7	5	5	2	2
15	8	7	6	6	2	2
16	8	8	6	6	2	2
17	8	8	7	6	2	2
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19	10	9	7	7	3	2
20	10	10	8	7	3	2
21	10	10	8	8	3	2
22	11	11	8	8	3	3
23	12	11	9	8	3	3
24	12	12	9	9	3	3
25	12	12	10	9	3	3
26	13	13	10	10	3	3
27	14	13	10	10	4	3
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35	18	17	13	13	5	4
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37	18	18	14	14	5	4
38	19	18	15	14	5	4
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42	21	20	16	15	6	5
43	22	21	16	16	6	5
44	22	21	17	16	6	5
45	22	22	17	16	6	5
46	23	22	18	17	6	5
47	24	23	18	17	6	5
48	24	23	18	18	6	6
49	24	24	19	18	7	6
50	25	24	19	18	7	6
51	26	25	20	19	7	6
52	26	25	20	19	7	6
53	26	26	20	19	7	6
54	27	26	21	20	7	6
55	28	27	21	20	7	6
56	28	27	21	21	7	7
57	28	28	22	21	8	7
58	29	28	22	21	8	7
59	30	29	23	22	8	7
60	30	29	23	22	8	7
″	30	29	23	22	8	7

Proportional Parts

′	*l* sin 9.	d 1′	*l* csc 10.	*l* tan 9.	d 1′	*l* cot 10.	*l* sec 10.	d 1′	*l* cos 9.	′
0	69897	22	30103	76144	29	23856	06247	7	93753	60
1	919	22	081	173	29	827	254	8	746	59
2	941	22	059	202	29	798	262	7	738	58
3	963	21	037	231	30	769	269	7	731	57
4	984	22	016	261	29	739	276	7	724	56
5	70006	22	29994	290	29	710	283	8	717	55
6	028	22	972	319	29	681	291	7	709	54
7	050	22	950	348	29	652	298	7	702	53
8	072	21	928	377	29	623	305	8	695	52
9	093	22	907	406	29	594	313	7	687	51
10	115	22	885	435	29	565	320	7	680	50
11	137	22	863	464	29	536	327	8	673	49
12	159	21	841	493	29	507	335	7	665	48
13	180	22	820	522	29	478	342	8	658	47
14	202	22	798	551	29	449	350	7	650	46
15	224	21	776	580	29	420	357	7	643	45
16	245	22	755	609	30	391	364	8	636	44
17	267	21	733	639	29	361	372	7	628	43
18	288	22	712	668	29	332	379	7	621	42
19	310	22	690	697	28	303	386	8	614	41
20	332	21	668	725	29	275	394	7	606	40
21	353	22	647	754	29	246	401	8	599	39
22	375	21	625	783	29	217	409	7	591	38
23	396	22	604	812	29	188	416	7	584	37
24	418	21	582	841	29	159	423	8	577	36
25	439	22	561	870	29	130	431	7	569	35
26	461	21	539	899	29	101	438	8	562	34
27	482	22	518	928	29	072	446	7	554	33
28	504	21	496	957	29	043	453	8	547	32
29	525	22	475	986	29	014	461	7	539	31
30	70547	21	29453	77015	29	22985	06468	7	93532	30
31	568	22	432	044	29	956	475	8	525	29
32	590	21	410	073	28	927	483	7	517	28
33	611	22	389	101	29	899	490	8	510	27
34	633	21	367	130	29	870	498	7	502	26
35	654	21	346	159	29	841	505	8	495	25
36	675	22	325	188	29	812	513	7	487	24
37	697	21	303	217	29	783	520	8	480	23
38	718	21	282	246	28	754	528	7	472	22
39	739	22	261	274	29	726	535	8	465	21
40	761	21	239	303	29	697	543	7	457	20
41	782	21	218	332	29	668	550	8	450	19
42	803	21	197	361	29	639	558	7	442	18
43	824	22	176	390	28	610	565	8	435	17
44	846	21	154	418	29	582	573	7	427	16
45	867	21	133	447	29	553	580	8	420	15
46	888	21	112	476	29	524	588	7	412	14
47	909	22	091	505	28	495	595	8	405	13
48	931	21	069	533	29	467	603	7	397	12
49	952	21	048	562	29	438	610	8	390	11
50	973	21	027	591	28	409	618	7	382	10
51	994	21	006	619	29	381	625	8	375	9
52	71015	21	28985	648	29	352	633	7	367	8
53	036	22	964	677	29	323	640	8	360	7
54	058	21	942	706	28	294	648	8	352	6
55	079	21	921	734	29	266	656	7	344	5
56	100	21	900	763	28	237	663	8	337	4
57	121	21	879	791	29	209	671	7	329	3
58	142	21	858	820	29	180	678	8	322	2
59	163	21	837	849	28	151	686	7	314	1
60	71184		28816	77877		22123	06693		93307	0
′	9. *l* cos	d 1′	10. *l* sec	9. *l* cot	d 1′	10. *l* tan	10. *l* csc	d 1′	9. *l* sin	′

Proportional Parts

″	30	29	28	22	21	8	7
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	1	1	1	1	1	0	0
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4	2	2	2	1	1	1	0
5	2	2	2	2	2	1	1
6	3	3	3	2	2	1	1
7	4	3	3	3	2	1	1
8	4	4	4	3	3	1	1
9	4	4	4	3	3	1	1
10	5	5	5	4	4	1	1
11	6	5	5	4	4	1	1
12	6	6	6	4	4	2	1
13	6	6	6	5	5	2	2
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15	8	7	7	6	5	2	2
16	8	8	7	6	6	2	2
17	8	8	8	6	6	2	2
18	9	9	8	7	6	2	2
19	10	9	9	7	7	3	2
20	10	10	9	7	7	3	2
21	10	10	10	8	7	3	2
22	11	11	10	8	8	3	3
23	12	11	11	8	8	3	3
24	12	12	11	9	8	3	3
25	12	12	12	9	9	3	3
26	13	13	12	10	9	3	3
27	14	13	13	10	9	4	3
28	14	14	13	10	10	4	3
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31	16	15	14	11	11	4	4
32	16	15	15	12	11	4	4
33	16	16	15	12	12	4	4
34	17	16	16	12	12	5	4
35	18	17	16	13	12	5	4
36	18	17	17	13	13	5	4
37	18	18	17	14	13	5	4
38	19	18	18	14	13	5	4
39	20	19	18	14	14	5	5
40	20	19	19	15	14	5	5
41	20	20	19	15	14	5	5
42	21	20	20	15	15	6	5
43	22	21	20	16	15	6	5
44	22	21	21	16	15	6	5
45	22	22	21	16	16	6	5
46	23	22	21	17	16	6	5
47	24	23	22	17	16	6	5
48	24	23	22	18	17	6	6
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52	26	25	24	19	18	7	6
53	26	26	25	19	19	7	6
54	27	26	25	20	19	7	6
55	28	27	26	20	19	7	6
56	28	27	26	21	20	7	7
57	28	28	27	21	20	8	7
58	29	28	27	21	20	8	7
59	30	29	28	22	21	8	7
60	30	29	28	22	21	8	7
″	30	29	28	22	21	8	7

Proportional Parts

120° 59°

′	l sin	d	l csc	l tan	d	l cot	l sec	d	l cos	′
	9.	1′	10.	9.	1′	10.	10.	1′	9.	
0	71184	21	28816	77877	29	22123	06693	8	93307	60
1	205	21	795	906	29	094	701	8	299	59
2	226	21	774	935	28	065	709	7	291	58
3	247	21	753	963	29	037	716	8	284	57
4	268	21	732	992	28	008	724	7	276	56
5	289	21	711	78020	29	21980	731	8	269	55
6	310	21	690	049	28	951	739	8	261	54
7	331	21	669	077	29	923	747	7	253	53
8	352	21	648	106	29	894	754	8	246	52
9	373	20	627	135	28	865	762	8	238	51
10	393	21	607	163	29	837	770	7	230	50
11	414	21	586	192	28	808	777	8	223	49
12	435	21	565	220	29	780	785	8	215	48
13	456	21	544	249	28	751	793	7	207	47
14	477	21	523	277	29	723	800	8	200	46
15	498	21	502	306	28	694	808	8	192	45
16	519	20	481	334	29	666	816	7	184	44
17	539	21	461	363	28	637	823	8	177	43
18	560	21	440	391	28	609	831	8	169	42
19	581	21	419	419	29	581	839	7	161	41
20	602	20	398	448	28	552	846	8	154	40
21	622	21	378	476	29	524	854	8	146	39
22	643	21	357	505	28	495	862	7	138	38
23	664	21	336	533	29	467	869	8	131	37
24	685	20	315	562	28	438	877	8	123	36
25	705	21	295	590	28	410	885	7	115	35
26	726	21	274	618	29	382	892	8	108	34
27	747	20	253	647	28	353	900	8	100	33
28	767	21	233	675	29	325	908	8	092	32
29	788	21	212	704	28	296	916	7	084	31
30	71809	20	28191	78732	28	21268	06923	8	93077	30
31	829	21	171	760	29	240	931	8	069	29
32	850	20	150	789	28	211	939	8	061	28
33	870	21	130	817	28	183	947	7	053	27
34	891	20	109	845	29	155	954	8	046	26
35	911	21	089	874	28	126	962	8	038	25
36	932	20	068	902	28	098	970	8	030	24
37	952	21	048	930	29	070	978	8	022	23
38	973	21	027	959	28	041	986	7	014	22
39	994	20	006	987	28	013	993	8	007	21
40	72014	20	27986	79015	28	20985	07001	8	92999	20
41	034	21	966	043	29	957	009	8	991	19
42	055	20	945	072	28	928	017	7	983	18
43	075	21	925	100	28	900	024	8	976	17
44	096	20	904	128	28	872	032	8	968	16
45	116	21	884	156	29	844	040	8	960	15
46	137	20	863	185	28	815	048	8	952	14
47	157	20	843	213	28	787	056	8	944	13
48	177	21	823	241	28	759	064	7	936	12
49	198	20	802	269	28	731	071	8	929	11
50	218	20	782	297	29	703	079	8	921	10
51	238	21	762	326	28	674	087	8	913	9
52	259	20	741	354	28	646	095	8	905	8
53	279	20	721	382	28	618	103	8	897	7
54	299	21	701	410	28	590	111	8	889	6
55	320	20	680	438	28	562	119	7	881	5
56	340	20	660	466	29	534	126	8	874	4
57	360	21	640	495	28	505	134	8	866	3
58	381	20	619	523	28	477	142	8	858	2
59	401	20	599	551	28	449	150	8	850	1
60	72421		27579	79579		20421	07158		92842	0
′	9.	d	10.	9.	d	10.	10.	d	9.	′
	l cos	1′	l sec	l cot	1′	l tan	l csc	1′	l sin	

″	Proportional Parts 29	28	21	20	8	7
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5	2	2	2	2	1	1
6	3	3	2	2	1	1
7	3	3	2	2	1	1
8	4	4	3	3	1	1
9	4	4	3	3	1	1
10	5	5	4	3	1	1
11	5	5	4	4	1	1
12	6	6	4	4	2	1
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41	20	19	14	14	5	5
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43	21	20	15	14	6	5
44	21	21	15	15	6	5
45	22	21	16	15	6	5
46	22	21	16	15	6	5
47	23	22	16	16	6	5
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52	25	24	18	17	7	6
53	26	25	19	18	7	6
54	26	25	19	18	7	6
55	27	26	19	18	7	6
56	27	26	20	19	7	7
57	28	27	20	19	8	7
58	28	27	20	19	8	7
59	29	28	21	20	8	7
60	29	28	21	20	8	7
″	29	28	21	20	8	7
	Proportional Parts					

′	*l* sin 9.	d 1′	*l* csc 10.	*l* tan 9.	d 1′	*l* cot 10.	*l* sec 10.	d 1′	*l* cos 9.	′
0	72421	20	27579	79579	28	20421	07158	8	92842	60
1	441	20	559	607	28	393	166	8	834	59
2	461	21	539	635	28	365	174	8	826	58
3	482	20	518	663	28	337	182	8	818	57
4	502	20	498	691	28	309	190	7	810	56
5	522	20	478	719	28	281	197	8	803	55
6	542	20	458	747	29	253	205	8	795	54
7	562	20	438	776	28	224	213	8	787	53
8	582	20	418	804	28	196	221	8	779	52
9	602	20	398	832	28	168	229	8	771	51
10	622	21	378	860	28	140	237	8	763	50
11	643	20	357	888	28	112	245	8	755	49
12	663	20	337	916	28	084	253	8	747	48
13	683	20	317	944	28	056	261	8	739	47
14	703	20	297	972	28	028	269	8	731	46
15	723	20	277	80000	28	000	277	8	723	45
16	743	20	257	028	28	19972	285	8	715	44
17	763	20	237	056	28	944	293	8	707	43
18	783	20	217	084	28	916	301	8	699	42
19	803	20	197	112	28	888	309	8	691	41
20	823	20	177	140	28	860	317	8	683	40
21	843	20	157	168	27	832	325	8	675	39
22	863	20	137	195	28	805	333	8	667	38
23	883	19	117	223	28	777	341	8	659	37
24	902	20	098	251	28	749	349	8	651	36
25	922	20	078	279	28	721	357	8	643	35
26	942	20	058	307	28	693	365	8	635	34
27	962	20	038	335	28	665	373	8	627	33
28	982	20	018	363	28	637	381	8	619	32
29	73002	20	26998	391	28	609	389	8	611	31
30	73022	19	26978	80419	28	19581	07397	8	92603	30
31	041	20	959	447	27	553	405	8	595	29
32	061	20	939	474	28	526	413	8	587	28
33	081	20	919	502	28	498	421	8	579	27
34	101	20	899	530	28	470	429	8	571	26
35	121	19	879	558	28	442	437	8	563	25
36	140	20	860	586	28	414	445	9	555	24
37	160	20	840	614	28	386	454	8	546	23
38	180	20	820	642	27	358	462	8	538	22
39	200	19	800	669	28	331	470	8	530	21
40	219	20	781	697	28	303	478	8	522	20
41	239	20	761	725	28	275	486	8	514	19
42	259	19	741	753	28	247	494	8	506	18
43	278	20	722	781	27	219	502	8	498	17
44	298	20	702	808	28	192	510	8	490	16
45	318	19	682	836	28	164	518	9	482	15
46	337	20	663	864	28	136	527	8	473	14
47	357	20	643	892	27	108	535	8	465	13
48	377	19	623	919	28	081	543	8	457	12
49	396	20	604	947	28	053	551	8	449	11
50	416	19	584	975	28	025	559	8	441	10
51	435	20	565	81003	27	18997	567	8	433	9
52	455	19	545	030	28	970	575	9	425	8
53	474	20	526	058	28	942	584	8	416	7
54	494	19	506	086	27	914	592	8	408	6
55	513	20	487	113	28	887	600	8	400	5
56	533	19	467	141	28	859	608	8	392	4
57	552	20	448	169	27	831	616	8	384	3
58	572	19	428	196	28	804	624	9	376	2
59	591	20	409	224	28	776	633	8	367	1
60	73611		26389	81252		18748	07641		92359	0
′	9. *l* cos	d 1′	10. *l* sec	9. *l* cot	d 1′	10. *l* tan	10. *l* csc	d 1′	9. *l* sin	′

Proportional Parts

″	29	28	27	21	20	19	9	8	7
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0
2	1	1	1	1	1	1	0	0	0
3	1	1	1	1	1	1	0	0	0
4	2	2	2	1	1	1	1	1	0
5	2	2	2	2	2	2	1	1	1
6	3	3	3	2	2	2	1	1	1
7	3	3	3	2	2	2	1	1	1
8	4	4	4	3	3	3	1	1	1
9	4	4	4	3	3	3	1	1	1
10	5	5	4	4	3	3	2	1	1
11	5	5	5	4	4	3	2	1	1
12	6	6	5	4	4	4	2	2	1
13	6	6	6	5	4	4	2	2	2
14	7	7	6	5	5	4	2	2	2
15	7	7	7	5	5	5	2	2	2
16	8	7	7	6	5	5	2	2	2
17	8	8	8	6	6	5	3	2	2
18	9	8	8	6	6	6	3	2	2
19	9	9	9	7	6	6	3	3	2
20	10	9	9	7	7	6	3	3	2
21	10	10	9	7	7	7	3	3	2
22	11	10	10	8	7	7	3	3	3
23	11	11	10	8	8	7	3	3	3
24	12	11	11	8	8	8	4	3	3
25	12	12	11	9	8	8	4	3	3
26	13	12	12	9	9	8	4	3	3
27	13	13	12	9	9	9	4	4	3
28	14	13	13	10	9	9	4	4	3
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32	15	15	14	11	11	10	5	4	4
33	16	15	15	12	11	10	5	4	4
34	16	16	15	12	11	11	5	5	4
35	17	16	16	12	12	11	5	5	4
36	17	17	16	13	12	11	5	5	4
37	18	17	17	13	12	12	6	5	4
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39	19	18	18	14	13	12	6	5	5
40	19	19	18	14	13	13	6	5	5
41	20	19	18	14	14	13	6	5	5
42	20	20	19	15	14	13	6	6	5
43	21	20	19	15	14	14	6	6	5
44	21	21	20	15	15	14	7	6	5
45	22	21	20	16	15	14	7	6	5
46	22	21	21	16	15	15	7	6	5
47	23	22	21	16	16	15	7	6	5
48	23	22	22	17	16	15	7	6	6
49	24	23	22	17	16	16	7	7	6
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51	25	24	23	18	17	16	8	7	6
52	25	24	23	18	17	16	8	7	6
53	26	25	24	19	18	17	8	7	6
54	26	25	24	19	18	17	8	7	6
55	27	26	25	19	18	17	8	7	6
56	27	26	25	20	19	18	8	7	7
57	28	27	26	20	19	18	9	8	7
58	28	27	26	20	19	18	9	8	7
59	29	28	27	21	20	19	9	8	7
60	29	28	27	21	20	19	9	8	7
″	29	28	27	21	20	19	9	8	7

Proportional Parts

′	l sin 9.	d 1′	l csc 10.	l tan 9.	d 1′	l cot 10.	l sec 10.	d 1′	l cos 9.	′
0	73611	19	26389	81252	27	18748	07641	8	92359	60
1	630	20	370	279	28	721	649	8	351	59
2	650	19	350	307	28	693	657	8	343	58
3	669	20	331	335	27	665	665	9	335	57
4	689	19	311	362	28	638	674	8	326	56
5	708	19	292	390	28	610	682	8	318	55
6	727	20	273	418	27	582	690	8	310	54
7	747	19	253	445	28	555	698	9	302	53
8	766	19	234	473	27	527	707	8	293	52
9	785	20	215	500	28	500	715	8	285	51
10	805	19	195	528	28	472	723	8	277	50
11	824	19	176	556	27	444	731	9	269	49
12	843	20	157	583	28	417	740	8	260	48
13	863	19	137	611	27	389	748	8	252	47
14	882	19	118	638	28	362	756	9	244	46
15	901	20	099	666	27	334	765	8	235	45
16	921	19	079	693	28	307	773	8	227	44
17	940	19	060	721	27	279	781	8	219	43
18	959	19	041	748	28	252	789	9	211	42
19	978	19	022	776	27	224	798	8	202	41
20	997	20	003	803	28	197	806	8	194	40
21	74017	19	25983	831	27	169	814	9	186	39
22	036	19	964	858	28	142	823	8	177	38
23	055	19	945	886	27	114	831	8	169	37
24	074	19	926	913	28	087	839	9	161	36
25	093	20	907	941	27	059	848	8	152	35
26	113	19	887	968	28	032	856	8	144	34
27	132	19	868	996	27	004	864	9	136	33
28	151	19	849	82023	28	17977	873	8	127	32
29	170	19	830	051	27	949	881	8	119	31
30	74189	19	25811	82078	28	17922	07889	9	92111	30
31	208	19	792	106	27	894	898	8	102	29
32	227	19	773	133	28	867	906	8	094	28
33	246	19	754	161	27	839	914	9	086	27
34	265	19	735	188	27	812	923	8	077	26
35	284	19	716	215	28	785	931	9	069	25
36	303	19	697	243	27	757	940	8	060	24
37	322	19	678	270	28	730	948	8	052	23
38	341	19	659	298	27	702	956	9	044	22
39	360	19	640	325	27	675	965	8	035	21
40	379	19	621	352	28	648	973	9	027	20
41	398	19	602	380	27	620	982	8	018	19
42	417	19	583	407	28	593	990	8	010	18
43	436	19	564	435	27	565	998	9	002	17
44	455	19	545	462	27	538	08007	8	91993	16
45	474	19	526	489	28	511	015	9	985	15
46	493	19	507	517	27	483	024	8	976	14
47	512	19	488	544	27	456	032	9	968	13
48	531	18	469	571	28	429	041	8	959	12
49	549	19	451	599	27	401	049	9	951	11
50	568	19	432	626	27	374	058	8	942	10
51	587	19	413	653	28	347	066	9	934	9
52	606	19	394	681	27	319	075	8	925	8
53	625	19	375	708	27	292	083	9	917	7
54	644	18	356	735	27	265	092	8	908	6
55	662	19	338	762	28	238	100	9	900	5
56	681	19	319	790	27	210	109	8	891	4
57	700	19	300	817	27	183	117	9	883	3
58	719	18	281	844	27	156	126	8	874	2
59	737	19	263	871	28	129	134	9	866	1
60	74756		25244	82899		17101	08143		91857	0
′	9. l cos	d 1′	10. l sec	9. l cot	d 1′	10. l tan	10. l csc	d 1′	9. l sin	′

″	Proportional Parts 28	27	20	19	18	9	8
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	1	1	1	1	1	0	0
3	1	1	1	1	1	0	0
4	2	2	1	1	1	1	1
5	2	2	2	2	2	1	1
6	3	3	2	2	2	1	1
7	3	3	2	2	2	1	1
8	4	4	3	3	2	1	1
9	4	4	3	3	3	1	1
10	5	4	3	3	3	2	1
11	5	5	4	3	3	2	1
12	6	5	4	4	4	2	2
13	6	6	4	4	4	2	2
14	7	6	5	4	4	2	2
15	7	7	5	5	4	2	2
16	7	7	5	5	5	2	2
17	8	8	6	5	5	3	2
18	8	8	6	6	5	3	2
19	9	9	6	6	6	3	3
20	9	9	7	6	6	3	3
21	10	9	7	7	6	3	3
22	10	10	7	7	7	3	3
23	11	10	8	7	7	3	3
24	11	11	8	8	7	4	3
25	12	11	8	8	8	4	3
26	12	12	9	8	8	4	3
27	13	12	9	9	8	4	4
28	13	13	9	9	8	4	4
29	14	13	10	9	9	4	4
30	14	14	10	10	9	4	4
31	14	14	10	10	9	5	4
32	15	14	11	10	10	5	4
33	15	15	11	10	10	5	4
34	16	15	11	11	10	5	5
35	16	16	12	11	10	5	5
36	17	16	12	11	11	5	5
37	17	17	12	12	11	6	5
38	18	17	13	12	11	6	5
39	18	18	13	12	12	6	5
40	19	18	13	13	12	6	5
41	19	18	14	13	12	6	5
42	20	19	14	13	13	6	6
43	20	19	14	14	13	6	6
44	21	20	15	14	13	7	6
45	21	20	15	14	14	7	6
46	21	21	15	15	14	7	6
47	22	21	16	15	14	7	6
48	22	22	16	15	14	7	6
49	23	22	16	16	15	7	7
50	23	22	17	16	15	8	7
51	24	23	17	16	15	8	7
52	24	23	17	16	16	8	7
53	25	24	18	17	16	8	7
54	25	24	18	17	16	8	7
55	26	25	18	17	16	8	7
56	26	25	19	18	17	8	7
57	27	26	19	18	17	9	8
58	27	26	19	18	17	9	8
59	28	27	20	19	18	9	8
60	28	27	20	19	18	9	8
″	28 Proportional Parts	27	20	19	18	9	8

′	l sin 9.	d 1′	l csc 10.	l tan 9.	d 1′	l cot 10.	l sec 10.	d 1′	l cos 9.	′
0	74756	19	25244	82899	27	17101	08143	8	91857	60
1	775	19	225	926	27	074	151	9	849	59
2	794	18	206	953	27	047	160	8	840	58
3	812	19	188	980	28	020	168	9	832	57
4	831	19	169	83008	27	16992	177	8	823	56
5	850	18	150	035	27	965	185	9	815	55
6	868	19	132	062	27	938	194	8	806	54
7	887	19	113	089	28	911	202	9	798	53
8	906	18	094	117	27	883	211	8	789	52
9	924	19	076	144	27	856	219	9	781	51
10	943	18	057	171	27	829	228	9	772	50
11	961	19	039	198	27	802	237	8	763	49
12	980	19	020	225	27	775	245	9	755	48
13	999	18	001	252	28	748	254	8	746	47
14	75017	19	24983	280	27	720	262	9	738	46
15	036	18	964	307	27	693	271	9	729	45
16	054	19	946	334	27	666	280	8	720	44
17	073	18	927	361	27	639	288	9	712	43
18	091	19	909	388	27	612	297	8	703	42
19	110	18	890	415	27	585	305	9	695	41
20	128	19	872	442	28	558	314	9	686	40
21	147	18	853	470	27	530	323	8	677	39
22	165	19	835	497	27	503	331	9	669	38
23	184	18	816	524	27	476	340	9	660	37
24	202	19	798	551	27	449	349	8	651	36
25	221	18	779	578	27	422	357	9	643	35
26	239	19	761	605	27	395	366	9	634	34
27	258	18	742	632	27	368	375	8	625	33
28	276	18	724	659	27	341	383	9	617	32
29	294	19	706	686	27	314	392	9	608	31
30	75313	18	24687	83713	27	16287	08401	8	91599	30
31	331	19	669	740	28	260	409	9	591	29
32	350	18	650	768	27	232	418	9	582	28
33	368	18	632	795	27	205	427	8	573	27
34	386	19	614	822	27	178	435	9	565	26
35	405	18	595	849	27	151	444	9	556	25
36	423	18	577	876	27	124	453	9	547	24
37	441	18	559	903	27	097	462	8	538	23
38	459	19	541	930	27	070	470	9	530	22
39	478	18	522	957	27	043	479	9	521	21
40	496	18	504	984	27	016	488	8	512	20
41	514	19	486	84011	27	15989	496	9	504	19
42	533	18	467	038	27	962	505	9	495	18
43	551	18	449	065	27	935	514	9	486	17
44	569	18	431	092	27	908	523	8	477	16
45	587	18	413	119	27	881	531	9	469	15
46	605	19	395	146	27	854	540	9	460	14
47	624	18	376	173	27	827	549	9	451	13
48	642	18	358	200	27	800	558	9	442	12
49	660	18	340	227	27	773	567	8	433	11
50	678	18	322	254	26	746	575	9	425	10
51	696	18	304	280	27	720	584	9	416	9
52	714	19	286	307	27	693	593	9	407	8
53	733	18	267	334	27	666	602	9	398	7
54	751	18	249	361	27	639	611	8	389	6
55	769	18	231	388	27	612	619	9	381	5
56	787	18	213	415	27	585	628	9	372	4
57	805	18	195	442	27	558	637	9	363	3
58	823	18	177	469	27	531	646	9	354	2
59	841	18	159	496	27	504	655	9	345	1
60	75859		24141	84523		15477	08664		91336	0
′	9. l cos	d 1′	10. l sec	9. l cot	d 1′	10. l tan	10. l csc	d 1′	9. l sin	′

″	28	27	26	19	18	9	8
	Proportional Parts						
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	1	1	1	1	1	0	0
3	1	1	1	1	1	0	0
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5	2	2	2	2	2	1	1
6	3	3	3	2	2	1	1
7	3	3	3	2	2	1	1
8	4	4	3	3	2	1	1
9	4	4	4	3	3	1	1
10	5	4	4	3	3	2	1
11	5	5	5	3	3	2	1
12	6	5	5	4	4	2	2
13	6	6	6	4	4	2	2
14	7	6	6	4	4	2	2
15	7	7	6	5	4	2	2
16	7	7	7	5	5	2	2
17	8	8	7	5	5	3	2
18	8	8	8	6	5	3	2
19	9	9	8	6	6	3	3
20	9	9	9	6	6	3	3
21	10	9	9	7	6	3	3
22	10	10	10	7	7	3	3
23	11	10	10	7	7	3	3
24	11	11	10	8	7	4	3
25	12	11	11	8	8	4	3
26	12	12	11	8	8	4	3
27	13	12	12	9	8	4	4
28	13	13	12	9	8	4	4
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30	14	14	13	10	9	4	4
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32	15	14	14	10	10	5	4
33	15	15	14	10	10	5	4
34	16	15	15	11	10	5	5
35	16	16	15	11	10	5	5
36	17	16	16	11	11	5	5
37	17	17	16	12	11	6	5
38	18	17	16	12	11	6	5
39	18	18	17	12	12	6	5
40	19	18	17	13	12	6	5
41	19	18	18	13	12	6	5
42	20	19	18	13	13	6	6
43	20	19	19	14	13	6	6
44	21	20	19	14	13	7	6
45	21	20	20	14	14	7	6
46	21	21	20	15	14	7	6
47	22	21	20	15	14	7	6
48	22	22	21	15	14	7	6
49	23	22	21	16	15	7	7
50	23	22	22	16	15	8	7
51	24	23	22	16	15	8	7
52	24	23	23	16	16	8	7
53	25	24	23	17	16	8	7
54	25	24	23	17	16	8	7
55	26	25	24	17	16	8	7
56	26	25	24	18	17	8	7
57	27	26	25	18	17	9	8
58	27	26	25	18	17	9	8
59	28	27	26	19	18	9	8
60	28	27	26	19	18	9	8
″	28	27	26	19	18	9	8
	Proportional Parts						

′	*l* sin 9.	d 1′	*l* csc 10.	*l* tan 9.	d 1′	*l* cot 10.	*l* sec 10.	d 1′	*l* cos 9.	′
0	75859	18	24141	84523	27	15477	08664	8	91336	60
1	877	18	123	550	26	450	672	9	328	59
2	895	18	105	576	27	424	681	9	319	58
3	913	18	087	603	27	397	690	9	310	57
4	931	18	069	630	27	370	699	9	301	56
5	949	18	051	657	27	343	708	9	292	55
6	967	18	033	684	27	316	717	9	283	54
7	985	18	015	711	27	289	726	8	274	53
8	76003	18	23997	738	26	262	734	9	266	52
9	021	18	979	764	27	236	743	9	257	51
10	039	18	961	791	27	209	752	9	248	50
11	057	18	943	818	27	182	761	9	239	49
12	075	18	925	845	27	155	770	9	230	48
13	093	18	907	872	27	128	779	9	221	47
14	111	18	889	899	26	101	788	9	212	46
15	129	17	871	925	27	075	797	9	203	45
16	146	18	854	952	27	048	806	9	194	44
17	164	18	836	979	27	021	815	9	185	43
18	182	18	818	85006	27	14994	824	9	176	42
19	200	18	800	033	26	967	833	9	167	41
20	218	18	782	059	27	941	842	9	158	40
21	236	17	764	086	27	914	851	8	149	39
22	253	18	747	113	27	887	859	9	141	38
23	271	18	729	140	26	860	868	9	132	37
24	289	18	711	166	27	834	877	9	123	36
25	307	17	693	193	27	807	886	9	114	35
26	324	18	676	220	27	780	895	9	105	34
27	342	18	658	247	26	753	904	9	096	33
28	360	18	640	273	27	727	913	9	087	32
29	378	17	622	300	27	700	922	9	078	31
30	76395	18	23605	85327	27	14673	08931	9	91069	30
31	413	18	587	354	26	646	940	9	060	29
32	431	17	569	380	27	620	949	9	051	28
33	448	18	552	407	27	593	958	9	042	27
34	466	18	534	434	26	566	967	10	033	26
35	484	17	516	460	27	540	977	9	023	25
36	501	18	499	487	27	513	986	9	014	24
37	519	18	481	514	26	486	995	9	005	23
38	537	17	463	540	27	460	09004	9	90996	22
39	554	18	446	567	27	433	013	9	987	21
40	572	18	428	594	26	406	022	9	978	20
41	590	17	410	620	27	380	031	9	969	19
42	607	18	393	647	27	353	040	9	960	18
43	625	17	375	674	26	326	049	9	951	17
44	642	18	358	700	27	300	058	9	942	16
45	660	17	340	727	27	273	067	9	933	15
46	677	18	323	754	26	246	076	9	924	14
47	695	17	305	780	27	220	085	9	915	13
48	712	18	288	807	27	193	094	10	906	12
49	730	17	270	834	26	166	104	9	896	11
50	747	18	253	860	27	140	113	9	887	10
51	765	17	235	887	26	113	122	9	878	9
52	782	18	218	913	27	087	131	9	869	8
53	800	17	200	940	27	060	140	9	860	7
54	817	18	183	967	26	033	149	9	851	6
55	835	17	165	993	27	007	158	10	842	5
56	852	18	148	86020	26	13980	168	9	832	4
57	870	17	130	046	27	954	177	9	823	3
58	887	17	113	073	27	927	186	9	814	2
59	904	18	096	100	26	900	195	9	805	1
60	76922		23078	86126		13874	09204		90796	0
′	9. *l* cos	d 1′	10. *l* sec	9. *l* cot	d 1′	10. *l* tan	10. *l* csc	d 1′	9. *l* sin	′

″	Proportional Parts 27	26	18	17	10	9	8
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	1	1	1	1	0	0	0
3	1	1	1	1	0	0	0
4	2	2	1	1	1	1	1
5	2	2	2	1	1	1	1
6	3	3	2	2	1	1	1
7	3	3	2	2	1	1	1
8	4	3	2	2	1	1	1
9	4	4	3	3	2	1	1
10	4	4	3	3	2	2	1
11	5	5	3	3	2	2	1
12	5	5	4	3	2	2	2
13	6	6	4	4	2	2	2
14	6	6	4	4	2	2	2
15	7	6	4	4	2	2	2
16	7	7	5	5	3	2	2
17	8	7	5	5	3	3	2
18	8	8	5	5	3	3	2
19	9	8	6	5	3	3	3
20	9	9	6	6	3	3	3
21	9	9	6	6	4	3	3
22	10	10	7	6	4	3	3
23	10	10	7	7	4	3	3
24	11	10	7	7	4	4	3
25	11	11	8	7	4	4	3
26	12	11	8	7	4	4	3
27	12	12	8	8	4	4	4
28	13	12	8	8	5	4	4
29	13	13	9	8	5	4	4
30	14	13	9	8	5	4	4
31	14	13	9	9	5	5	4
32	14	14	10	9	5	5	4
33	15	14	10	9	6	5	4
34	15	15	10	10	6	5	5
35	16	15	10	10	6	5	5
36	16	16	11	10	6	5	5
37	17	16	11	10	6	6	5
38	17	16	11	11	6	6	5
39	18	17	12	11	6	6	5
40	18	17	12	11	7	6	5
41	18	18	12	12	7	6	5
42	19	18	13	12	7	6	6
43	19	19	13	12	7	6	6
44	20	19	13	12	7	7	6
45	20	20	14	13	8	7	6
46	21	20	14	13	8	7	6
47	21	20	14	13	8	7	6
48	22	21	14	14	8	7	6
49	22	21	15	14	8	7	7
50	22	22	15	14	8	8	7
51	23	22	15	14	8	8	7
52	23	23	16	15	9	8	7
53	24	23	16	15	9	8	7
54	24	23	16	15	9	8	7
55	25	24	16	16	9	8	7
56	25	24	17	16	9	8	7
57	26	25	17	16	10	9	8
58	26	25	17	16	10	9	8
59	27	26	18	17	10	9	8
60	27	26	18	17	10	9	8
″	27 Proportional Parts	26	18	17	10	9	8

′	*l* sin 9.	d 1′	*l* csc 10.	*l* tan 9.	d 1′	*l* cot 10.	*l* sec 10.	d 1′	*l* cos 9.	′
0	76922	17	23078	86126	27	13874	09204	9	90796	60
1	939	18	061	153	26	847	213	10	787	59
2	957	17	043	179	27	821	223	9	777	58
3	974	17	026	206	26	794	232	9	768	57
4	991	18	009	232	27	768	241	9	759	56
5	77009	17	22991	259	26	741	250	9	750	55
6	026	17	974	285	27	715	259	10	741	54
7	043	18	957	312	26	688	269	9	731	53
8	061	17	939	338	27	662	278	9	722	52
9	078	17	922	365	27	635	287	9	713	51
10	095	17	905	392	26	608	296	10	704	50
11	112	18	888	418	27	582	306	9	694	49
12	130	17	870	445	26	555	315	9	685	48
13	147	17	853	471	27	529	324	9	676	47
14	164	17	836	498	26	502	333	10	667	46
15	181	18	819	524	27	476	343	9	657	45
16	199	17	801	551	26	449	352	9	648	44
17	216	17	784	577	26	423	361	9	639	43
18	233	17	767	603	27	397	370	10	630	42
19	250	18	750	630	26	370	380	9	620	41
20	268	17	732	656	27	344	389	9	611	40
21	285	17	715	683	26	317	398	10	602	39
22	302	17	698	709	27	291	408	9	592	38
23	319	17	681	736	26	264	417	9	583	37
24	336	17	664	762	27	238	426	9	574	36
25	353	17	647	789	26	211	435	10	565	35
26	370	17	630	815	27	185	445	9	555	34
27	387	18	613	842	26	158	454	9	546	33
28	405	17	595	868	26	132	463	10	537	32
29	422	17	578	894	27	106	473	9	527	31
30	77439	17	22561	86921	26	13079	09482	9	90518	30
31	456	17	544	947	27	053	491	10	509	29
32	473	17	527	974	26	026	501	9	499	28
33	490	17	510	87000	27	000	510	10	490	27
34	507	17	493	027	26	12973	520	9	480	26
35	524	17	476	053	26	947	529	9	471	25
36	541	17	459	079	27	921	538	10	462	24
37	558	17	442	106	26	894	548	9	452	23
38	575	17	425	132	26	868	557	9	443	22
39	592	17	408	158	27	842	566	10	434	21
40	609	17	391	185	26	815	576	9	424	20
41	626	17	374	211	27	789	585	10	415	19
42	643	17	357	238	26	762	595	9	405	18
43	660	17	340	264	26	736	604	10	396	17
44	677	17	323	290	27	710	614	9	386	16
45	694	17	306	317	26	683	623	9	377	15
46	711	17	289	343	26	657	632	10	368	14
47	728	16	272	369	27	631	642	9	358	13
48	744	17	256	396	26	604	651	10	349	12
49	761	17	239	422	26	578	661	9	339	11
50	778	17	222	448	27	552	670	10	330	10
51	795	17	205	475	26	525	680	9	320	9
52	812	17	188	501	26	499	689	10	311	8
53	829	17	171	527	27	473	699	9	301	7
54	846	16	154	554	26	446	708	10	292	6
55	862	17	138	580	26	420	718	9	282	5
56	879	17	121	606	27	394	727	10	273	4
57	896	17	104	633	26	367	737	9	263	3
58	913	17	087	659	26	341	746	10	254	2
59	930	16	070	685	26	315	756	9	244	1
60	77946		22054	87711		12289	09765		90235	0
′	9. *l* cos	d 1′	10. *l* sec	9. *l* cot	d 1′	10. *l* tan	10. *l* csc	d 1′	9. *l* sin	′

″	27	26	18	17	16	10	9
	Proportional Parts						
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	1	1	1	1	1	0	0
3	1	1	1	1	1	0	0
4	2	2	1	1	1	1	1
5	2	2	2	1	1	1	1
6	3	3	2	2	2	1	1
7	3	3	2	2	2	1	1
8	4	3	2	2	2	1	1
9	4	4	3	3	2	2	1
10	4	4	3	3	3	2	2
11	5	5	3	3	3	2	2
12	5	5	4	3	3	2	2
13	6	6	4	4	3	2	2
14	6	6	4	4	4	2	2
15	7	6	4	4	4	2	2
16	7	7	5	5	4	3	2
17	8	7	5	5	5	3	3
18	8	8	5	5	5	3	3
19	9	8	6	5	5	3	3
20	9	9	6	6	5	3	3
21	9	9	6	6	6	4	3
22	10	10	7	6	6	4	3
23	10	10	7	7	6	4	3
24	11	10	7	7	6	4	4
25	11	11	8	7	7	4	4
26	12	11	8	7	7	4	4
27	12	12	8	8	7	4	4
28	13	12	8	8	7	5	4
29	13	13	9	8	8	5	4
30	14	13	9	8	8	5	4
31	14	13	9	9	8	5	5
32	14	14	10	9	9	5	5
33	15	14	10	9	9	6	5
34	15	15	10	10	9	6	5
35	16	15	10	10	9	6	5
36	16	16	11	10	10	6	5
37	17	16	11	10	10	6	6
38	17	16	11	11	10	6	6
39	18	17	12	11	10	6	6
40	18	17	12	11	11	7	6
41	18	18	12	12	11	7	6
42	19	18	13	12	11	7	6
43	19	19	13	12	11	7	6
44	20	19	13	12	12	7	7
45	20	20	14	13	12	8	7
46	21	20	14	13	12	8	7
47	21	20	14	13	13	8	7
48	22	21	14	14	13	8	7
49	22	21	15	14	13	8	7
50	22	22	15	14	13	8	8
51	23	22	15	14	14	8	8
52	23	23	16	15	14	9	8
53	24	23	16	15	14	9	8
54	24	23	16	15	14	9	8
55	25	24	16	16	15	9	8
56	25	24	17	16	15	9	8
57	26	25	17	16	15	10	9
58	26	25	17	16	15	10	9
59	27	26	18	17	16	10	9
60	27	26	18	17	16	10	9
″	27	26	18	17	16	10	9
	Proportional Parts						

′	*l* sin 9.	d 1′	*l* csc 10.	*l* tan 9.	d 1′	*l* cot 10.	*l* sec 10.	d 1′	*l* cos 9.	′
0	77946	17	22054	87711	27	12289	09765	10	90235	60
1	963	17	037	738	26	262	775	9	225	59
2	980	17	020	764	26	236	784	10	216	58
3	997	16	003	790	27	210	794	9	206	57
4	78013	17	21987	817	26	183	803	10	197	56
5	030	17	970	843	26	157	813	9	187	55
6	047	16	953	869	26	131	822	10	178	54
7	063	17	937	895	27	105	832	9	168	53
8	080	17	920	922	26	078	841	10	159	52
9	097	16	903	948	26	052	851	10	149	51
10	113	17	887	974	26	026	861	9	139	50
11	130	17	870	88000	27	000	870	10	130	49
12	147	16	853	027	26	11973	880	9	120	48
13	163	17	837	053	26	947	889	10	111	47
14	180	17	820	079	26	921	899	10	101	46
15	197	16	803	105	26	895	909	9	091	45
16	213	17	787	131	27	869	918	10	082	44
17	230	16	770	158	26	842	928	9	072	43
18	246	17	754	184	26	816	937	10	063	42
19	263	17	737	210	26	790	947	10	053	41
20	280	16	720	236	26	764	957	9	043	40
21	296	17	704	262	27	738	966	10	034	39
22	313	16	687	289	26	711	976	10	024	38
23	329	17	671	315	26	685	986	9	014	37
24	346	16	654	341	26	659	995	10	005	36
25	362	17	638	367	26	633	10005	10	89995	35
26	379	16	621	393	27	607	015	9	985	34
27	395	17	605	420	26	580	024	10	976	33
28	412	16	588	446	26	554	034	10	966	32
29	428	17	572	472	26	528	044	9	956	31
30	78445	16	21555	88498	26	11502	10053	10	89947	30
31	461	17	539	524	26	476	063	10	937	29
32	478	16	522	550	27	450	073	9	927	28
33	494	16	506	577	26	423	082	10	918	27
34	510	17	490	603	26	397	092	10	908	26
35	527	16	473	629	26	371	102	10	898	25
36	543	17	457	655	26	345	112	9	888	24
37	560	16	440	681	26	319	121	10	879	23
38	576	16	424	707	26	293	131	10	869	22
39	592	17	408	733	26	267	141	10	859	21
40	609	16	391	759	27	241	151	9	849	20
41	625	17	375	786	26	214	160	10	840	19
42	642	16	358	812	26	188	170	10	830	18
43	658	16	342	838	26	162	180	10	820	17
44	674	17	326	864	26	136	190	9	810	16
45	691	16	309	890	26	110	199	10	801	15
46	707	16	293	916	26	084	209	10	791	14
47	723	16	277	942	26	058	219	10	781	13
48	739	17	261	968	26	032	229	10	771	12
49	756	16	244	994	26	006	239	9	761	11
50	772	16	228	89020	26	10980	248	10	752	10
51	788	17	212	046	27	954	258	10	742	9
52	805	16	195	073	26	927	268	10	732	8
53	821	16	179	099	26	901	278	10	722	7
54	837	16	163	125	26	875	288	10	712	6
55	853	16	147	151	26	849	298	9	702	5
56	869	17	131	177	26	823	307	10	693	4
57	886	16	114	203	26	797	317	10	683	3
58	902	16	098	229	26	771	327	10	673	2
59	918	16	082	255	26	745	337	10	663	1
60	78934		21066	89281		10719	10347		89653	0
′	9. *l* cos	d 1′	10. *l* sec	9. *l* cot	d 1′	10. *l* tan	10. *l* csc	d 1′	9. *l* sin	′

″	Proportional Parts 27	26	17	16	10	9
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	1	1	1	1	0	0
3	1	1	1	1	0	0
4	2	2	1	1	1	1
5	2	2	1	1	1	1
6	3	3	2	2	1	1
7	3	3	2	2	1	1
8	4	3	2	2	1	1
9	4	4	3	2	2	1
10	4	4	3	3	2	2
11	5	5	3	3	2	2
12	5	5	3	3	2	2
13	6	6	4	3	2	2
14	6	6	4	4	2	2
15	7	6	4	4	2	2
16	7	7	5	4	3	2
17	8	7	5	5	3	3
18	8	8	5	5	3	3
19	9	8	5	5	3	3
20	9	9	6	5	3	3
21	9	9	6	6	4	3
22	10	10	6	6	4	3
23	10	10	7	6	4	3
24	11	10	7	6	4	4
25	11	11	7	7	4	4
26	12	11	7	7	4	4
27	12	12	8	7	4	4
28	13	12	8	7	5	4
29	13	13	8	8	5	4
30	14	13	8	8	5	4
31	14	13	9	8	5	5
32	14	14	9	9	5	5
33	15	14	9	9	6	5
34	15	15	10	9	6	5
35	16	15	10	9	6	5
36	16	16	10	10	6	5
37	17	16	10	10	6	6
38	17	16	11	10	6	6
39	18	17	11	10	6	6
40	18	17	11	11	7	6
41	18	18	12	11	7	6
42	19	18	12	11	7	6
43	19	19	12	11	7	6
44	20	19	12	12	7	7
45	20	20	13	12	8	7
46	21	20	13	12	8	7
47	21	20	13	13	8	7
48	22	21	14	13	8	7
49	22	21	14	13	8	7
50	22	22	14	13	8	8
51	23	22	14	14	8	8
52	23	23	15	14	9	8
53	24	23	15	14	9	8
54	24	23	15	14	9	8
55	25	24	16	15	9	8
56	25	24	16	15	9	8
57	26	25	16	15	10	9
58	26	25	16	15	10	9
59	27	26	17	16	10	9
60	27	26	17	16	10	9
″	27 Proportional Parts	26	17	16	10	9

′	*l* sin 9.	d 1′	*l* csc 10.	*l* tan 9.	d 1′	*l* cot 10.	*l* sec 10.	d 1′	*l* cos 9.	′
0	78934	16	21066	89281	26	10719	10347	10	89653	60
1	950	17	050	307	26	693	357	10	643	59
2	967	16	033	333	26	667	367	9	633	58
3	983	16	017	359	26	641	376	10	624	57
4	999	16	001	385	26	615	386	10	614	56
5	79015	16	20985	411	26	589	396	10	604	55
6	031	16	969	437	26	563	406	10	594	54
7	047	16	953	463	26	537	416	10	584	53
8	063	16	937	489	26	511	426	10	574	52
9	079	16	921	515	26	485	436	10	564	51
10	095	16	905	541	26	459	446	10	554	50
11	111	17	889	567	26	433	456	10	544	49
12	128	16	872	593	26	407	466	10	534	48
13	144	16	856	619	26	381	476	10	524	47
14	160	16	840	645	26	355	486	10	514	46
15	176	16	824	671	26	329	496	9	504	45
16	192	16	808	697	26	303	505	10	495	44
17	208	16	792	723	26	277	515	10	485	43
18	224	16	776	749	26	251	525	10	475	42
19	240	16	760	775	26	225	535	10	465	41
20	256	16	744	801	26	199	545	10	455	40
21	272	16	728	827	26	173	555	10	445	39
22	288	16	712	853	26	147	565	10	435	38
23	304	15	696	879	26	121	575	10	425	37
24	319	16	681	905	26	095	585	10	415	36
25	335	16	665	931	26	069	595	10	405	35
26	351	16	649	957	26	043	605	10	395	34
27	367	16	633	983	26	017	615	10	385	33
28	383	16	617	90009	26	09991	625	11	375	32
29	399	16	601	035	26	965	636	10	364	31
30	79415	16	20585	90061	25	09939	10646	10	89354	30
31	431	16	569	086	26	914	656	10	344	29
32	447	16	553	112	26	888	666	10	334	28
33	463	15	537	138	26	862	676	10	324	27
34	478	16	522	164	26	836	686	10	314	26
35	494	16	506	190	26	810	696	10	304	25
36	510	16	490	216	26	784	706	10	294	24
37	526	16	474	242	26	758	716	10	284	23
38	542	16	458	268	26	732	726	10	274	22
39	558	15	442	294	26	706	736	10	264	21
40	573	16	427	320	26	680	746	10	254	20
41	589	16	411	346	25	654	756	11	244	19
42	605	16	395	371	26	629	767	10	233	18
43	621	15	379	397	26	603	777	10	223	17
44	636	16	364	423	26	577	787	10	213	16
45	652	16	348	449	26	551	797	10	203	15
46	668	16	332	475	26	525	807	10	193	14
47	684	15	316	501	26	499	817	10	183	13
48	699	16	301	527	26	473	827	11	173	12
49	715	16	285	553	25	447	838	10	162	11
50	731	15	269	578	26	422	848	10	152	10
51	746	16	254	604	26	396	858	10	142	9
52	762	16	238	630	26	370	868	10	132	8
53	778	15	222	656	26	344	878	10	122	7
54	793	16	207	682	26	318	888	11	112	6
55	809	16	191	708	26	292	899	10	101	5
56	825	15	175	734	25	266	909	10	091	4
57	840	16	160	759	26	241	919	10	081	3
58	856	16	144	785	26	215	929	11	071	2
59	872	15	128	811	26	189	940	10	060	1
60	79887		20113	90837		09163	10950		89050	0
′	9. *l* cos	d 1′	10. *l* sec	9. *l* cot	d 1′	10. *l* tan	10. *l* csc	d 1′	9. *l* sin	′

″	26	25	17	16	15	11	10	9
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	1	1	1	1	0	0	0	0
3	1	1	1	1	1	1	0	0
4	2	2	1	1	1	1	1	1
5	2	2	1	1	1	1	1	1
6	3	2	2	2	2	1	1	1
7	3	3	2	2	2	1	1	1
8	3	3	2	2	2	1	1	1
9	4	4	3	2	2	2	2	1
10	4	4	3	3	2	2	2	2
11	5	5	3	3	3	2	2	2
12	5	5	3	3	3	2	2	2
13	6	5	4	3	3	2	2	2
14	6	6	4	4	4	3	2	2
15	6	6	4	4	4	3	2	2
16	7	7	5	4	4	3	3	2
17	7	7	5	5	4	3	3	3
18	8	8	5	5	4	3	3	3
19	8	8	5	5	5	3	3	3
20	9	8	6	5	5	4	3	3
21	9	9	6	6	5	4	4	3
22	10	9	6	6	6	4	4	3
23	10	10	7	6	6	4	4	3
24	10	10	7	6	6	4	4	4
25	11	10	7	7	6	5	4	4
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28	12	12	8	7	7	5	5	4
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30	13	12	8	8	8	6	5	4
31	13	13	9	8	8	6	5	5
32	14	13	9	9	8	6	5	5
33	14	14	9	9	8	6	6	5
34	15	14	10	9	8	6	6	5
35	15	15	10	9	9	6	6	5
36	16	15	10	10	9	7	6	5
37	16	15	10	10	9	7	6	6
38	16	16	11	10	10	7	6	6
39	17	16	11	10	10	7	6	6
40	17	17	11	11	10	7	7	6
41	18	17	12	11	10	8	7	6
42	18	18	12	11	10	8	7	6
43	19	18	12	11	11	8	7	6
44	19	18	12	12	11	8	7	7
45	20	19	13	12	11	8	8	7
46	20	19	13	12	12	8	8	7
47	20	20	13	13	12	9	8	7
48	21	20	14	13	12	9	8	7
49	21	20	14	13	12	9	8	7
50	22	21	14	13	12	9	8	8
51	22	21	14	14	13	9	8	8
52	23	22	15	14	13	10	9	8
53	23	22	15	14	13	10	9	8
54	23	22	15	14	14	10	9	8
55	24	23	16	15	14	10	9	8
56	24	23	16	15	14	10	9	8
57	25	24	16	15	14	10	10	9
58	25	24	16	15	14	11	10	9
59	26	25	17	16	15	11	10	9
60	26	25	17	16	15	11	10	9
″	26	25	17	16	15	11	10	9

Proportional Parts

128° 51°

′	l sin 9.	d 1′	l csc 10.	l tan 9.	d 1′	l cot 10.	l sec 10.	d 1′	l cos 9.	′
0	79887	16	20113	90837	26	09163	10950	10	89050	60
1	903	15	097	863	26	137	960	10	040	59
2	918	16	082	889	25	111	970	10	030	58
3	934	16	066	914	26	086	980	11	020	57
4	950	15	050	940	26	060	991	10	009	56
5	965	16	035	966	26	034	11001	10	88999	55
6	981	15	019	992	26	008	011	11	989	54
7	996	16	004	91018	25	08982	022	10	978	53
8	80012	15	19988	043	26	957	032	10	968	52
9	027	16	973	069	26	931	042	10	958	51
10	043	15	957	095	26	905	052	11	948	50
11	058	16	942	121	26	879	063	10	937	49
12	074	15	926	147	25	853	073	10	927	48
13	089	16	911	172	26	828	083	11	917	47
14	105	15	895	198	26	802	094	10	906	46
15	120	16	880	224	26	776	104	10	896	45
16	136	15	864	250	26	750	114	11	886	44
17	151	15	849	276	25	724	125	10	875	43
18	166	16	834	301	26	699	135	10	865	42
19	182	15	818	327	26	673	145	11	855	41
20	197	16	803	353	26	647	156	10	844	40
21	213	15	787	379	25	621	166	10	834	39
22	228	16	772	404	26	596	176	11	824	38
23	244	15	756	430	26	570	187	10	813	37
24	259	15	741	456	26	544	197	10	803	36
25	274	16	726	482	25	518	207	11	793	35
26	290	15	710	507	26	493	218	10	782	34
27	305	15	695	533	26	467	228	11	772	33
28	320	16	680	559	26	441	239	10	761	32
29	336	15	664	585	25	415	249	10	751	31
30	80351	15	19649	91610	26	08390	11259	11	88741	30
31	366	16	634	636	26	364	270	10	730	29
32	382	15	618	662	26	338	280	11	720	28
33	397	15	603	688	25	312	291	10	709	27
34	412	16	588	713	26	287	301	11	699	26
35	428	15	572	739	26	261	312	10	688	25
36	443	15	557	765	26	235	322	10	678	24
37	458	15	542	791	25	209	332	11	668	23
38	473	16	527	816	26	184	343	10	657	22
39	489	15	511	842	26	158	353	11	647	21
40	504	15	496	868	25	132	364	10	636	20
41	519	15	481	893	26	107	374	11	626	19
42	534	16	466	919	26	081	385	10	615	18
43	550	15	450	945	26	055	395	11	605	17
44	565	15	435	971	25	029	406	10	594	16
45	580	15	420	996	26	004	416	11	584	15
46	595	15	405	92022	26	07978	427	10	573	14
47	610	15	390	048	25	952	437	11	563	13
48	625	16	375	073	26	927	448	10	552	12
49	641	15	359	099	26	901	458	11	542	11
50	656	15	344	125	25	875	469	10	531	10
51	671	15	329	150	26	850	479	11	521	9
52	686	15	314	176	26	824	490	11	510	8
53	701	15	299	202	25	798	501	10	499	7
54	716	15	284	227	26	773	511	11	489	6
55	731	15	269	253	26	747	522	10	478	5
56	746	16	254	279	25	721	532	11	468	4
57	762	15	238	304	26	696	543	10	457	3
58	777	15	223	330	26	670	553	11	447	2
59	792	15	208	356	25	644	564	11	436	1
60	80807		19193	92381		07619	11575		88425	0
′	9. l cos	d 1′	10. l sec	9. l cot	d 1′	10. l tan	10. l csc	d 1′	9. l sin	′

″	26	25	16	15	11	10
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	1	1	1	0	0	0
3	1	1	1	1	1	0
4	2	2	1	1	1	1
5	2	2	1	1	1	1
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16	7	7	4	4	3	3
17	7	7	5	4	3	3
18	8	8	5	4	3	3
19	8	8	5	5	3	3
20	9	8	5	5	4	3
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22	10	9	6	6	4	4
23	10	10	6	6	4	4
24	10	10	6	6	4	4
25	11	10	7	6	5	4
26	11	11	7	6	5	4
27	12	11	7	7	5	4
28	12	12	7	7	5	5
29	13	12	8	7	5	5
30	13	12	8	8	6	5
31	13	13	8	8	6	5
32	14	13	9	8	6	5
33	14	14	9	8	6	6
34	15	14	9	8	6	6
35	15	15	9	9	6	6
36	16	15	10	9	7	6
37	16	15	10	9	7	6
38	16	16	10	10	7	6
39	17	16	10	10	7	6
40	17	17	11	10	7	7
41	18	17	11	10	8	7
42	18	18	11	10	8	7
43	19	18	11	11	8	7
44	19	18	12	11	8	7
45	20	19	12	11	8	8
46	20	19	12	12	8	8
47	20	20	13	12	9	8
48	21	20	13	12	9	8
49	21	20	13	12	9	8
50	22	21	13	12	9	8
51	22	21	14	13	9	8
52	23	22	14	13	10	9
53	23	22	14	13	10	9
54	23	22	14	14	10	9
55	24	23	15	14	10	9
56	24	23	15	14	10	9
57	25	24	15	14	10	10
58	25	24	15	14	11	10
59	26	25	16	15	11	10
60	26	25	16	15	11	10
″	26	25	16	15	11	10

Proportional Parts

′	*l* sin 9.	d 1′	*l* csc 10.	*l* tan 9.	d 1′	*l* cot 10.	*l* sec 10.	d 1′	*l* cos 9.	′
0	80807	15	19193	92381	26	07619	11575	10	88425	60
1	822	15	178	407	26	593	585	11	415	59
2	837	15	163	433	25	567	596	10	404	58
3	852	15	148	458	26	542	606	11	394	57
4	867	15	133	484	26	516	617	11	383	56
5	882	15	118	510	25	490	628	10	372	55
6	897	15	103	535	26	465	638	11	362	54
7	912	15	088	561	26	439	649	11	351	53
8	927	15	073	587	25	413	660	10	340	52
9	942	15	058	612	26	388	670	11	330	51
10	957	15	043	638	25	362	681	11	319	50
11	972	15	028	663	26	337	692	10	308	49
12	987	15	013	689	26	311	702	11	298	48
13	81002	15	18998	715	25	285	713	11	287	47
14	017	15	983	740	26	260	724	10	276	46
15	032	15	968	766	26	234	734	11	266	45
16	047	14	953	792	25	208	745	11	255	44
17	061	15	939	817	26	183	756	10	244	43
18	076	15	924	843	25	157	766	11	234	42
19	091	15	909	868	26	132	777	11	223	41
20	106	15	894	894	26	106	788	11	212	40
21	121	15	879	920	25	080	799	10	201	39
22	136	15	864	945	26	055	809	11	191	38
23	151	15	849	971	25	029	820	11	180	37
24	166	14	834	996	26	004	831	11	169	36
25	180	15	820	93022	26	06978	842	10	158	35
26	195	15	805	048	25	952	852	11	148	34
27	210	15	790	073	26	927	863	11	137	33
28	225	15	775	099	25	901	874	11	126	32
29	240	14	760	124	26	876	885	10	115	31
30	81254	15	18746	93150	25	06850	11895	11	88105	30
31	269	15	731	175	26	825	906	11	094	29
32	284	15	716	201	26	799	917	11	083	28
33	299	15	701	227	25	773	928	11	072	27
34	314	14	686	252	26	748	939	10	061	26
35	328	15	672	278	25	722	949	11	051	25
36	343	15	657	303	26	697	960	11	040	24
37	358	14	642	329	25	671	971	11	029	23
38	372	15	628	354	26	646	982	11	018	22
39	387	15	613	380	26	620	993	11	007	21
40	402	15	598	406	25	594	12004	11	87996	20
41	417	14	583	431	26	569	015	10	985	19
42	431	15	569	457	25	543	025	11	975	18
43	446	15	554	482	26	518	036	11	964	17
44	461	14	539	508	25	492	047	11	953	16
45	475	15	525	533	26	467	058	11	942	15
46	490	15	510	559	25	441	069	11	931	14
47	505	14	495	584	26	416	080	11	920	13
48	519	15	481	610	26	390	091	11	909	12
49	534	15	466	636	25	364	102	11	898	11
50	549	14	451	661	26	339	113	10	887	10
51	563	15	437	687	25	313	123	11	877	9
52	578	14	422	712	26	288	134	11	866	8
53	592	15	408	738	25	262	145	11	855	7
54	607	15	393	763	26	237	156	11	844	6
55	622	14	378	789	25	211	167	11	833	5
56	636	15	364	814	26	186	178	11	822	4
57	651	14	349	840	25	160	189	11	811	3
58	665	15	335	865	26	135	200	11	800	2
59	680	14	320	891	25	109	211	11	789	1
60	81694		18306	93916		06084	12222		87778	0
′	9. *l* cos	d 1′	10. *l* sec	9. *l* cot	d 1′	10. *l* tan	10. *l* csc	d 1′	9. *l* sin	′

″	Proportional Parts 26	25	15	14	11	10
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	1	1	0	0	0	0
3	1	1	1	1	1	0
4	2	2	1	1	1	1
5	2	2	1	1	1	1
6	3	2	2	1	1	1
7	3	3	2	2	1	1
8	3	3	2	2	1	1
9	4	4	2	2	2	2
10	4	4	2	2	2	2
11	5	5	3	3	2	2
12	5	5	3	3	2	2
13	6	5	3	3	2	2
14	6	6	4	3	3	2
15	6	6	4	4	3	2
16	7	7	4	4	3	3
17	7	7	4	4	3	3
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25	11	10	6	6	5	4
26	11	11	6	6	5	4
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29	13	12	7	7	5	5
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31	13	13	8	7	6	5
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33	14	14	8	8	6	6
34	15	14	8	8	6	6
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36	16	15	9	8	7	6
37	16	15	9	9	7	6
38	16	16	10	9	7	6
39	17	16	10	9	7	6
40	17	17	10	9	7	7
41	18	17	10	10	8	7
42	18	18	10	10	8	7
43	19	18	11	10	8	7
44	19	18	11	10	8	7
45	20	19	11	10	8	8
46	20	19	12	11	8	8
47	20	20	12	11	9	8
48	21	20	12	11	9	8
49	21	20	12	11	9	8
50	22	21	12	12	9	8
51	22	21	13	12	9	8
52	23	22	13	12	10	9
53	23	22	13	12	10	9
54	23	22	14	13	10	9
55	24	23	14	13	10	9
56	24	23	14	13	10	9
57	25	24	14	13	10	10
58	25	24	14	14	11	10
59	26	25	15	14	11	10
60	26	25	15	14	11	10
″	26	25	15	14	11	10
	Proportional Parts					

′	*l* sin 9.	d 1′	*l* csc 10.	*l* tan 9.	d 1′	*l* cot 10.	*l* sec 10.	d 1′	*l* cos 9.	′
0	81694	15	18306	93916	26	06084	12222	11	87778	60
1	709	14	291	942	25	058	233	11	767	59
2	723	15	277	967	26	033	244	11	756	58
3	738	14	262	993	25	007	255	11	745	57
4	752	15	248	94018	26	05982	266	11	734	56
5	767	14	233	044	25	956	277	11	723	55
6	781	15	219	069	26	931	288	11	712	54
7	796	14	204	095	25	905	299	11	701	53
8	810	15	190	120	26	880	310	11	690	52
9	825	14	175	146	25	854	321	11	679	51
10	839	15	161	171	26	829	332	11	668	50
11	854	14	146	197	25	803	343	11	657	49
12	868	14	132	222	26	778	354	11	646	48
13	882	15	118	248	25	752	365	11	635	47
14	897	14	103	273	26	727	376	11	624	46
15	911	15	089	299	25	701	387	12	613	45
16	926	14	074	324	26	676	399	11	601	44
17	940	15	060	350	25	650	410	11	590	43
18	955	14	045	375	26	625	421	11	579	42
19	969	14	031	401	25	599	432	11	568	41
20	983	15	017	426	26	574	443	11	557	40
21	998	14	002	452	25	548	454	11	546	39
22	82012	14	17988	477	26	523	465	11	535	38
23	026	15	974	503	25	497	476	11	524	37
24	041	14	959	528	26	472	487	12	513	36
25	055	14	945	554	25	446	499	11	501	35
26	069	15	931	579	25	421	510	11	490	34
27	084	14	916	604	26	396	521	11	479	33
28	098	14	902	630	25	370	532	11	468	32
29	112	14	888	655	26	345	543	11	457	31
30	82126	15	17874	94681	25	05319	12554	12	87446	30
31	141	14	859	706	26	294	566	11	434	29
32	155	14	845	732	25	268	577	11	423	28
33	169	15	831	757	26	243	588	11	412	27
34	184	14	816	783	25	217	599	11	401	26
35	198	14	802	808	26	192	610	12	390	25
36	212	14	788	834	25	166	622	11	378	24
37	226	14	774	859	25	141	633	11	367	23
38	240	15	760	884	26	116	644	11	356	22
39	255	14	745	910	25	090	655	11	345	21
40	269	14	731	935	26	065	666	12	334	20
41	283	14	717	961	25	039	678	11	322	19
42	297	14	703	986	26	014	689	11	311	18
43	311	15	689	95012	25	04988	700	12	300	17
44	326	14	674	037	25	963	712	11	288	16
45	340	14	660	062	26	938	723	11	277	15
46	354	14	646	088	25	912	734	11	266	14
47	368	14	632	113	26	887	745	12	255	13
48	382	14	618	139	25	861	757	11	243	12
49	396	14	604	164	26	836	768	11	232	11
50	410	14	590	190	25	810	779	12	221	10
51	424	15	576	215	25	785	791	11	209	9
52	439	14	561	240	26	760	802	11	198	8
53	453	14	547	266	25	734	813	12	187	7
54	467	14	533	291	26	709	825	11	175	6
55	481	14	519	317	25	683	836	11	164	5
56	495	14	505	342	26	658	847	12	153	4
57	509	14	491	368	25	632	859	11	141	3
58	523	14	477	393	25	607	870	11	130	2
59	537	14	463	418	26	582	881	12	119	1
60	82551		17449	95444		04556	12893		87107	0
′	9. *l* cos	d 1′	10. *l* sec	9. *l* cot	d 1′	10. *l* tan	10. *l* csc	d 1′	9. *l* sin	′

″	Proportional Parts 26	25	15	14	12	11
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	1	1	0	0	0	0
3	1	1	1	1	1	1
4	2	2	1	1	1	1
5	2	2	1	1	1	1
6	3	2	2	1	1	1
7	3	3	2	2	1	1
8	3	3	2	2	2	1
9	4	4	2	2	2	2
10	4	4	2	2	2	2
11	5	5	3	3	2	2
12	5	5	3	3	2	2
13	6	5	3	3	3	2
14	6	6	4	3	3	3
15	6	6	4	4	3	3
16	7	7	4	4	3	3
17	7	7	4	4	3	3
18	8	8	4	4	4	3
19	8	8	5	4	4	3
20	9	8	5	5	4	4
21	9	9	5	5	4	4
22	10	9	6	5	4	4
23	10	10	6	5	5	4
24	10	10	6	6	5	4
25	11	10	6	6	5	5
26	11	11	6	6	5	5
27	12	11	7	6	5	5
28	12	12	7	7	6	5
29	13	12	7	7	6	5
30	13	12	8	7	6	6
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38	16	16	10	9	8	7
39	17	16	10	9	8	7
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41	18	17	10	10	8	8
42	18	18	10	10	8	8
43	19	18	11	10	9	8
44	19	18	11	10	9	8
45	20	19	11	10	9	8
46	20	19	12	11	9	8
47	20	20	12	11	9	9
48	21	20	12	11	10	9
49	21	20	12	11	10	9
50	22	21	12	12	10	9
51	22	21	13	12	10	9
52	23	22	13	12	10	10
53	23	22	13	12	11	10
54	23	22	14	13	11	10
55	24	23	14	13	11	10
56	24	23	14	13	11	10
57	25	24	14	13	11	10
58	25	24	14	14	12	11
59	26	25	15	14	12	11
60	26	25	15	14	12	11
″	26 Proportional Parts	25	15	14	12	11

′	l sin 9.	d 1′	l csc 10.	l tan 9.	d 1′	l cot 10.	l sec 10.	d 1′	l cos 9.	′
0	82551	14	17449	95444	25	04556	12893	11	87107	60
1	565	14	435	469	26	531	904	11	096	59
2	579	14	421	495	25	505	915	12	085	58
3	593	14	407	520	25	480	927	11	073	57
4	607	14	393	545	26	455	938	12	062	56
5	621	14	379	571	25	429	950	11	050	55
6	635	14	365	596	26	404	961	11	039	54
7	649	14	351	622	25	378	972	12	028	53
8	663	14	337	647	25	353	984	11	016	52
9	677	14	323	672	26	328	995	12	005	51
10	691	14	309	698	25	302	13007	11	86993	50
11	705	14	295	723	25	277	018	12	982	49
12	719	14	281	748	26	252	030	11	970	48
13	733	14	267	774	25	226	041	12	959	47
14	747	14	253	799	26	201	053	11	947	46
15	761	14	239	825	25	175	064	12	936	45
16	775	13	225	850	25	150	076	11	924	44
17	788	14	212	875	26	125	087	11	913	43
18	802	14	198	901	25	099	098	12	902	42
19	816	14	184	926	26	074	110	11	890	41
20	830	14	170	952	25	048	121	12	879	40
21	844	14	156	977	25	023	133	12	867	39
22	858	14	142	96002	26	03998	145	11	855	38
23	872	13	128	028	25	972	156	12	844	37
24	885	14	115	053	25	947	168	11	832	36
25	899	14	101	078	26	922	179	12	821	35
26	913	14	087	104	25	896	191	11	809	34
27	927	14	073	129	26	871	202	12	798	33
28	941	14	059	155	25	845	214	11	786	32
29	955	13	045	180	25	820	225	12	775	31
30	82968	14	17032	96205	26	03795	13237	11	86763	30
31	982	14	018	231	25	769	248	12	752	29
32	996	14	004	256	25	744	260	12	740	28
33	83010	13	16990	281	26	719	272	11	728	27
34	023	14	977	307	25	693	283	12	717	26
35	037	14	963	332	25	668	295	11	705	25
36	051	14	949	357	26	643	306	12	694	24
37	065	13	935	383	25	617	318	12	682	23
38	078	14	922	408	25	592	330	11	670	22
39	092	14	908	433	26	567	341	12	659	21
40	106	14	894	459	25	541	353	12	647	20
41	120	13	880	484	26	516	365	11	635	19
42	133	14	867	510	25	490	376	12	624	18
43	147	14	853	535	25	465	388	12	612	17
44	161	13	839	560	26	440	400	11	600	16
45	174	14	826	586	25	414	411	12	589	15
46	188	14	812	611	25	389	423	12	577	14
47	202	13	798	636	26	364	435	11	565	13
48	215	14	785	662	25	338	446	12	554	12
49	229	13	771	687	25	313	458	12	542	11
50	242	14	758	712	26	288	470	12	530	10
51	256	14	744	738	25	262	482	11	518	9
52	270	13	730	763	25	237	493	12	507	8
53	283	14	717	788	26	212	505	12	495	7
54	297	13	703	814	25	186	517	11	483	6
55	310	14	690	839	25	161	528	12	472	5
56	324	14	676	864	26	136	540	12	460	4
57	338	13	662	890	25	110	552	12	448	3
58	351	14	649	915	25	085	564	11	436	2
59	365	13	635	940	26	060	575	12	425	1
60	83378		16622	96966		03034	13587		86413	0
′	9. l cos	d 1′	10. l sec	9. l cot	d 1′	10. l tan	10. l csc	d 1′	9. l sin	′

″	Proportional Parts 26	25	14	13	12	11
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	1	1	0	0	0	0
3	1	1	1	1	1	1
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5	2	2	1	1	1	1
6	3	2	1	1	1	1
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8	3	3	2	2	2	1
9	4	4	2	2	2	2
10	4	4	2	2	2	2
11	5	5	3	2	2	2
12	5	5	3	3	2	2
13	6	5	3	3	3	2
14	6	6	3	3	3	3
15	6	6	4	3	3	3
16	7	7	4	3	3	3
17	7	7	4	4	3	3
18	8	8	4	4	4	3
19	8	8	4	4	4	3
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22	10	9	5	5	4	4
23	10	10	5	5	5	4
24	10	10	6	5	5	4
25	11	10	6	5	5	5
26	11	11	6	6	5	5
27	12	11	6	6	5	5
28	12	12	7	6	6	5
29	13	12	7	6	6	5
30	13	12	7	6	6	6
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37	16	15	9	8	7	7
38	16	16	9	8	8	7
39	17	16	9	8	8	7
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41	18	17	10	9	8	8
42	18	18	10	9	8	8
43	19	18	10	9	9	8
44	19	18	10	10	9	8
45	20	19	10	10	9	8
46	20	19	11	10	9	8
47	20	20	11	10	9	9
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55	24	23	13	12	11	10
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57	25	24	13	12	11	10
58	25	24	14	13	12	11
59	26	25	14	13	12	11
60	26	25	14	13	12	11
″	26	25	14	13	12	11
	Proportional Parts					

′	l sin 9.	d 1′	l csc 10.	l tan 9.	d 1′	l cot 10.	l sec 10.	d 1′	l cos 9.	′
0	83378	14	16622	96966	25	03034	13587	12	86413	60
1	392	13	608	991	25	009	599	12	401	59
2	405	14	595	97016	26	02984	611	12	389	58
3	419	13	581	042	25	958	623	11	377	57
4	432	14	568	067	25	933	634	12	366	56
5	446	13	554	092	26	908	646	12	354	55
6	459	14	541	118	25	882	658	12	342	54
7	473	13	527	143	25	857	670	12	330	53
8	486	14	514	168	25	832	682	12	318	52
9	500	13	500	193	26	807	694	11	306	51
10	513	14	487	219	25	781	705	12	295	50
11	527	13	473	244	25	756	717	12	283	49
12	540	14	460	269	26	731	729	12	271	48
13	554	13	446	295	25	705	741	12	259	47
14	567	14	433	320	25	680	753	12	247	46
15	581	13	419	345	26	655	765	12	235	45
16	594	14	406	371	25	629	777	12	223	44
17	608	13	392	396	25	604	789	11	211	43
18	621	13	379	421	26	579	800	12	200	42
19	634	14	366	447	25	553	812	12	188	41
20	648	13	352	472	25	528	824	12	176	40
21	661	13	339	497	26	503	836	12	164	39
22	674	14	326	523	25	477	848	12	152	38
23	688	13	312	548	25	452	860	12	140	37
24	701	14	299	573	25	427	872	12	128	36
25	715	13	285	598	26	402	884	12	116	35
26	728	13	272	624	25	376	896	12	104	34
27	741	14	259	649	25	351	908	12	092	33
28	755	13	245	674	26	326	920	12	080	32
29	768	13	232	700	25	300	932	12	068	31
30	83781	14	16219	97725	25	02275	13944	12	86056	30
31	795	13	205	750	26	250	956	12	044	29
32	808	13	192	776	25	224	968	12	032	28
33	821	13	179	801	25	199	980	12	020	27
34	834	14	166	826	25	174	992	12	008	26
35	848	13	152	851	26	149	14004	12	85996	25
36	861	13	139	877	25	123	016	12	984	24
37	874	13	126	902	25	098	028	12	972	23
38	887	14	113	927	26	073	040	12	960	22
39	901	13	099	953	25	047	052	12	948	21
40	914	13	086	978	25	022	064	12	936	20
41	927	13	073	98003	26	01997	076	12	924	19
42	940	14	060	029	25	971	088	12	912	18
43	954	13	046	054	25	946	100	12	900	17
44	967	13	033	079	25	921	112	12	888	16
45	980	13	020	104	26	896	124	12	876	15
46	993	13	007	130	25	870	136	13	864	14
47	84006	14	15994	155	25	845	149	12	851	13
48	020	13	980	180	26	820	161	12	839	12
49	033	13	967	206	25	794	173	12	827	11
50	046	13	954	231	25	769	185	12	815	10
51	059	13	941	256	25	744	197	12	803	9
52	072	13	928	281	26	719	209	12	791	8
53	085	13	915	307	25	693	221	13	779	7
54	098	14	902	332	25	668	234	12	766	6
55	112	13	888	357	26	643	246	12	754	5
56	125	13	875	383	25	617	258	12	742	4
57	138	13	862	408	25	592	270	12	730	3
58	151	13	849	433	25	567	282	12	718	2
59	164	13	836	458	26	542	294	13	706	1
60	84177		15823	98484		01516	14307		85693	0
′	9. l cos	d 1′	10. l sec	9. l cot	d 1′	10. l tan	10. l csc	d 1′	9. l sin	′

Proportional Parts

″	26	25	14	13	12	11
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	1	1	0	0	0	0
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5	2	2	1	1	1	1
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7	3	3	2	2	1	1
8	3	3	2	2	2	1
9	4	4	2	2	2	2
10	4	4	2	2	2	2
11	5	5	3	2	2	2
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22	10	9	5	5	4	4
23	10	10	5	5	5	4
24	10	10	6	5	5	4
25	11	10	6	5	5	5
26	11	11	6	6	5	5
27	12	11	6	6	5	5
28	12	12	7	6	6	5
29	13	12	7	6	6	5
30	13	12	7	6	6	6
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56	24	23	13	12	11	10
57	25	24	13	12	11	10
58	25	24	14	13	12	11
59	26	25	14	13	12	11
60	26	25	14	13	12	11
″	26	25	14	13	12	11

Proportional Parts

′	l sin 9.	d 1′	l csc 10.	l tan 9.	d 1′	l cot 10.	l sec 10	d 1′	l cos 9.	′
0	84177	13	15823	98484	25	01516	14307	12	85693	60
1	190	13	810	509	25	491	319	12	681	59
2	203	13	797	534	26	466	331	12	669	58
3	216	13	784	560	25	440	343	12	657	57
4	229	13	771	585	25	415	355	13	645	56
5	242	13	758	610	25	390	368	12	632	55
6	255	14	745	635	26	365	380	12	620	54
7	269	13	731	661	25	339	392	12	608	53
8	282	13	718	686	25	314	404	13	596	52
9	295	13	705	711	26	289	417	12	583	51
10	308	13	692	737	25	263	429	12	571	50
11	321	13	679	762	25	238	441	12	559	49
12	334	13	666	787	25	213	453	13	547	48
13	347	13	653	812	26	188	466	12	534	47
14	360	13	640	838	25	162	478	12	522	46
15	373	12	627	863	25	137	490	13	510	45
16	385	13	615	888	25	112	503	12	497	44
17	398	13	602	913	26	087	515	12	485	43
18	411	13	589	939	25	061	527	13	473	42
19	424	13	576	964	25	036	540	12	460	41
20	437	13	563	989	26	011	552	12	448	40
21	450	13	550	99015	25	00985	564	13	436	39
22	463	13	537	040	25	960	577	12	423	38
23	476	13	524	065	25	935	589	12	411	37
24	489	13	511	090	26	910	601	13	399	36
25	502	13	498	116	25	884	614	12	386	35
26	515	13	485	141	25	859	626	13	374	34
27	528	12	472	166	25	834	639	12	361	33
28	540	13	460	191	26	809	651	12	349	32
29	553	13	447	217	25	783	663	13	337	31
30	84566	13	15434	99242	25	00758	14676	12	85324	30
31	579	13	421	267	26	733	688	13	312	29
32	592	13	408	293	25	707	701	12	299	28
33	605	13	395	318	25	682	713	13	287	27
34	618	12	382	343	25	657	726	12	274	26
35	630	13	370	368	26	632	738	12	262	25
36	643	13	357	394	25	606	750	13	250	24
37	656	13	344	419	25	581	763	12	237	23
38	669	13	331	444	25	556	775	13	225	22
39	682	12	318	469	26	531	788	12	212	21
40	694	13	306	495	25	505	800	13	200	20
41	707	13	293	520	25	480	813	12	187	19
42	720	13	280	545	25	455	825	13	175	18
43	733	12	267	570	26	430	838	12	162	17
44	745	13	255	596	25	404	850	13	150	16
45	758	13	242	621	25	379	863	12	137	15
46	771	13	229	646	26	354	875	13	125	14
47	784	12	216	672	25	328	888	12	112	13
48	796	13	204	697	25	303	900	13	100	12
49	809	13	191	722	25	278	913	13	087	11
50	822	13	178	747	26	253	926	12	074	10
51	835	12	165	773	25	227	938	13	062	9
52	847	13	153	798	25	202	951	12	049	8
53	860	13	140	823	25	177	963	13	037	7
54	873	12	127	848	26	152	976	12	024	6
55	885	13	115	874	25	126	988	13	012	5
56	898	13	102	899	25	101	15001	13	84999	4
57	911	12	089	924	25	076	014	12	986	3
58	923	13	077	949	26	051	026	13	974	2
59	936	13	064	975	25	025	039	12	961	1
60	84949		15051	00000		00000	15051		84949	0
′	9. l cos	d 1′	10. l sec	10. l cot	d 1′	10. l tan	10. l csc	d 1′	9. l sin	′

″	Proportional Parts 26	25	14	13	12
0	0	0	0	0	0
1	0	0	0	0	0
2	1	1	0	0	0
3	1	1	1	1	1
4	2	2	1	1	1
5	2	2	1	1	1
6	3	2	1	1	1
7	3	3	2	2	1
8	3	3	2	2	2
9	4	4	2	2	2
10	4	4	2	2	2
11	5	5	3	2	2
12	5	5	3	3	2
13	6	5	3	3	3
14	6	6	3	3	3
15	6	6	4	3	3
16	7	7	4	3	3
17	7	7	4	4	3
18	8	8	4	4	4
19	8	8	4	4	4
20	9	8	5	4	4
21	9	9	5	5	4
22	10	9	5	5	4
23	10	10	5	5	5
24	10	10	6	5	5
25	11	10	6	5	5
26	11	11	6	6	5
27	12	11	6	6	5
28	12	12	7	6	6
29	13	12	7	6	6
30	13	12	7	6	6
31	13	13	7	7	6
32	14	13	7	7	6
33	14	14	8	7	7
34	15	14	8	7	7
35	15	15	8	8	7
36	16	15	8	8	7
37	16	15	9	8	7
38	16	16	9	8	8
39	17	16	9	8	8
40	17	17	9	9	8
41	18	17	10	9	8
42	18	18	10	9	8
43	19	18	10	9	9
44	19	18	10	10	9
45	20	19	10	10	9
46	20	19	11	10	9
47	20	20	11	10	9
48	21	20	11	10	10
49	21	20	11	11	10
50	22	21	12	11	10
51	22	21	12	11	10
52	23	22	12	11	10
53	23	22	12	11	11
54	23	22	13	12	11
55	24	23	13	12	11
56	24	23	13	12	11
57	25	24	13	12	11
58	25	24	14	13	12
59	26	25	14	13	12
60	26	25	14	13	12
″	26 Proportional Parts	25	14	13	12